Closed Form Expressions of Linear Continuous Time System Responses ⋆

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Abstract:
General closed form expressions of linear continuous time system responses of an arbitrary order are derived, by first relating them to basic responses, i.e., responses corresponding to unity numerator transfer functions. Those are then related to the fundamental solutions of the underlying differential equations. These expressions apply to all systems without any restrictions on the poles or the zeros, further the systems may be noncausal. We derive responses for all regular types of inputs, impulse, step, ramp, parabola, etc., in addition for all generalized derivatives of the impulse. All the presented results have a direct counterpart in results presented in Sigurðsson et al. (2017) on discrete time systems based on the fundamental solution of the associated difference equation. Efficient evaluations of the fundamental solutions along with their derivatives and integrals can thus be extended to the responses and are readily implemented, e.g., as Matlab functions. Such results may be presented symbolically as functions of time or evaluated numerically at any sequence of times without time stepping.

Keywords: Control education, continuous time systems, time domain responses, linear control systems.

1. INTRODUCTION

Closed form expressions for transfer functions of continuous time systems receive some attention in textbooks on control systems, see e.g. Dorf et al. (2017), Ogata (2010), Franklin et al. (2014), Kailath (1980), and signals and systems, see e.g., Roberts (2011), Chen (2004), Oppenheim (1997). Typically, the task of finding the system time response includes some basic steps. First, linear differential equations are introduced in order to describe the dynamics of a physical system from the laws of physics or from system identification procedures. Then, a Laplace transformation of the differential equations is presented in order to generate the transfer function of the system. Once the transfer function has been found some techniques for direct inverse Laplace transform can be used to obtain the time response for simple transfer functions. A more general solution is then presented using partial fraction expansion in order to simplify more complicated transfer functions into sums of simpler terms, where inverse Laplace transforms can be applied directly.

Little attention has, however, been given to the task of presenting general expressions for continuous time system responses in a unified manner, that may be used as a basis for general algorithms. The main contribution of this paper is to relate such expressions to those of basic responses, referring to the case when we have unity in the numerator of the transfer function, and subsequently to the fundamental solution of the underlying differential equation. As such, these expressions provide a closer analogue to the responses of discrete time systems, than typically attained by treating the latter by $\mathcal{Z}-$transforms. By relating responses of continuous time systems to the fundamental solution of the differential equation, all the results of this paper have a direct counterpart in the results presented in Sigurðsson et al. (2017) on discrete time system responses. The closed forms result in efficient computation of the responses at any sequence of times without timestepping - as well as in symbolic responses. These are readily implemented, e.g., as Matlab functions. As illustrated by an example in this paper, the computational efficiency of such a Matlab function, is comparable to that of the Matlab function `lsim`, when the number of output points is the same. Moreover, when the input is not linear and the number of chosen output values is smaller than that required to obtain sufficient accuracy with `lsim`, the approach in this paper offers a potential computational gain over `lsim`.

Earlier closed form expressions were initially developed by the first author in Hauksdóttir (1996), for systems with non-repeated poles. Since then they have been extended to handle repeated poles and applied, e.g., in model reduction, see Herjólfsson et al. (2009), as well as to the optimal computation of PID zeros in Herjólfsson et al. (2005) and Herjólfsson et al. (2012). Other potential applications

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related to PID control are, e.g., in response evaluations related to tracking and disturbance rejection characteristics, when we wish to concentrate on the response at critical time intervals.

The closed form expressions in this paper are related more explicitly to the fundamental solution of the associated differential equation, simplifying the proofs. They are also extended so that there are no restrictions on either the poles or the zeros, i.e., those may be real, complex, repeated, stable, marginally stable or unstable. The forcing functions are restricted to power products of a given order, i.e., step, ramp, parabola, etc., but also include impulse and have been extended to the generalized derivatives of the delta function in this paper, allowing the inclusion of noncausal systems for the sake of completeness. In all these cases, that include polynomial inputs of any order by linearity, we get a closed form expression for the output. Moreover, as shown in Hauksdóttir et al. (2018), where a similar approach to responses of a general MIMO-system is presented for a general input $u(t)$, the output in this case can be expressed in terms of a convolution between $u(t)$ and a vector of the fundamental solution and its derivatives and integrals. Combining such general expressions with the results of this paper will clearly increase their scope with respect to efficient response evaluation.

Assumptions and the forms of possible forcing functions are introduced in Sections 2 and 3. The relationship between basic responses and fundamental solutions is derived in Section 4. The result for general responses is presented in Section 5. Computational issues including an example and response interpretation are discussed in 6 and 7, respectively. Conclusions are discussed in 8.

2. ASSUMPTIONS

Consider the n-th order linear continuous-time system

$$\sum_{i=0}^{n} a_i y^{(i)}(t) = \sum_{i=0}^{m} b_i u^{(i)}(t), \quad a_n = 1, \quad a_0 \neq 0, \tag{1}$$

assuming that $y(t) = 0$ for $t < 0$. Here $u(t)$ is a forcing function of a given order γ and the system corresponds to the transfer function

$$\frac{Y(s)}{U(s)} = \frac{b_m s^m + b_{m-1}s^{m-1} + \to + b_0}{s^n + a_{n-1}s^{n-1} + \to + a_0} = \frac{b(s)}{a(s)}. \tag{2}$$

Note that $b(s)$ and $a(s)$ may have common factors that do not need to be cancelled.

It turns out to be convenient to assume that the characteristic equation has no zero roots, i.e., $a_0 \neq 0$, for the sake of analysis. This is no real restriction, because if we were to add a zero root of order n_0, so that the denominator in (2), i.e., the characteristic equation, becomes $s^{n_0}\left(s^n + a_{n-1}s^{n-1} + \to + a_0\right)$, this can be taken care of simply by increasing the order of the forcing function to $\gamma + n_0$.

It means, though, that we do not include systems that only have zero poles, i.e., $a_i = 0,\ i = 0, 1, \ldots, n-1$. However, such a system, takes the form

$$y^{(n)}(t) = \sum_{i=0}^{m} b_i u^{(i)}(t) \tag{3}$$

and is readily solved by repeated integration, e.g., in the case of PID controllers.

3. FORCING FUNCTIONS

As stated in the introduction, we are restricting the attention to forcing functions that are power products of a given order, as well as the Dirac delta function, $\delta(t)$. We choose to define

$$\begin{aligned} I_0(t) &= \delta(t) \\ I_\gamma(t) &= \frac{t^{\gamma-1}}{(\gamma-1)!}, \quad \gamma \geq 1, \quad t \geq 0. \end{aligned} \tag{4}$$

Further, $I_\gamma(t) = 0$, $\forall t < 0$, $\gamma \geq 0$. Thus, $I_1(t)$ is the Heaviside step function.

The forcing functions $I_\gamma(t), \gamma = 0, 1, 2, \cdots$ thus correspond to the impulse input, the step input, the ramp input, the parabolic input, etc., with the Laplace transforms

$$\mathcal{L}\{I_\gamma(t)\} = \frac{1}{s^\gamma}, \tag{5}$$

explaining the fact why a zero pole of order n_0 effectively corresponds to an increase in the degree of the forcing function by n_0.

We shall also make use of generalized derivatives, so that $\frac{d}{dt}I_1(t) = I_0(t)$, and denote the generalized $\gamma - th$ derivative of $\delta(t)$ with $I_{-\gamma}(t)$, i.e.,

$$I_{-\gamma}(t) = I_0^{(\gamma)}(t), \quad \gamma \geq 0. \tag{6}$$

We also allow (6) as possible forcing functions for the sake of generality, noting that (5) also holds true for $\gamma < 0$.

It is common to define the forcing function of order $\gamma \geq 1$ simply as $t^{\gamma-1}$. The reason for adopting the choice (4) is, apart from the correspondence (5), the fact that

$$\begin{aligned} \frac{d}{dt}I_{\gamma+1}(t) &= I_\gamma(t), \\ I_{\gamma+1}(t) &= \int_0^t I_\gamma(\tau)d\tau. \end{aligned} \tag{7}$$

We further note that in the sense of generalized functions, (7) holds for all integer values of γ.

Denoting the response when the forcing function $I_\gamma(t)$ is applied to (1) by $y_\gamma(t)$, it follows by linearity, that

$$\begin{aligned} \frac{d}{dt}y_{\gamma+1}(t) &= y_\gamma(t), \\ y_{\gamma+1}(t) &= \int_0^t y_\gamma(\tau)d\tau. \end{aligned} \tag{8}$$

Thus, as well known, the impulse response, the step response, the ramp response, etc., are related in the same manner as the inputs themselves.

More generally, it also follows by linearity, that if the forcing function is a linear combination of the functions in (4) and (6)

$$u(t) = \sum_\sigma f_\sigma I_\sigma(t), \tag{9}$$

then the response will be the corresponding linear combination

$$y(t) = \sum_\sigma f_\sigma y_\sigma(t). \tag{10}$$

4. BASIC RESPONSES AND FUNDAMENTAL SOLUTIONS

We introduce the notation $y_b(t)$ for the basic impulse response, i.e., $y_b(t)$ is the response of the system

$$\sum_{i=0}^{n} a_i y^{(i)}(t) = I_0(t). \tag{11}$$

We note that for $t \geq 0$, $y_f(t)$, also referred to as the fundamental solution, can be characterized as the solution to the differential equation

$$\sum_{i=0}^{n} a_i y^{(i)}(t) = 0 \tag{12}$$

satisfying the initial conditions

$$y^{(i)}(0) = 0, \quad i = 0, 1, \ldots, n-2, \quad y^{(n-1)}(0) = 1. \tag{13}$$

However, whereas the fundamental solution $y_f(t)$ with all its derivatives extend continuously to $t < 0$, the $(n-1)$st derivative of the basic impulse response $y_b(t)$ has a discontinuous jump from 0 to 1 at $t = 0$. We denote the fundamental solution by $y_f(t)$ in order to make this distinction and we have the following result in terms of generalized derivatives (6) at $t = 0$.

Lemma 1: Consider the γ-th differential of the basic impulse response $y_b^{(\gamma)}(t)$. Then

$$y_b^{(\gamma)}(t) = y_f^{(\gamma)}(t) I_1(t) + \sum_{i=n}^{\gamma} y_f^{(i-1)}(0) I_{i-\gamma}(t), \quad \gamma \geq 0. \tag{14}$$

Proof: We prove this by induction.

The result clearly holds for $\gamma = 0, 1, \ldots, n-1$, since then $y_b^{(\gamma)}(t) = y_f^{(\gamma)}(t) I_1(t)$.

Assume that it holds for $\gamma = k \geq n-1$. Then it follows from (7) that

$$\begin{aligned} y_b^{(k+1)}(t) &= y_f^{(k+1)}(t) I_1(t) + y_f^{(k)}(t) I_0(t) \\ &\quad + \sum_{i=n}^{k} y_f^{(i-1)}(0) I_{i-k-1}(t) \\ &= y_f^{(k+1)}(t) I_1(t) + \sum_{i=n}^{k+1} y_f^{(i-1)}(0) I_{i-k-1}(t), \end{aligned} \tag{15}$$

i.e., the result also holds for $\gamma = k+1$ and hence for all $\gamma \geq 0$. q.e.d.

We observe that

$$y_b^{(\gamma)}(t) = \mathcal{L}^{-1}\left\{ \frac{1}{a(s)} s^{\gamma} \right\}. \tag{16}$$

We now introduce the notation $y_{b,\gamma}(t)$ for the basic response to forcing function $I_\gamma(t)$. For $\gamma \geq 1$, $y_{b,\gamma}(t)$ can be characterized as the solution to the differential equation

$$\sum_{i=0}^{n} a_i y^{(i)}(t) = I_\gamma(t), \tag{17}$$

satisfying the initial conditions

$$y^{(i)}(0) = 0, \quad i = 0, 1, \ldots, n-1. \tag{18}$$

The basic response $y_{b,\gamma}(t)$ is composed of two components, a fundamental component $y_{f,\gamma}(t)$, also termed the transient response, satisfying (12) and a particular component, also termed the forced response satisfying (17). We then have the following result.

Lemma 2: Consider the basic response with the forcing function $I_\gamma(t)$, $y_{b,\gamma}(t)$. Then

$$y_{b,\gamma}(t) = y_{f,\gamma}(t) I_1(t) - \sum_{i=0}^{\gamma-1} y_{f,i+1}(0) I_{\gamma-i}(t), \quad \gamma \geq 0. \tag{19}$$

Proof: We proof this by induction.

First note that from (8)

$$\frac{d}{dt} y_{b,\gamma+1}(t) = y_{b,\gamma}(t) \quad \text{for } \gamma \geq 0, \quad t \geq 0. \tag{20}$$

It follows that

$$\frac{d}{dt} y_{f,\gamma+1}(t) = y_{f,\gamma}(t) \quad \text{for } \gamma \geq 0, \quad t \geq 0, \tag{21}$$

since our assumption that the system has no zero poles implies that the derivative of the particular/forced component of $y_{b,\gamma+1}(t)$ cannot contribute to the fundamental/transient component of $y_{b,\gamma}(t)$.

The result clearly holds for $\gamma = 0$. Assume that it holds for $\gamma = k \geq 0$. Then it follows from (21), (7) and (8) that

$$\begin{aligned} y_{b,k+1}(t) &= \int_0^t \frac{d}{dt} y_{f,k+1}(\tau) d\tau I_1(t) \\ &\quad - \sum_{i=0}^{k-1} y_{f,i+1}(0) \int_0^t I_{k-i}(\tau) d\tau \\ &= \left(y_{f,k+1}(t) - y_{f,k+1}(0) \right) I_1(t) \\ &\quad - \sum_{i=0}^{k-1} y_{f,i+1}(0) I_{k+1-i}(t) \\ &= y_{f,k+1}(t) I_1(t) - \sum_{i=0}^{k} y_{f,i+1}(0) I_{k+1-i}(t), \end{aligned} \tag{22}$$

i.e., the result also holds for $\gamma = k+1$ and hence for all $\gamma \geq 0$. q.e.d.

We observe that

$$y_{b,\gamma}(t) = \mathcal{L}^{-1}\left\{ \frac{1}{a(s)} \frac{1}{s^{\gamma}} \right\}. \tag{23}$$

We now introduce the notation that

$$y_f^{(-\gamma)}(t) = y_{f,\gamma}(t), \quad \gamma \geq 0, \quad t \geq 0 \tag{24}$$

and applying this to (21)

$$\frac{d}{dt} y_f^{(-\gamma-1)}(t) = y_f^{(-\gamma)}(t) \quad \gamma \geq 0. \tag{25}$$

It then follows that

$$\frac{d}{dt} y_f^{(\gamma)}(t) = y_f^{(\gamma+1)}(t) \quad \forall \gamma. \tag{26}$$

Further, note that by introducing generalized forcing functions $I_{-\gamma}(t) = I_0^{(\gamma)}(t)$ as in (6), it follows from (7) that

$$y_{b,-\gamma}(t) = y_b^{(\gamma)}(t) \quad \gamma \geq 0, \quad t \geq 0. \tag{27}$$

Theorem 1: The basic response of system (1) to forcing function $I_\gamma(t)$, (4) or (6) is given by

$$\begin{aligned} y_{b,\gamma}(t) &= y_f^{(-\gamma)}(t) I_1(t) - \sum_{i=0}^{\gamma-1} y_f^{(-i-1)}(0) I_{\gamma-i}(t) \\ &\quad + \sum_{i=n}^{-\gamma} y_f^{(i-1)}(0) I_{i+\gamma}(t) \quad \forall \gamma. \end{aligned} \tag{28}$$

Proof: We combine (14) in Lemma 1, reversing the sign on γ, and (19) in Lemma 2 into a single result, valid for all γ. q.e.d.

We note that the sum in Theorem 1 with the negative sign (integrating effect from Lemma 2) is only present if $\gamma > 0$. The sum with the positive sign (differentiating effect from Lemma 1) is only present if $\gamma \leq -n$. Thus at most only one of the sums is present for any one value of γ and no sum is present if $-n < \gamma \leq 0$.

5. GENERAL RESPONSES

Introduce the l-vector function including differentials and/or integrals of the basic impulse response

$$\mathcal{Y}_l^{(j)}(t) = \left[\, y_f^{(j)}(t)\ y_f^{(j+1)}(t)\ \cdots\ y_f^{(j+l-1)}(t) \,\right]^T, \forall j, \quad (29)$$

and the $(j-i+1)-$vector, for any integers i and j, $i \leq j$,

$$\mathcal{B}_{i,j} = [\, b_i\ b_{i+1} \rightarrow b_j \,]^T, \quad (30)$$

where $b_l = 0$ if $l > m$ or $l < 0$. Then we can state:

Theorem 2: Denote by $y_\gamma(t)$, for any integer value γ, positive, negative or zero, the response of the linear system (1) where $n > 0$ and $m \geq 0$, with possibly an additional zero pole of order $n_0 \geq 0$ and with $u(t) = I_\gamma(t)$. Then with $\hat{\gamma} = \gamma + n_0$

$$\begin{aligned} y_\gamma(t) &= \mathcal{B}_{0,m}^T \mathcal{Y}_{m+1}^{(-\hat{\gamma})}(t) I_1(t) \\ &\quad - \sum_{i=0}^{\hat{\gamma}-1} \mathcal{B}_{0,i}^T \mathcal{Y}_{i+1}^{(-i-1)}(0) I_{\hat{\gamma}-i}(t) \\ &\quad + \sum_{i=n}^{m-\hat{\gamma}} \mathcal{B}_{m+n-i,m}^T \mathcal{Y}_{i-n+1}^{(n-1)}(0) I_{-m+i+\hat{\gamma}}(t), \\ &\quad \text{for } t \geq 0. \end{aligned} \quad (31)$$

Proof: Assume first that $n_0 = 0$. We then have by linearity from (28) that

$$\begin{aligned} y_\gamma(t) &= \sum_{l=0}^{m} b_l y_{b,\gamma-l}(t) \\ &= \sum_{l=0}^{m} b_l y_f^{(-\gamma+l)}(t) \\ &\quad - \sum_{l=0}^{\min(\gamma-1,m)} b_l \sum_{i=0}^{\gamma-l-1} y_f^{(-i-1)}(0) I_{\gamma-l-i}(t) \\ &\quad + \sum_{l=\max(0,n+\gamma)}^{m} b_l \sum_{i=n}^{-\gamma+l} y_f^{(i-1)}(0) I_{-l+i+\gamma}(t) \end{aligned} \quad (32)$$

where $min(\gamma-1, m)$ is due to the fact that the inner sum becomes empty if $l > \gamma - 1$ in the forcing function part and $max(0, n+\gamma)$ is due to the fact that the inner sum becomes empty for $l < n + \gamma$ in the noncausal part. Then

$$\begin{aligned} y_\gamma(t) &= \sum_{l=0}^{m} b_l y_f^{(-\gamma+l)}(t) \\ &\quad - \sum_{i=0}^{\gamma-1} \left(\sum_{l=0}^{\min(i,m)} b_l y_f^{(l-i-1)}(0) \right) I_{\gamma-i}(t) \\ &\quad + \sum_{i=n}^{m-\gamma} \left(\sum_{l=0}^{i-n} b_{m-l} y_f^{(i-1-l)}(0) \right) I_{-m+i+\gamma}(t), \end{aligned} \quad (33)$$

which is equivalent to the stated result when $n_0 = 0$.

If the characteristic equation has an additional root of order n_0 at $s = 0$, the denominator in (2) or equivalently the characteristic equation becomes

$$s^{n_0} a(s) = s^{n_0} \left(s^n + a_{n-1} s^{n-1} + \rightarrow + a_0 \right). \quad (34)$$

The factor s^{n_0} can according to (5) be incorporated into the forcing function $u(t)$, by changing it from $I_\gamma(t)$ to $I_{\gamma+n_0}(t)$. Thus (31) also holds for $n_0 > 0$. q.e.d.

Theorem 2 does not include systems of form (3). It follows, however, from (7) and (8) by repeated integration or differentiation, that for any integer value γ the response when $u(t) = I_\gamma(t)$ can be expressed by

$$y_\gamma(t) = \sum_{i=0}^{m} b_i I_{-i+\hat{\gamma}}(t). \quad (35)$$

The usefulness of Theorem 2 is twofold. Firstly, it shows explicitly how the system response relates to the solution of the underlying fundamental solution. Secondly, we have closed form expressions for the fundamental solution, $y_f(t)$ along with its repeated derivatives and the transient part of its integrals, that are readily evaluated, given the poles of the system, as shown in the next section. Thus, the theorem extends such expressions to the general responses of systems of form (1).

6. COMPUTATIONAL ISSUES

In order to compute $y_\gamma(t)$ from Theorem 2 we need to be able to compute the vector function $\mathcal{Y}_{m+1}^{(-\hat{\gamma})}(t)$ as well as the vectors $\mathcal{Y}_{\hat{\gamma}}^{(-\hat{\gamma})}(0)$ if $\hat{\gamma} \geq 1$ and $\mathcal{Y}_{m-n-\hat{\gamma}+1}^{(n-1)}(0)$ if $t = 0$ and $\hat{\gamma} \leq m - n$ and we are interested in the coefficients of the generalized functions $I_{i-m+\hat{\gamma}}(t)$, $i = n, n+1, \ldots, m - \hat{\gamma}$ at $t = 0$. We shown in this section how this can be done in a manner that is readily implemented, e.g., in Matlab.

Assume that we know the roots of the characteristic equation corresponding to (1) denoted by $\lambda_1, \lambda_2, \ldots, \lambda_\nu$ and their multiplicities denoted by $d_1, d_2, \ldots, d_\nu$, respectively, i.e.,

$$\begin{aligned} \frac{1}{a(s)} &= \frac{1}{(s-\lambda_1)^{d_1}(s-\lambda_2)^{d_2} \rightarrow (s-\lambda_\nu)^{d_\nu}} \\ &= \sum_{i=1}^{\nu} \sum_{j=1}^{d_i} \frac{\kappa_{ij}}{(s-\lambda_i)^j}, \end{aligned} \quad (36)$$

where κ_{ij} denotes the basic partial fraction coefficients. We further introduce the n-vector

$$\begin{aligned} \kappa = [\, & \kappa_{11} \leftrightarrow \kappa_{1d_1}\ \kappa_{21} \leftrightarrow \kappa_{2d_2} \\ & \rightarrow \kappa_{\nu 1} \leftrightarrow \kappa_{\nu d_\nu} \,]^T. \end{aligned} \quad (37)$$

Next we introduce the n-vector function

$$\mathcal{E}(t) = \left[\mathcal{E}_1^T(t), \mathcal{E}_2^T(t), \ldots, \mathcal{E}_\nu^T(t) \right]^T, \quad (38)$$

where

$$\mathcal{E}_i(t) = \begin{bmatrix} e^{\lambda_i t} \\ \left.\dfrac{d}{d\lambda} e^{\lambda t}\right|_{\lambda=\lambda_i} \\ \downarrow \\ \left.\dfrac{1}{(d_i-1)!} \dfrac{d^{d_i-1}}{d\lambda^{d_i-1}} e^{\lambda t}\right|_{\lambda=\lambda_i} \end{bmatrix} = \begin{bmatrix} e^{\lambda_i t} \\ t e^{\lambda_i t} \\ \downarrow \\ \dfrac{t^{d_i-1}}{(d_i-1)!} e^{\lambda_i t} \end{bmatrix}. \quad (39)$$

Then it follows

$$y_f(t) = \kappa^T \mathcal{E}(t). \quad (40)$$

Now we introduce the $n \times n$ Jordan matrix

$$J = \begin{bmatrix} J_1 & 0 & \leftrightarrow & 0 \\ 0 & J_2 & \leftarrow & 0 \\ \updownarrow & \ddots & \ddots & \downarrow \\ 0 & \leftrightarrow & 0 & J_\nu \end{bmatrix} \tag{41}$$

with the diagonal blocks

$$J_i = \begin{bmatrix} \lambda_i & 1 & \leftarrow & 0 \\ 0 & \ddots & \ddots & \downarrow \\ \updownarrow & \ddots & \ddots & 1 \\ 0 & \leftrightarrow & 0 & \lambda_i \end{bmatrix} \tag{42}$$

each a $d_i \times d_i$ matrix.

Here we note that

$$\frac{d}{dt}\mathcal{E}(t) = J^T \mathcal{E}(t) \tag{43}$$

and hence from (26) that

$$y_f^{(j)}(t) = \left(J^j \kappa\right)^T \mathcal{E}(t) \quad \forall j. \tag{44}$$

Finally, introduce the $l \times n$ matrix

$$\mathcal{K}_l^{(j)} = \begin{bmatrix} \left(J^j \kappa\right)^T \\ \left(J^{j+1}\kappa\right)^T \\ \downarrow \\ \left(J^{j+l-1}\kappa\right)^T \end{bmatrix}, \tag{45}$$

noting that the assumption that the system has no zero poles, implies that J is non-singular and hence that $\mathcal{K}_k^{(j)}$ is well defined for all j. Then it follows directly from (44) that

$$\mathcal{Y}_l^{(j)}(t) = \mathcal{K}_l^{(j)} \mathcal{E}(t). \tag{46}$$

We can now compute $y_\gamma(t)$ from Theorem 2 by first computing $\mathcal{K}^{(-\hat{\gamma})}_{max(m+1,\hat{\gamma})}$, starting with the $(\hat{\gamma}+1)-$st row that is equal to κ^T and the proceeding recursively downwards with the Matlab operation $u = vJ^T$ and upwards with the operation $u = v\backslash J^T$. These operations can in fact be separated into block calculations, the blocks corresponding to separate, possibly repeated, eigenvalues. The basic partial fraction coefficients are evaluated by a procedure given in Ævarsson (2005) and Herjólfsson et al. (2009). Since $\mathcal{E}(t)$ and $I_j(t)$ are readily calculated for any sequence of $t-$values, the same holds true for the calculation of $y_\gamma(t)$.

This procedure has been implemented as a Matlab function which has been compared with the Matlab function `lsim`. Considering arbitrarily a system of the following transfer function

$$\frac{(s+0.69)(s+0.51)(s+0.36)(s+0.22)}{(s+0.67\pm j0.20)(s+0.46\pm j0.32)(s+0.16\pm j0.36)(s+0.11)^2} \tag{47}$$

over the interval $[0, 100]$, the relative difference between the output values obtained by `lsim` and by our function, with respect to the latter, are shown in the table below. The results are shown for $t = 10$ (the first value within brackets) and $t = 100$ (the second value within brackets), for the linear forcing function $I_2(t)$ (ramp) and the quadratic function $I_3(t)$, and for the stepsizes $\Delta t = 1, 10^{-2}$ and 10^{-4} in the case of `lsim`.

	$\Delta t = 1$	$\Delta t = 10^{-2}$	$\Delta t = 10^{-4}$
$I_2(t)$	[8.0e-15,6.7e-16]	[5.0e-14,-4.4e-15]	[-7.7e-14,1.3e-14]
$I_3(t)$	[1.2e-02,2.3e-05]	[1.2e-06,2.3e-09]	1.2e-10,1.4e-13]

We note that the difference increases with the stepsize for $I_3(t)$, whereas for $I_2(t)$, when one expects `lsim` to be accurate, it decreases slightly with stepsize due to the smaller number of timesteps. This also indicates that the output from our function is more accurate. Moreover, since timing tests show that the computing times for both functions are very similar when the number of output values is the same, the implication is that our function offers potential computational savings when the number of output values is smaller than that required to obtain sufficient accuracy with `lsim`. Tests with other systems reflect a similar behaviour.

7. RESPONSE INTERPRETATION

In this section we expand the result of Theorem 2 such as to bring out more clearly the structure of the various terms.

The fundamental component or equivalently the transient response is given by the first term in (31)

$$\mathcal{B}_m^T \mathcal{Y}_{m+1}^{(-\hat{\gamma})}(t) I_1(t) = \left[\, b_0 \;\; b_1 \; \rightarrow \; b_m \,\right] \begin{bmatrix} \left(J^{-\hat{\gamma}}\kappa\right)^T \\ \left(J^{-\hat{\gamma}+1}\kappa\right)^T \\ \downarrow \\ \left(J^{-\hat{\gamma}+m}\kappa\right)^T \end{bmatrix} \mathcal{E}(t) I_1(t), \tag{48}$$

which shows the integrating effects of $\hat{\gamma} > 0$ from Lemma 2 or differentiating effects if $\hat{\gamma} < 0$ from Lemma 1. In addition we have the differentiating effects of the numerator elements in $\mathcal{B}$ from the proof of Theorem 2. These simply reside in the power of the Jordan matrix.

The forced type part of the response is given by the first sum in (31) for $\hat{\gamma} \geq 1$. The sum expands as

$$\begin{aligned} -\sum_{i=0}^{\hat{\gamma}-1} \mathcal{B}_i^T \mathcal{Y}_{i+1}^{(-i-1)}(0) I_{\hat{\gamma}-i}(t) &= -\left[\, b_0 \;\; b_1 \; \rightarrow \; b_{\hat{\gamma}-1} \,\right] \\ \times &\begin{bmatrix} \left(J^{-\hat{\gamma}}\kappa\right)^T \mathcal{E}(0) & \left(J^{-\hat{\gamma}+1}\kappa\right)^T \mathcal{E}(0) & \rightarrow & \left(J^{-1}\kappa\right)^T \mathcal{E}(0) \\ \left(J^{-\hat{\gamma}+1}\kappa\right)^T \mathcal{E}(0) & \left(J^{-\hat{\gamma}+2}\kappa\right)^T \mathcal{E}(0) & \rightarrow & 0 \\ \downarrow & \iddots & \iddots & \updownarrow \\ \left(J^{-1}\kappa\right)^T \mathcal{E}(0) & 0 & \leftrightarrow & 0 \end{bmatrix} \begin{bmatrix} I_1(t) \\ I_2(t) \\ \downarrow \\ I_{\hat{\gamma}}(t) \end{bmatrix}, \end{aligned} \tag{49}$$

We note the relation to the zero pattern in the initial conditions from (13) which is mirrored here, as it is in the noncausal type part of the response, given by the second sum in (31), only active when $m - \hat{\gamma} \geq n$, which has been included for the sake of completeness. The sum expands as

$$\begin{aligned} \sum_{i=n}^{m-\hat{\gamma}} \mathcal{B}_{m+n-i,m}^T \mathcal{Y}_{i-n+1}^{(n-1)}(0) I_{-m+i+\hat{\gamma}}(t) &= \left[\, b_{n+\hat{\gamma}} \; \rightarrow \; b_{m-1} \;\; b_m \,\right] \\ \times &\begin{bmatrix} 0 & \leftrightarrow & 0 & \left(J^{n-1}\kappa\right)^T \mathcal{E}(0) \\ \updownarrow & \rightarrow & \left(J^{n-1}\kappa\right)^T \mathcal{E}(0) & \left(J^{n}\kappa\right)^T \mathcal{E}(0) \\ 0 & & \iddots & \downarrow \\ \left(J^{n-1}\kappa\right)^T \mathcal{E}(0) & \rightarrow & \rightarrow & \left(J^{m-\hat{\gamma}-1}\kappa\right)^T \mathcal{E}(0) \end{bmatrix} \\ \times &\begin{bmatrix} I_{-m+n+\hat{\gamma}}(t) \\ \downarrow \\ I_{-1}(t) \\ I_0(t) \end{bmatrix}. \end{aligned} \tag{50}$$

The impulse response for a causal system with $n_0 = 0$, thus $\hat{\gamma} = 0$ and $m < n$ can be expressed as

$$y_0(t) = [\, b_0 \;\, b_1 \;\rightarrow\; b_m \,] \begin{bmatrix} \kappa^T \\ (J\kappa)^T \\ \downarrow \\ (J^m\kappa)^T \end{bmatrix} \mathcal{E}(t) I_1(t). \tag{51}$$

The step response with $n_0 = 0$, such that $\hat{\gamma} = 1$ and $m < n + 1$ is given by

$$\begin{aligned} y_1(t) = [\, b_1 \;\, b_2 \;\rightarrow\; b_m \,] & \begin{bmatrix} \kappa^T \\ (J\kappa)^T \\ \rightarrow \\ \left(J^{-1+m}\kappa\right)^T \end{bmatrix} \mathcal{E}(t) I_1(t) \\ + b_0 \left(J^{-1}\kappa\right)^T & \left(\mathcal{E}(t) - \mathcal{E}(0)\right) I_1(t). \end{aligned} \tag{52}$$

The ramp response with $n_0 = 0$, then $\hat{\gamma} = 2$ and $m < n+2$ is given by

$$\begin{aligned} y_2(t) = [\, b_2 \;\, b_3 \;\rightarrow\; b_m \,] & \begin{bmatrix} \kappa^T \\ (J\kappa)^T \\ \downarrow \\ \left(J^{-2+m}\kappa\right)^T \end{bmatrix} \mathcal{E}(t) I_1(t) \\ + [\, b_0 \;\, b_1 \,] & \begin{bmatrix} \left(J^{-2}\kappa\right)^T \\ \left(J^{-1}\kappa\right)^T \end{bmatrix} \left(\mathcal{E}(t) - \mathcal{E}(0)\right) I_1(t) \\ - b_0 \left(J^{-1}\kappa\right)^T & \mathcal{E}(0) I_2(t) \\ & \text{for } t \geq 0. \end{aligned} \tag{53}$$

8. CONCLUSIONS

The derivation of the results in this paper is guided by various aims. First, to clarify how the responses relate to the solutions of the underlying differential equation, in particular the relationship between the fundamental solution of the differential equation and the basic response, when the forcing function is a Dirac delta function. One observes a structure in the expressions for the responses by focusing on the fundamental solution, that is less evident when deriving corresponding expressions from Laplace transforms. It also implies that they are readily presented in elementary texts on linear differential equations, as well as in textbooks on, e.g., control systems and on signals and systems.

Second, to relate the responses of continous time systems to those of discrete time systems. By basing the expressions in the latter case on the fundamental solution of underlying difference equations as is done in Sigurðsson et al. (2017), one obtains a complete correspondence to the continuous time results in this paper. Again, this is less evident, if one derives the expressions for continuous time systems by the Laplace transform on one hand, and for discrete time systems by the $\mathcal{Z}$-transform on the other.

Third, to obtain expressions that are fully general with respect to the nature of the roots of the characteristic equation and are at the same time transparently and efficiently implemented in a program environment like Matlab, either numerically or symbolically.

The closed form responses may be used in simulation, at fixed or varying intervals in time and may as such have applications in hybrid system simulation. They also turn out to play a useful role in the computation of general MIMO responses as shown in Hauksdóttir et al. (2018). Finally, they lend themselves well in various types of analysis and optimization involved in the computation of PID-type controllers, e.g., closed form Gramians, see Herjólfsson et al. (2005) and Herjólfsson et al. (2009).

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