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Robust Optimal Feedback Control for Periodic Biochemical Processes

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Abstract: This paper is concerned with optimal feedback control synthesis for periodic processes with economic control objectives. The focus is on tube-based methods which optimize over robust forward invariant tubes (RFITs) in order to determine the nonlinear feedback law. The main contribution is an approach to conservatively approximating this set-based periodic feedback control optimization problem by a tractable optimal control problem, which can be solved with existing optimal control solvers. The approach is applied to an uncertain periodic biochemical production process, where the objective is to maximize the profit subject to robust safety constraints.

Keywords: Robust Feedback Control, Robust Forward Invariant Tubes, Periodic Systems.

1. INTRODUCTION

It is well known that periodic operation of chemical and biochemical processes can lead to improved performance compared to steady-state operation (Bailey, 1974). This has sparked a great interest in finding and analyzing optimal periodic (open-loop) operating policies for such processes (Parulekar, 1998). These systems and processes are highly nonlinear, typically uncertain, and often subject to operational and economic constraints. Furthermore, optimal operating regimes are often determined by the operational constraints (Telen et al., 2015). Thus, in the presence of disturbances, an open-loop optimal policy may lead to suboptimal or even worse, unsafe operation. This has motivated the need for robust control strategies, which provide a guarantee for constraint satisfaction. Optimal robust open-loop control strategies, have become widespread (see e.g. Houska et al. (2012); Telen et al. (2015)). In a periodic setting, system stability is a natural requirement. Hence a significant effort has been devoted to devising methods to compute robust open-loop stable orbits for periodic systems (Mombaur et al., 2004, 2005).

Robust optimal closed-loop control has been studied extensively, typically from a receding-horizon—model predictive control—perspective (see e.g. Bemporad and Morari, 1999, for a survey). This problem is significantly harder to solve than its open-loop counterpart, since the optimization variables are now state-dependent feedback functions. Tube-based approaches (Langson et al., 2004; Raković et al., 2005) have emerged as state-of-the-art tools to construct conservative but tractable approximations of the general optimal feedback control problem. These methods are set-theoretical in nature and find their roots in viability theory (Aubin, 1991). Informally, tube-based approaches optimize over robust forward invariant tubes (RFITs), namely set-valued maps containing all uncertain state trajectories under a given feedback law.

This paper presents a method for designing robust optimal feedback controllers for periodic systems. The problem is addressed using set-based computing techniques. This method is based on recent results for constructing and optimizing over RFITs with ellipsoidal cross-sections. This formulation leads to an optimal control problem with periodicity constraints for the center and shape matrix of the tube cross-sections. As a byproduct we obtain a nonlinear feedback control law which is not parameterized a priori, and can be used to control the system inside the RFIT. The paper is organized as follows: Section 2 presents the problem formulation. Section 3 introduces a tractable reformulation, based on ellipsoidal RFITs, of the robust optimal feedback control problem. In Section 4 the ellipsoidal RFIT approach is used to synthesize a robust feedback controller for a periodic bioreactor with economic objective. Section 5 concludes the paper.

Notation: L_2^n denotes the set of n-dimensional Lebesgue integrable functions, and $W_{1,2}^n$ the Sobolev space of weakly differentiable functions with square-integrable derivatives. The set of compact and convex compact sets in $\mathbb{R}^n$ are denoted respectively by $\mathbb{K}^n$ and $\mathbb{K}_\mathrm{C}^n$. The set of $n \times n$ symmetric positive semidefinite and definite matrices are denoted by $\mathbb{S}_+^n$ and $\mathbb{S}_{++}^n$ respectively. An ellipsoid with center $q \in \mathbb{R}^n$ and shape matrix $Q \in \mathbb{S}_+^n$ is given by

$$\mathcal{E}(q,Q) = \left\{ q + Q^{\frac{1}{2}} v \;\middle|\; v^\intercal v \leq 1 \right\} ,$$

where $Q^{\frac{1}{2}}$ is the positive semidefinite square root of Q. The ith row of a matrix A is denoted by A_i. For matrices and vectors, the symbols $\geq, \leq$, are understood componentwise.

2. ROBUST OPTIMAL FEEDBACK CONTROL FOR PERIODIC SYSTEMS

2.1 Problem Setting and Inf-Sup Formulation

We consider uncertain control systems of the form

$$\forall t \in \mathbb{R}, \quad \dot{x}(t) = f(x(t), w(t)) + Gu(t) \, . \tag{1}$$

The state trajectory, $x : \mathbb{R} \to \mathbb{R}^{n_x}$, is required to satisfy state constraints,

$$\forall t \in \mathbb{R}, \quad x(t) \in \mathbb{X} \subseteq \mathbb{R}^{n_x} \, . \tag{2}$$

The control $u : \mathbb{R} \to \mathbb{R}^{n_u}$ and disturbance $w : \mathbb{R} \to \mathbb{R}^{n_w}$ signals are assumed to be bounded, i.e.

$$\forall t \in \mathbb{R}, \quad u(t) \in \mathbb{U} \in \mathbb{K}_{\mathrm{C}}^{n_u} \quad \text{and} \quad w(t) \in \mathbb{W} \in \mathbb{K}_{\mathrm{C}}^{n_w} \, .$$

The function $f : \mathbb{R}^{n_x} \times \mathbb{R}^{n_w} \to \mathbb{R}^{n_x}$ is nonlinear, but assumed to be integrable in all its arguments and Lipschitz continuous in its first argument. Although the theory in this paper allows for a nonlinear (Lipschitz continuous) function $G : \mathbb{R}^{n_x} \to \mathbb{R}^{n_x \times n_u}$, we will only consider a constant matrix $G \in \mathbb{R}^{n_x \times n_u}$ to simplify the presentation.

The framework of robust optimal feedback control, is concerned with searching for a function $\mu : \mathbb{R} \times \mathbb{X} \to \mathbb{U}$, such that the closed-loop system

$$\forall t \in \mathbb{R} : \quad \dot{x}(t) = f(x(t), w(t)) + G\mu(t, x(t)) \tag{3}$$

with $x(0) = x_0$, satisfies (2) for every $w \in L^{n_w}$ valued in $\mathbb{W}$. Furthermore, we consider that the system operates in a periodic mode, i.e. we search over the set of feedback laws which satisfy

$$\exists T \in \mathbb{R} : \quad \mu(t + T, x(t)) = \mu(t, x(t)) \, , \tag{4}$$

for all $t \in \mathbb{R}_+$ and all $x(t) \in \mathbb{X}$. In particular this feedback is chosen to minimize an economic criterion in Mayer form, $m(x(T))$ with a given continuous function $m : \mathbb{R}^n \to \mathbb{R}$.

Let $\xi(t, x_0, w, \mu) = x(t)$ be the solution of (3), for a given initial condition $x_0 \in \mathbb{X}$, disturbance $w : \mathbb{R} \to \mathbb{W}$, and feedback $\mu : \mathbb{R} \times \mathbb{X} \to \mathbb{U}$. The robust optimal feedback control problem can be formulated as

$$\begin{aligned}
&\inf_{\substack{T \in \mathbb{R} \\ \phi_0 : \mathbb{X} \to \mathbb{R} \\ \mu : \mathbb{R} \times \mathbb{X} \to \mathbb{U}}} \ \sup_{\substack{w : [0,T] \to \mathbb{W} \\ x_0 \in \mathbb{X}, \, \phi_0(x_0) \le 0}} \ m\left(\xi(T, x_0, w, \mu)\right) \\
&\text{s.t.} \ \begin{cases} \sup\limits_{\substack{w : [0,T] \to \mathbb{W} \\ x_0 \in \mathbb{X}, \, \phi_0(x_0) \le 0}} \ \xi(t, x_0, w, \mu) \in \mathbb{X}, \ \forall t \in [0, T] \\ \sup\limits_{\substack{w : [0,T] \to \mathbb{W} \\ x_0 \in \mathbb{X}, \, \phi_0(x_0) \le 0}} \ \phi_0(\xi(T, x_0, w, \mu)) \le 0 \, , \end{cases}
\end{aligned} \tag{5}$$

A couple of remarks regarding (5) are in order. First, it has a bilevel structure similar to that of finite-time robust optimal feedback control problems with fixed initial condition. The added complexity comes from enforcing the periodicity requirement for the solution. Here, the solutions of (3) at the final time T, must satisfy

$$\xi(T, x_0, w, \mu) \in \mathbb{X}_0 = \{\xi_0 \in \mathbb{X} \,|\, \phi_0(\xi_0) \le 0\},$$

for all disturbance function $w : [0, T] \to \mathbb{W}$ and all $x_0 \in \mathbb{X}_0$. The set $\mathbb{X}_0$—or equivalently the function $\phi_0 : \mathbb{X} \to \mathbb{R}$—is an optimization variable. Also notice that condition (4) is not directly enforced in (5), but is enforced a posteriori. This can be done since modifying μ on a set of Lebesgue measure zero does not alter the optimal value of (5).

Remark 1. Lagrange terms of the form

$$\int_0^T \ell(x(t)) \mathrm{d}t \, ,$$

with stage cost $\ell : \mathbb{R}^n_x \to \mathbb{R}^{n_x}$, can be eliminated by setting $m(x(T)) = x_{n_x+1}$ and stacking the ODE

$$\dot{x}_{n_x+1}(t) = \ell(x(t)) \, , \quad \text{with} \quad x_{n_x+1}(0) = 0$$

to (1). This formulation also allows to consider Lagrange costs over an infinite horizon, by constraining the system to be periodic and optimizing the cost at the end of cycle.

2.2 Robust Forward Invariant Tube Formulation

This section introduces a conservative reformulation of (5) by means of robust forward invariant tubes. The reach set of (3)—assumed to be compact— for a given initial value x_0 and feedback law μ is denoted by

$$Y(t, x_0, \mu) = \left\{ \xi_t \;\middle|\; \begin{array}{l} \exists x \in W_{1,2}^{n_x}, \, \exists w \in L_2^{n_w} : \forall \tau \in [0, t] \\ \dot{x}(t) = f(x(\tau), w(\tau)) + G\mu(\tau, x(\tau)) \\ x(0) = x_0, \, x(t) = \xi_t, \, w(\tau) \in \mathbb{W} \end{array} \right\} .$$

Although we could directly formulate (5) in terms of reach sets, this would not help us devising a solution strategy. Instead, we take a more indirect approach using RFITs. A set-valued function $X(t) : [0, T] \to \mathbb{K}^{n_x}$ is an RFIT for (1) on $[0, T]$ if there exists a function $\mu : \mathbb{R} \times \mathbb{X} \to \mathbb{U}$ such that

$$X(t_2) \supseteq \bigcup_{x_1 \in X(t_1)} Y(t_2 - t_1, x_1, \mu) \, ,$$

for all $[t_1, t_2] \subseteq [0, T]$.

and $\mathcal{M} : \mathbb{K}^n \to \mathbb{R}$ being a Mayer objective. The following lemma formalizes the relation between (5) and (6).

Lemma 2. Let the function $\mathcal{M} : \mathbb{K}^{n_x} \to \mathbb{R}$ be given by $\mathcal{M} : Y \mapsto \sup_{y \in Y} \ m(y)$, for all $Y \in \mathbb{K}^{n_x}$. Consider the optimization problem

$$\inf_{X \in \mathcal{X}, \, T \in \mathbb{R}} \ \mathcal{M}(X(T)) \quad \text{s.t.} \ \begin{cases} X(t) \subseteq \mathbb{X}, \, \forall t \in [0, T] \\ X(T) \subseteq X(0) \, , \end{cases} \tag{6}$$

with $\mathcal{X}$ denoting the set of all RFITs of (1) on $[0, T]$. Any solution of Problem (6) is also a feasible solution of (5).

Proof. Let (X^*, T^*) be a solution of (6). The existence of a function $\xi(t, x_0, w, \mu^*)$ satisfying the state constraints in (5), for all $w \in \mathbb{W}$ and all $x_0 \in X^*(0)$, follows by the definition of an RFIT and the feasibility of X^*. By the periodicity of X^* and compactness of its image for every t, and particularly at $t = 0$, we can construct a continuous function ϕ_0, such that the periodicity constraint in (5) is also satisfied. The point (T^*, ϕ_0, μ^*) is, by construction, a feasible point of (5). □

Problem (6) is intractable, in all but the simplest of cases. Fortunately, restricting the search of RFITs to those with convex images, we can aim to construct conservative but tractable reformulations to this problem. In fact, Theorem 1 in Villanueva et al. (2017) presents sufficient conditions for a convex set-valued function to be an RFIT. Although checking these conditions involves solving an optimal control problem with semi-infinite differential inequality constraints, its discretization leads to a linear growth of its complexity with respect to the length of the time horizon. Furthermore, its proof is constructive, thus providing an explicit expression for a feedback law generating the tube.

3. ROBUST OPTIMAL CONTROL SYNTHESIS FOR PERIODIC SYSTEMS USING ELLIPSOIDAL RFITS

3.1 Ellipsoidal Robust Forward Invariant Tubes

We focus on the construction of RFITs with ellipsoidal cross-sections, i.e. $X : t \mapsto \mathcal{E}(q_x(t), Q_x(t))$. This tube is parameterized by a central path $q_x : \mathbb{R} \to \mathbb{R}^{n_x}$ and a time-varying shape matrix $Q_x : \mathbb{R} \to \mathbb{S}_+^{n_x}$. Sufficient conditions for choosing the functions q_x and Q_x are given in Theorem 3.

In the following, we will use the shorthand notation

$$A(t) = \frac{\partial f}{\partial x}(q_x(t), q_w) , \quad B(t) = \frac{\partial f}{\partial w}(q_x(t), q_w) , \quad \text{and}$$
$$\begin{aligned}\Phi(q_x(t), Q_x(t), \lambda(t), \kappa(t), K(t)) &= (A(t) - GK(t))Q_x(t) \\ &+ Q_x(A(t) - GK(t))^\intercal + \left(\frac{1}{\lambda(t)} + \frac{1}{\kappa(t)}\right) Q_x(t) \\ &+ \lambda(t)B(t)Q_w B(t)^\intercal + \kappa(t)\Omega_n(q_x(t), Q_x(t)) .\end{aligned}$$

The nonlinearity bounder $\Omega_n : \mathbb{R}^{n_x} \times \mathbb{S}_+^{n_x} \to \mathbb{S}_+^{n_x}$ is required to satisfy

$$\begin{aligned} f(\xi, \omega) - f(q_x(t), q_w) - A(t)(\xi - q_x) \\ - B(t)(\omega - q_w) \in \mathcal{E}(0, \Omega_n(q_x(t), Q_x(t))) ,\end{aligned} \tag{7}$$

for each $(\xi, \omega) \in \mathcal{E}(q_x(t), Q_x(t)) \times \mathcal{E}(q_w, Q_w)$.

Theorem 3. Consider (1) with control and uncertainty sets $\mathbb{U} := \{\nu \in \mathbb{R}^{n_u} \,|\, \underline{u} \le \nu \le \overline{u}\}$ and $\mathbb{W} := \mathcal{E}(q_w, Q_w)$, with $q_w \in \mathbb{R}^{n_w}$, $Q_w \in \mathbb{S}_+^{n_w}$ and $\underline{u}, \overline{u} \in \mathbb{R}^{n_u}$. If q_x and Q_x satisfy

$$\begin{aligned}\dot{q}_x(t) &= f(q_x(t), q_w) + Gu_x(t) \\ \dot{Q}_x(t) &= \Phi(q_x(t), Q_x(\tau), \lambda(t), \kappa(t), K(t))\end{aligned} \tag{8}$$

on $[0, T]$, for some integrable functions $\lambda, \kappa : \mathbb{R} \to \mathbb{R}_{++}$, $u_x : \mathbb{R} \to \mathbb{U}$ and $K : \mathbb{R} \to \mathbb{R}^{n_u \times n_x}$ satisfying

$$\begin{aligned} u_{x_i}(t) + \sqrt{K_i(t)Q_x(t)K_i(t)^\intercal} \le \overline{u}_i \ , \ i \in \{1, \ldots, n_u\} \\ u_{x_i}(t) - \sqrt{K_i(t)Q_x(t)K_i(t)^\intercal} \ge \underline{u}_i \ , \ i \in \{1, \ldots, n_u\} \ ,\end{aligned} \tag{9}$$

then $X : t \mapsto \mathcal{E}(q_x(t), Q_x(t))$ is an RFIT for (1) on $[0, T]$.

Proof. See Villanueva et al. (2017, Thm. 5) for a proof.

3.2 Conservative Approximation of the Periodic ROFCP

A periodic RFIT can be constructed by checking the conditions of Theorem 3 by solving a nonlinear OCP over T, q_x, Q_x, u_x, λ, κ, and K, with the periodicity constraint

$$q_x(0) = q_x(T) \quad \text{and} \quad Q_x(0) = Q_x(T) . \tag{10}$$

For the state constraints, we consider a sublevel set

$$\mathbb{X} = \{\xi \in \mathbb{R}^{n_x} \,|\, \phi(\xi) \le 0\} \ ,$$

of a given function $\phi : \mathbb{R}^{n_x} \to \mathbb{R}^{n_c}$. The path constraint $\mathcal{E}(q_x(t), Q_x(t)) \subseteq \mathbb{X}$ is satisfied on $[0, T]$ if and only if,

$$\forall i \in \{1, \ldots, n_c\}, \ \forall t \in [0, T], \qquad \sup_{\xi \in \mathcal{E}(q_x(t), Q_x(t))} \phi_i(\xi) \le 0. \tag{11}$$

In the polytopic setting, e.g. $\phi(\xi) = C\xi - c$ with $c \in \mathbb{R}^{n_c}$ and $C \in \mathbb{R}^{n_c \times n_x}$, (11) is satisfied if and only if,

$$\forall i \in \{1, \ldots, n_c\}, \ \forall t \in [0, T], \qquad C_i^\intercal q_x(t) + \sqrt{C_i^\intercal Q_x(t) C_i} \le c_i. \tag{12}$$

In the case of nonlinear functions ϕ_i, it may not be possible to solve the maximization in (11) exactly. Nevertheless, we can replace (11) with the conservative condition

$$\begin{aligned}&\forall i \in \{1, \ldots, n_c\}, \forall t \in [0, T], \\ &d_i(t) + \sqrt{D_i(t)Q_x(t)D_i(t)^\intercal} + \Omega_{\phi_i}(q_x(t), Q_x(t)) \le 0 \, ,\end{aligned} \tag{13}$$

Here, $d_{\phi_i}(t) = \phi_i(q_x(t))$, $D_i(t) = \frac{\partial \phi_i}{\partial x}(q_x(t))$, and the function $\Omega_{\phi_i} : \mathbb{R}^{n_x} \times \mathbb{S}_+^{n_x} \to \mathbb{R}$ must satisfy

$$|\phi_i(\xi) - d_i(t) - D_i(t)(\xi - q_x(t))| \le \Omega_{\phi_i}(q_x(t), Q_x(t))$$

for all $\xi \in \mathcal{E}(q_x(t), Q_x(t))$.

Now, a periodic RFIT can be constructed by solving

$$\begin{aligned} &\inf_{\substack{T, q_x, Q_x, \\ u_x, \lambda, \kappa, K}} \quad \mathcal{M}\left(\mathcal{E}(q_x(T), Q_x(T)\right) \\ &\text{s.t.} \quad \begin{cases} \text{ODEs (8) with periodic conditions (10)} \\ \text{State constraints (11) (via (12) or (13))} \\ \lambda(t), \, \kappa(t) > 0, \ Q_x(t) \in \mathbb{S}_+^{n_x} \ \forall t \in [0, T] \\ \underline{u} \le u_x(t) \le \overline{u}, \qquad \qquad \forall t \in [0, T] \ . \end{cases}\end{aligned} \tag{14}$$

The Mayer objective function

$$\mathcal{M}\left(\mathcal{E}(q_x(T), Q_x(T))\right) = \max_{\xi_T \in \mathcal{E}(q_x(T), Q_x(T))} m(\xi_T) \ , \tag{15}$$

can then be constructed using the approaches discussed previously in the context of robustifying the function ϕ_i.

3.3 Robust Feedback Control using Ellipsoidal RFITs

Once an RFIT has been constructed, the next task is to find a valid feedback function associated to it. Corollary 4 provides a feedback law inducing an ellipsoidal RFIT.

Corollary 4. Let $X : t \mapsto \mathcal{E}(q_x(t), Q_x(t))$ satisfy the conditions of Theorem 3. Then, a feedback law inducing this RFIT is given by

$$\mu^*(t, x(t)) = \begin{cases} u_x(t), & \text{if } x(t) = q_x(t) \\ u_x(t) - \dfrac{\nu(t, x(t))}{\|\nu(t, x(t))\|_2}, & \text{otherwise} \end{cases}$$

with $\nu(t, x(t)) = (K(\tau)Q_x(\tau)K(\tau)^\intercal)\, G^\intercal \gamma(x(\tau))$, and

$$\gamma(x(\tau)) = \frac{Q_x(\tau)^\dagger (x(\tau) - q_x(\tau))}{\|Q_x(\tau)^\dagger (x(\tau) - q_x(\tau))\|_2} \ ,$$

where $(\cdot)^\dagger$ denotes the Moore-Penrose pseudoinverse.

Proof. The construction is a byproduct of the proof of Thm. 3. It can be found in Villanueva et al. (2017, Cor. 6).

Since the tube is periodic by construction, the control feedback law is periodic. Furthermore, the control signal $u(t) = \mu^*(t, x(t))$ can be sent to the system to keep it inside the tube, whenever $x(t)$ is in the tube.

4. CASE STUDY: A PERIODIC BIOREACTOR

In this section we consider the problem of finding a robust optimal feedback law for a biochemical process. More precisely, we consider a continuous stirred tank bioreactor (see, e.g. (Parulekar, 1998)), with uncertain dynamics

$$\begin{aligned}\dot{Z}(t) &= -DZ(t) + r(S(t), P(t))Z(t) + w_1(t) \\ \dot{S}(t) &= -DS(t) - \frac{r(S(t), P(t))Z(t)}{Y_{Z/S}(t)} + DS_\mathrm{F}(t) \\ \dot{P}(t) &= -DP(t) - (\alpha r(S(t), P(t)) + \beta)\, Z(t) + w_3(t) \ .\end{aligned} \tag{16}$$

Here, $Z(t)$, $S(t)$, and $P(t)$ denote the biomass, substrate, and product concentrations respectively. The process is controlled by the input $S_{\mathrm{F}}(t)$ representing the substrate feed concentration, with bounds $[\underline{S}_{\mathrm{F}}, \overline{S}_{\mathrm{F}}]$. The process is affected by three uncertain inputs w_1, w_2, and w_3. While w_1 and w_3 enter the system affinely, w_2 perturbs the system nonlinearly through the biomass yield $Y_{Z/S}(t) = Y_0 + w_2(t)$. The product yield parameters α, β and the dilution rate D are assumed to be constant. The biomass growth rate is given by a generalized Monod kinetic model

$$r(S(t), P(t)) = r_{\mathrm{m}} \frac{\left(1 - \frac{P(t)}{K_P}\right) S(t)}{K_{\mathrm{m}} + S(t) + \left(\frac{S(t)^2}{K_S}\right)} ,$$

which accounts for both substrate and product inhibitions. The parameters K_S and K_P are substrate and product inhibition constants, while K_{m} is the substrate saturation constant. The maximum growth rate is denoted by r_{m}.

The problem is to maximize the profit of the process. For this, we assume that we can sell our product for a price p_P, while the only cost of the process is that of the substrate, which is bought at a price $p_{S_{\mathrm{F}}}$. We assume the process is continuous and operated periodically, i.e.

$$(Z(0),\, S(0),\, P(0))^{\intercal} = (Z(T),\, S(T),\, P(T))^{\intercal} , \tag{17}$$

for some $T > 0$. The control objective is to maximize

$$p_P \int_0^{\infty} DP(t)\mathrm{d}t - p_{S_{\mathrm{F}}} \int_0^{\infty} DS_{\mathrm{F}}(t)\mathrm{d}t .$$

In view of (17), this is equivalent to maximizing the profit at the end of a period $[0, T]$, i.e.,

$$m(I(T)) = I(T) . \tag{18}$$

where I is an auxiliary states satisfying

$$\dot{I}(t) = p_P DP(t) - p_{S_{\mathrm{F}}} DS_{\mathrm{F}}(t) \tag{19}$$

Lastly, the maximum average of the biomass concentration over one period is assumed to be constrained by $\overline{Z}_{\mathrm{M}}$, i.e.

$$Z_{\mathrm{M}}(T) \leq T\overline{Z}_{\mathrm{M}} . \tag{20}$$

Here, $Z_{\mathrm{M}}(t)$ is another auxiliary state satisfying

$$\dot{Z}_{\mathrm{M}}(t) = Z(t) . \tag{21}$$

Notice that the extra integrating states satisfy

$$(I(0),\, Z_{\mathrm{M}}(0))^{\intercal} = (0,\, 0)^{\intercal} . \tag{22}$$

4.1 Nominal Open-Loop Optimization

First, we consider finding an open-loop control for the periodic operation of the bioreactor by solving

$$\begin{aligned} \inf_{\substack{Z,\, S,\, P,\, I,\, Z_{\mathrm{M}} \\ S_{\mathrm{F}}:[0,T]\to[\underline{S}_{\mathrm{F}}, \overline{S}_{\mathrm{F}}]}} \quad & m(I(T)) \\ \text{s.t.} \quad & \begin{cases} \text{ODEs (16), (19), and (21)} \\ \text{Periodicity conditions (17)} \\ \text{Initial conditions (22)} \\ \text{State constraint (20)} , \end{cases} \end{aligned} \tag{23}$$

with disturbances at their nominal value $q_w = (0, 0, 0)^{\intercal}$.

Problem (23) is solved using `ACADO Toolkit`. The problem is discretized using multiple shooting (default option), with a piecewise constant discretization (30 pieces) and a Runge-Kutta integrator of order 4 with discretization error control of order 5. The discretized problem is solved using a sequential quadratic programming algorithm, with `qpOASES` as the QP solver. The problem is solved to a KKT tolerance of 1×10^{-7} with integrator and absolute tolerances of 1×10^{-7} and 1×10^{-8} respectively.

Table 1. Biochemical Process Parameters.

Parameter	Symbol	Value
Dilution rate	D	$0.15\,\mathrm{h}^{-1}$
Substrate inhibition constant	K_S	$22\,\frac{\mathrm{g}}{\mathrm{L}}$
Substrate saturation constant	K_{m}	$1.2\,\frac{\mathrm{g}}{\mathrm{L}}$
Product inhibition constant	K_P	$50\,\frac{\mathrm{g}}{\mathrm{L}}$
Nominal biomass yield	Y_0	0.4
Product to substrate yield (slope)	α	2.2
Product yield (constant)	β	$0.2\,\mathrm{h}^{-1}$
Minimum feed substrate	$\underline{S}_{\mathrm{F}}$	$26\,\frac{\mathrm{g}}{\mathrm{L}}$
Maximum feed substrate	$\overline{S}_{\mathrm{F}}$	$42\,\frac{\mathrm{g}}{\mathrm{L}}$
Maximum average biomass concentration	$\overline{Z}_{\mathrm{M}}$	$5.9\,\frac{\mathrm{g}}{\mathrm{L}}$
Product price (per gram)	p_P	\$ 250
Feed price (per gram)	$p_{S_{\mathrm{F}}}$	\$ 100
Period time	T	48 h

As expected, the state trajectories depicted as blue solid lines in Figs. 1-(a), (b), and (c), are indeed periodic. The optimal profit (per reactor volume [L]) is

$$m\,(I^*(T)) \approx \$215 .$$

At the optimal solution of the discretized problem, the state constraint is not active,

$$\frac{1}{T}\int_0^T Z^*(t)\mathrm{d}t \approx 5.6\frac{\mathrm{g}}{\mathrm{L}} \leq 5.9\frac{\mathrm{g}}{\mathrm{L}} .$$

The optimal input S_{F}^*, depicted as a blue solid line in Fig. 1-(d), touches both the upper and lower bounds in a partial bang-bang structure. In the first phase the substrate concentration decreases, since the feed is at its lowest and the existing substrate is being used for biomass growth. In a second phase, the substrate begins to accumulate, as the feed increases and the biomass and product reach their peak and start decreasing. The final two phases repeat with a lower peak amplitude.

4.2 Ellipsoidal Robust Optimal Feedback Control

Next, we are interested in synthesizing a robust feedback controller, using ellipsoidal robust forward invariant tubes. We assume that the uncertainty belongs to the ellipsoid $\mathcal{E}(q_w, Q_w)$ with $Q_w = \mathrm{diag}(1 \times 10^{-4}, 4 \times 10^{-4}, 1 \times 10^{-4})$. Notice that the biomass yield is being perturbed by 5% from its nominal value Y_0.

We distinguish between the true state trajectories Z, S, and P, which are stacked into $x(t) = (Z(t), S(t), P(t))^{\intercal}$ and the auxiliary states. The states $x(t)$ will be enclosed by an ellipsoidal tube parameterized by $q_x : [0, T] \to \mathbb{R}^3$ and $Q_x : [0, T] \to \mathbb{S}^3_+$, while the auxiliary states will be treated separately. A conservative approximate of the worst case profit, is given by

$$\mathcal{M}(\mathcal{E}(q_x, Q_x)) = \overline{I}(T). \tag{24}$$

The auxiliary variable $\overline{I}(t)$ satisfies

$$\begin{aligned} \dot{\overline{I}}(t) = \;& p_P D\left(q_{x_3}(t) - \sqrt{Q_{x_{(3,3)}(t)}}\right) \\ & - p_{S_{\mathrm{F}}} D\left(S_{\mathrm{F}}(t) + \sqrt{K(t)Q_x(t)K(t)}\right) . \end{aligned} \tag{25}$$

The first term in brackets integrates the minimum possible product concentration in the bioreactor over the cycle. The second term in brackets integrates the maximum control input exerted on the system over the cycle. The last additional state is $\overline{Z}(t)$, with

$$\dot{\overline{Z}}(t) = q_{x_1}(t) + \sqrt{Q_{x_{(1,1)}(t)}} \ , \qquad (26)$$

and integrates the maximum biomass concentration over the cycle. Thus the robustified state constraint is given by

$$\overline{Z}(T) \le T\overline{Z}_{\mathrm{M}} \ . \qquad (27)$$

Remark 1. A simpler approach would be to stack both the original and auxiliary states and construct an ellipsoidal enclosure in the extended state-space. Our experience is that this approach often leads to unnecessary over conservatism, at least in our implementation. In this case it may be caused by the fact that the auxiliary integrating states $\overline{I}$, and $\overline{Z}$ are always increasing and cause a growth in the dynamics of the ellipsoidal states in all directions.

The last ingredient needed to formulate Problem (14) is the nonlinearity estimate Ω_n. This can be computed numerically as in Villanueva et al. (2017) or an explicit expression can be constructed analytically (see Houska et al. (2012)). Lemma 5 in Appendix A provides an explicit nonlinearity bounder for the bioreactor model (16).

A solution of the optimal control problem

$$\begin{aligned} \inf_{\substack{q_x, Q_x, \lambda, \kappa, K, \\ u_x:[0,T]\to[\underline{S}_{\mathrm{F}},\overline{S}_{\mathrm{F}}], \\ \overline{I}, \overline{Z}_{\mathrm{M}}}} \quad & \mathcal{M}(\mathcal{E}(q_x(T), Q_x(T))) \\ \text{s.t.} \quad & \begin{cases} \text{ODEs (8), (25) and (26)} \\ \text{Periodicity conditions (10)} \\ \text{Initial conditions (22)} \\ \text{State constraint (20)} \ , \end{cases} \end{aligned} \qquad (28)$$

parameterizes a periodic ellipsoidal RFIT (16).

Problem (28) is solved using the same settings as the nominal OCP (23). The tube—whose projections onto the state-time dimensions are depicted as shaded red areas in Figs. 1-(a), (b), and (c)—are also periodic. The worst-case profit (per reactor volume [L]) is

$$\mathcal{M}(\mathcal{E}(q_x(T), Q_x(T))) \approx \$103 \ .$$

The conservatism of the optimal solution can be analyzed based on the profit obtained from the central path, with nominal control u_x. In this case, the profit is around \$130, which represents a loss of nearly 40%.

Here, the robustified constraint is active. Applying the control u_x to a system with nominal uncertainty gives a constraint value of 5.8 $\frac{\mathrm{g}}{\mathrm{L}}$. The state profiles show a similar trend as the nominal open-loop solution, but with a lower peak amplitude. Figure 1-(d) shows the nominal input u_x (dashed red line) as well as the evolution of the control set $u_x(t)+[-\sqrt{K(t)Q_x(t)K(t)^\intercal}, \sqrt{K(t)Q_x(t)K(t)^\intercal}]$ for all $t \in [0,T]$ (shaded red area). We can see that in the worst case, the solution is also partially bang bang.

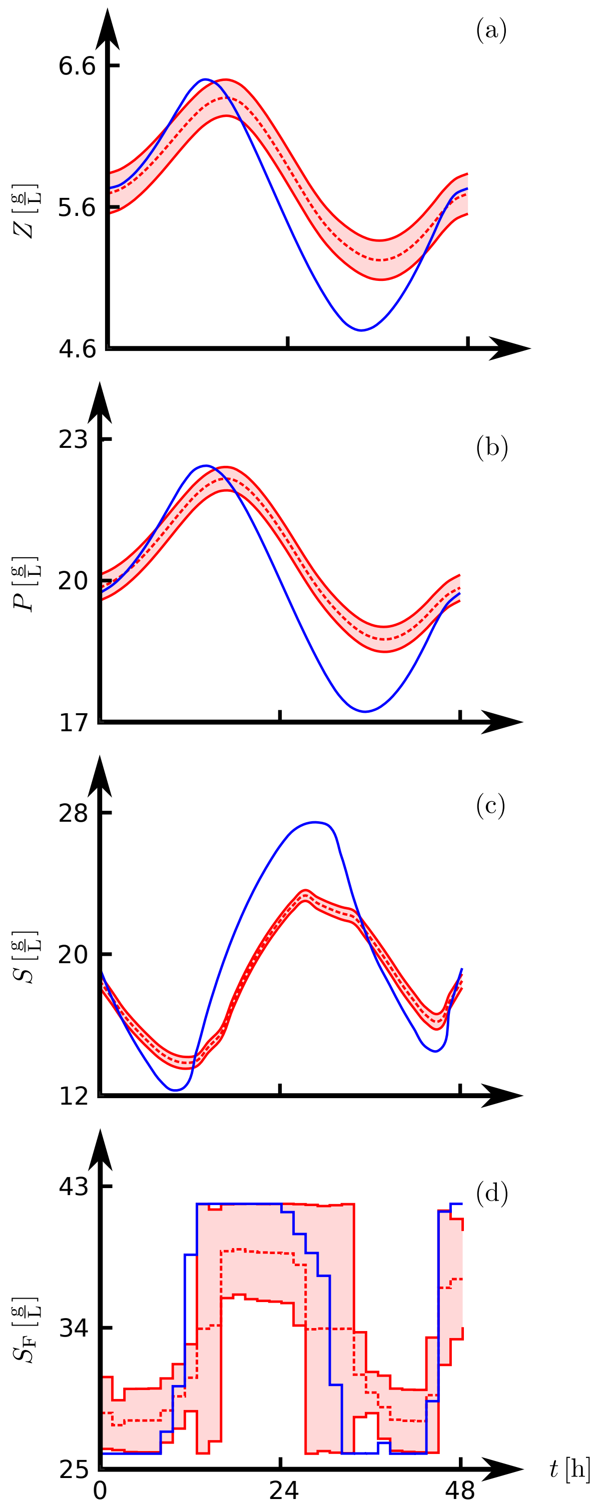

Fig. 1. Nominal (blue) and robust state and control profiles (red). Projections of the RFIT into the: (a) (Z,t), (b) (P,t), and (c) (S,t) planes are shown as a red area with the q_x as a dashed red line. The control set (d) is shown as a red area, with u_x as a dashed red line

5. CONCLUSION

This paper has presented an off-line design method for robust closed-loop feedback control laws systems with periodicity constraints. We have reviewed methods for constructing ellipsoidal robust forward invariant tubes and we have extended the corresponding framework to periodic systems. Moreover, we have discussed how to recover a nonlinear feedback control law, which keeps the system in the optimized set-valued tube. This approach has been applied to the design of a robust closed-loop control law maximizing the economic profit of a periodically operated biochemical reactor.

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Appendix A. NONLINEARITY BOUNDER FOR (16)

Lemma 5. Consider the functions $\omega_i : \mathbb{R}^{n_x} \times \mathbb{R}^{n_x} \to \mathbb{R}$, with $i \in \{1, \ldots, 3\}$, given by

$$\omega_1(q,\Delta) = \frac{a_1(q,\Delta)}{b_1(q,\Delta)}\ , \quad \omega_3(q,\Delta) = \alpha\omega_1(q,\Delta)\ ,$$

$$\text{and} \quad \omega_2(q,\Delta) = \frac{a_2(q,\Delta) + (q_{w_2}+Y_0)^3 a_1(q,\Delta)}{b_2(q,\Delta) + (q_{w_2}+Y_0)^2 b_1(q,\Delta)}\ ,$$

with $a_i, b_i : \mathbb{R}^{n_x} \times \mathbb{R}^{n_x} \to \mathbb{R}$ defined in (A.1). Then, the function $\Omega_n : \mathbb{R}^3 \times \mathbb{S}^3_+ \to \mathbb{S}^3_+$ given by

$$\Omega_n(q_x(t), Q_x(t)) = \mathrm{diag}\left((\omega_i(q_x(t), \Delta_x(t)))^2\right)_{1\le i\le 3}$$

with $\Delta_x(t) = \left(\sqrt{Q_{x_{(1,1)}}(t)}, \ldots, \sqrt{Q_{x_{(n_x,n_x)}}(t)}\right)^\intercal$ is a nonlinearity bounder in the sense of (7) for system (16).

Proof. We only provide a sketch of the proof, as the computations are straightforward. First, we decompose the state and disturbance into a nominal and perturbed part, e.g. $y = q_y + \delta_y$. Here, we introduced $y = (x, w)^\intercal$ and dropped the time dependency. Then, we evaluate

$$\eta_i(q_y, \delta_y) = f_i(q_y + \delta_y) - f_i(q_y) - \frac{\partial f_i}{\partial x}_i(q_y)\delta_y$$

We want to construct a bounder $\omega_i(q_y, \Delta_y)$, such that

$$\forall \delta_y \in [-\Delta_y, \Delta_y], \quad |\eta_i(q_y, \delta_y)| \le \omega_i(q_y, \Delta_y)\ .$$

One strategy is to employ addition theorems—globally valid formulas expressing a function $g(\xi + \delta\xi)$ as a rational function of $g(\xi)$ and $g(\delta_\xi)$. In our case, the appropriate addition theorem is just the binomial theorem $(y+\delta_y)^2 = y^2 + 2y\delta_y + \delta_y^2$. After expanding the terms in η_i, we identify a rational expression in terms of d_y with the coefficients given with respect to q_y. Now, it is straightforward to construct ω_i, since all coefficients are nonnegative, except those including $q_p - K_P$. Taking absolute values proves the formula

$$\begin{aligned}
c_0 &= (q_{w_2} + Y_0)\\
c_1(q) &= q_2^2 + K_S(K_\mathrm{m} + q_2)\\
c_2(q) &= K_S + 2q_2\\
c_3(q) &= (K_S r_\mathrm{m})|(-K_P + q_3)|\\
c_4(q) &= K_S(K_S - K_\mathrm{m}) + 3(K_S + q_2)\\
b_1(q,\Delta) &= \Delta_2^2 K_P c_1(q)^2 + \Delta_2 K_P c_2(q) c_1(q)^2 + K_P c_1(q)^3\\
a_1(q,\Delta) &= \Delta_2^2\Delta_1 c_3(q)c_1(q) + \Delta_2\Delta_1 c_3(q)c_2(q)c_1(q)\\
&\quad + \Delta_3\Delta_1(K_S r_\mathrm{m})c_1(q)^2 + \Delta_2^3 c_3(q)c_2(q)q_1\\
&\quad + \Delta_3\Delta_2^2(K_S r_\mathrm{m})c_1(q)q_1 + \Delta_2^2 c_3(q)c_4(q)q_1\\
&\quad + \Delta_3\Delta_2(K_S r_\mathrm{m})c_2(q)c_1(q)q_1\\
b_2(q,\Delta) &= \Delta_2^2\Delta_w K_P c_1(q)^2 c_0^2 + \Delta_w K_P c_1(q)^3 c_0^2\\
&\quad + \Delta_2\Delta_w K_P c_2(q)c_1(q)^2 c_0^2 + \Delta_w^2 c_3(q)c_1(q)^2 q_1\\
a_2(q,\Delta) &= \Delta_2^2\Delta_w^2 c_3(q)c_1(q)q_1\\
&\quad + \Delta_2\Delta_w^2 c_3(q)c_2(q)c_1(q)q_1\\
&\quad + \Delta_w 2\Delta_1 c_3(q)c_1(q)^2 c_0\\
&\quad + \Delta_2^2\Delta_w\Delta_1 c_3(q)c_1(q)c_0\\
&\quad + \Delta_2^3\Delta_w c_3(q)c_2(q)q_1 c_0\\
&\quad + \Delta_2^2\Delta_w c_3(q)c_2(q)^2 q_1 c_0\\
&\quad + \Delta_3\Delta_2^2\Delta_w 2(K_S r_\mathrm{m})c_1(q)q_1 c_0\\
&\quad + \Delta_3\Delta_2\Delta_w 2(K_S r_\mathrm{m})c_2(q)c_1(q)q_1 c_0\\
&\quad + \Delta_2\Delta_w c_3(q)c_2(q)c_1(q)q_1 c_0\\
&\quad + \Delta_3\Delta_w(K_S r_\mathrm{m})c_1(q)^2 q_1 c_0\\
&\quad + \Delta_2\Delta_w\Delta_1 c_3(q)c_2(q)c_1(q)c_0\ .
\end{aligned} \tag{A.1}$$

Notice that Δ_w is the only disturbance bound in (A.1). Since w_2 is the only disturbance entering the nonlinearly in (16), we have $\Delta_w = \sqrt{Q_{w_{(2,2)}}}$. □