# Optimization-based Investigations of a Thermofluidic Engine for Low-grade Heat Recovery

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**Abstract:** This paper presents an analysis of the non-inertive-feedback thermofluidic engine (NIFTE) under cyclic steady-state conditions. The analysis is based on a nonlinear model of NIFTE that had previously been validated experimentally, and applies an optimization-based approach to detect the cyclic steady states (CSS). The stability of the CSS is furthermore determined by analyzing their monodromy matrix. It is found that NIFTE can exhibit multiple CSS for certain values of the design parameters, which may be either stable or unstable, a result that had not been reported before. Subsequently, a parametric study is conducted by varying key design parameters, revealing that higher efficiencies could be achieved by controlling the engine at different CSS, including unstable ones. Lastly, the paper investigates the trade-offs between efficiency and work output in NIFTE.

*Keywords:* low-grade heat, thermofluidic oscillator, NIFTE, exergetic efficiency, cyclic steady state, limit cycle, stability analysis, bifurcation analysis

### 1. INTRODUCTION

High-temperature heat engines are ubiquitous in the power generation industry, by virtue of their high maximal efficiency. The high demand for high-temperature heat sources worldwide has led to a fast depletion of fossil fuels, meanwhile the combustion of these fuels releases a large quantity of greenhouse gases to the atmosphere. It is therefore urgent to exploit alternative heat sources. Any heat available at low temperature, typically below 250°C is classified as low-grade. Such low-grade heat is abundant in the solar and geothermal energy industry as well as the process and chemical industry in the form of "waste" heat. Nevertheless, the available low-grade heat recovery technologies are hindered by their low thermal efficiencies (Ammar et al., 2012), as a consequence of the second law of thermodynamics.

Thermofluidic oscillators are a class of engines that convert low-grade heat into useful work via the creation of oscillatory fluid motion. The non-inertive-feedback thermofluidic engine (NIFTE), as depicted in Fig. 1, is a particular type of two-phase thermofluidic oscillator (Smith, 2004, 2005, 1996). The working fluid exists both in the liquid phase and vapor phase. Low-grade heat from an external source is supplied to the system via the hot heat exchanger (HHX) [4] to vaporize the working fluid. The pressure in the adiabatic vapor region [9] thus increases, causing the liquid level in the power cylinder [1] to fall below the equilibrium level. The liquid level in the displacer cylinder [3] drops as a result of the hydrostatic pressure. The vapor comes into contact with the cold heat exchanger (CHX) [5], where heat is removed from the system, causing



Fig. 1. NIFTE fluid pump schematic (adapted from Markides et al., 2013). 1: liquid level in the power cylinder; 2: point of attachment of the load; 3: liquid level in the displacer cylinder; 4: hot heat exchanger; 5: cold heat exchanger; 6: feedback valve in the feedback line (connection); 7: power cylinder; 8: displacer cylinder; 9: adiabatic vapor region.

the vapor to condense and the liquid height to rise back above the equilibrium position. Sustained oscillations in the load [2] are thereby created by the cyclic condensation and evaporation of the working fluid in the cylinders. NIFTE is simple to construct and has few moving parts in comparison with conventional engines, making it both a reliable and cheap technology. Experimental prototypes have furthermore demonstrated that NIFTE is able to operate with a temperature difference as small as  $30^{\circ}$ C between the heat source and the cold sink (Markides and Smith, 2011).

Nearly all of the mathematical models of NIFTE proposed so far to analyze the dynamical behavior of NIFTE feature linear dynamics. Smith (2004, 2005, 1996) first developed a linear model, which builds on earlier work by Backhaus and Swift (2000) on thermoacoustic engines, and by Ceperley (1979) and Huang and Chuang (1996) on electrical analogies for thermal and fluid processes. In this model, the NIFTE components are described by linear submodels in analogy with electrical components; namely, resistors to account for fluid drag and thermal resistance, and capacitors to represent hydrostatic pressure and vapor compressibility. Moreover, the heat exchange rate between the hot and cold elements and the working fluid is assumed to be either linear or constant. The complete engine model then simply connects these electrical components according to the actual thermal-fluid process layout. This early model of NIFTE did not account for the inertive effect of the fluid, leading to rather inaccurate predictions. Markides and Smith (2011) and Solanki et al. (2012) later introduced inductors in the electrical analogy to account for the fluid inertia.

Notwithstanding their ability to provide accurate quantitative predictions of the NIFTE performance when tuned at their marginal stability limit, these linear models cannot describe the sustained, robust periodic oscillations of the working fluid, which is by essence a nonlinear phenomenon. Subject to a constant temperature difference between the two heat exchangers, the engine reaches a cyclic steady state after a sufficiently long settling time, where the amplitude and frequency of the oscillations remain constant. Moreover, when variations in the heat exchanger temperature difference or other external disturbances, which are inevitable in practice, occur the engine shall return to a cyclic steady state after some time. This is the signature of an asymptotically stable limit cycle in the dynamics, which may only occur in nonlinear dynamical systems.

Inspired by the earlier work of Daw et al. (1998), Chen et al. (1997), Hoffmann et al. (1997) and Sun et al. (1997) on exploiting the nonlinearity of core physical elements to predict the efficiency and robustness of thermodynamic engines, Markides et al. (2013) were the first to propose a nonlinear inertive model of NIFTE. The key nonlinearity in this model, referred to as NIFTE-NTP hereafter, corresponds to the inherent physical saturation in heat transfer rate between the hot and cold elements and the working fluid, whereas the rest of the engine components are described using the same electrical analogy as in the linear dynamic model by Markides and Smith (2011) and Solanki et al. (2012). It is noteworthy that the NIFTE-NTP has been validated experimentally.

This paper presents an analysis of NIFTE under cyclic steady-state (CSS) conditions, based on the NIFTE-NTP model. Our approach entails an dynamic optimization formulation to detect the CSS, and determines the stability of these CSS by analyzing their monodromy matrix. We also conduct a parametric analysis by varying key design parameters, in order to better understand the trade-offs between efficiency and work output in NIFTE. In the rest of the paper, Sec. 2 reviews the NIFTE-NTP model and presents the optimization-based methodology. The results of the cyclic steady-state analysis are presented and discussed in Sec. 3. Finally, Sec. 4 concludes the paper.

## 2. BACKGROUND AND METHODOLOGY

# 2.1 NIFTE Modeling

Two major driving forces at play in a NIFTE device are the pressure difference driving the mass flow of the working fluid, and the temperature difference responsible for the heat flow. The respective submodels for the fluid and thermal domains are described first, before giving the full NIFTE-NTP model equations; refer to Markides et al. (2013) for further details on the NIFTE-NTP model development.

Fluid Domain The following electrical analogies are used to model the various components of the engine. The drag experienced by the working fluid in the load [2] (due to viscosity) and in the feedback connection [6] (due to the values) are represented by resistors, denoted by  $R_i$ . The inertive effect experienced by the oscillating liquid flow in the feedback tube [6], load, displacer cylinder [3] and power cylinder [7], are modeled by inductors, denoted by  $L_i$ . The hydrostatic pressures at the bottom of the displacer cylinder [3] and power cylinder [7], and the adiabatic compression/expansion in the vapor region [9] are represented by capacitors, denoted by  $C_i$ . Notice that the processes undergone by the working fluid in the vapor region [9], i.e. the vapor compression and expansion, are assumed to be both adiabatic and reversible (i.e. isentropic). Linear first-order ordinary differential equations (ODEs) can be derived from these analogies by applying the principle of superposition in the case of multiple effects occurring within one particular component.

Thermal Domain A key feature of the NIFTE-NTP model is accouting for the nonlinearity in heat transfer rate between the heat exchangers and the working fluid. The fact that the wall temperature of the heat exchanger,  $T_{hx}$  cannot increase or decrease indefinitely with the liquid level in the displacer cylinder [3] is captured via the following saturation relationship between  $T_{hx}$  and the liquid level z,

$$T_{hx}(z) = \alpha \tanh(\beta z) \,,$$

as depicted in Fig. 2. The equilibrium position z = 0 is taken to be halfway between the two heat exchangers; saturation amplitude and the slope at the origin are  $\alpha$ and  $\alpha\beta$ , respectively.

Complete MIFTE-NTP Model The differential equations describing the NIFTE device in analogy to a closed electric RLC circuit are given in Eqs. (1a-e) below. The thermodynamic states represent the pressure in the adiabatic vapor region,  $\hat{P}_{ad}$ ; the hydrostatic pressure in the displacer cylinder,  $\hat{P}_d$  and in the power cylinders,  $\hat{P}_p$ ; and the volumetric flowrates in the feedback tube,  $\hat{U}_f$  and in the power cylinder  $\hat{U}_p$ . All five states are dimensionless and express deviations with respect to a reference for z = 0.

$$\frac{\mathrm{d}\hat{P}_{ad}}{\mathrm{d}\hat{t}} = \frac{\tau[K\tanh(\Lambda\hat{P}_d) - \hat{P}_{ad}]}{R_{tb}C_{ad}} + \frac{\tau U_0(\hat{U}_f + \hat{U}_p)}{P_0C_{ad}} \tag{1a}$$

$$\tau U_0 \hat{U}_f$$
 (1b)

$$\frac{\mathrm{d}t}{\mathrm{d}\hat{t}} = \frac{\tau U_0 \hat{U}_p}{P_0 C_p} \tag{1c}$$

$$\frac{\mathrm{d}\hat{U}_f}{\mathrm{d}\hat{t}} = \frac{(\tau P_0/U_0)[L_l\hat{P}_p - L_p\hat{P}_{ad} - (L_p + L_l)\hat{P}_d] - \tau[L_lR_f + L_p(R_f + R_l)]\hat{U}_f - \tau L_pR_l\hat{U}_p}{L_l(L_d + L_f) + L_p(L_d + L_f + L_l)} \tag{1d}$$

$$\frac{\mathrm{d}\hat{U}_p}{\mathrm{d}\hat{t}} = \frac{(\tau P_0/U_0)[L_l\hat{P}_d - )(L_d + L_f)\hat{P}_{ad} - (L_d + L_f + L_l)\hat{P}_p] + \tau[L_lR_f - (L_d + L_f)R_l]\hat{U}_f - \tau(L_d + L_f)R_l\hat{U}_p}{L_l(L_d + L_f) + L_p(L_d + L_f + L_l)} \tag{1e}$$



Fig. 2. Saturation relationship between the temperature of the heat exchanger wall and the liquid level in the displacer cylinder.

The nominal parameter values for  $R_i$ ,  $L_i$  and  $C_i$  are reported in Table 1 below, whereas the values of the corresponding physical and geometrical parameters can be found in Markides et al. (2013).

 

 Table 1. Nominal parameter values in NIFTE-NTP model (source: Markides and Smith, 2011).

Parameter	Nominal values	Units	
$R_{f}$	$2.13 \times 10^6$	kg/m <sup>4</sup> s	
$R_l$	$4.08  imes 10^6$	$ m kg/m^4s$	
$R_{th}$	$5.02 \times 10^8$	$\rm kg/m^4 s$	
$L_d$	$1.80  imes 10^5$	$kg/m^4$	
$L_{f}$	$4.74  imes 10^6$	$kg/m^4$	
$L_l$	$1.27 \times 10^7$	$kg/m^4$	
$L_p$	$3.77  imes 10^5$	$kg/m^4$	
$C_{ad}$	$1.76  imes 10^{-9}$	$m^4 s^2/kg$	
$C_d$	$7.35 \times 10^{-8}$	$m^4 s^2/kg$	
$C_p$	$7.43 \times 10^{-8}$	$m^4 s^2/kg$	
$\dot{K}$	1.37	_	
$P_0$	$1.013 \times 10^5$	$\mathbf{Pa}$	
$U_0$	$8.00 \times 10^{-4}$	$m^3/s$	
au	5	s	
Λ	330	-	

Performance Criteria A key performance indicator is the useful work output from the load,  $W_l$ , given by

$$W_l := P_l(t)U_l(t)$$

where  $U_l$  and  $P_l$  denote the volumetric flowrate in the load and the pressure inside the load, respectively; and the average value is calculated over a steady-state cycle of the engine (see Sec. 2.2 below). Another two meaningful performance criteria are the device exergetic efficiency,  $\eta_{dev}$  and the system exergetic efficiency,  $\eta_{sys}$ , defined as

$$\eta_{dev} := \frac{P_l(t)U_l(t)}{P_{th}(t)U_{th}(t)}, \qquad (2a)$$

$$\eta_{sys} := \frac{\overline{P_l(t)U_l(t)} + \overline{P_f(t)U_f(t)}}{\overline{P_{th}(t)U_{th}(t)}}, \qquad (2b)$$

with  $P_f$  and  $P_{th}$ , the pressure in the feedback tube and the input thermal pressure; and  $U_f$  and  $U_{th}$ , the volumetric flowrates through the feedback tube and associated with the phase change. All of these auxiliary variables can be expressed from the RLC circuit analogy by applying Kirchoff's circuit laws (Markides et al., 2013).

### 2.2 Cyclic Steady-State Detection and Analysis

The search for a CSS of NIFTE can be automated by solving the following constrained dynamic optimization problem,

$$\min_{\mathbf{x}^{0},T} T$$
s.t.  $\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{p}),$ 

$$\mathbf{x}(0) = \mathbf{x}(T) = \mathbf{x}^{0},$$

$$(3)$$

where T denotes a cycle period; the state vector  $\mathbf{x}$  comprises the thermodynamic states  $\hat{P}_{ad}, \hat{P}_d, \hat{P}_p, \hat{U}_f, \hat{U}_p$ ;  $\mathbf{p}$  is the vector of model parameters; and  $\mathbf{f}$  comprises the right-hand-side functions in Eqs. (1a-e). Observe that this problem is over-parameterized, and one of the initial states can be forced to zero, e.g.  $\hat{P}_{ad}(0) = 0$ , without any loss of generality.

The optimization problem (3) is nonconvex and may therefore present multiple (local) optima. Herein, we use a simple single-shooting approach as implemented in our inhouse optimisation toolkit CRONOS<sup>1</sup> to compute such local optima, and we apply a multi-start search heuristic based on Sobol sampling to detect the presence of multiple local optima. CRONOS provides an interface to the NLP solver IPOPT (Wächter and Biegler, 2006) coupled with the ODE integrator CVODES in SUNDIALS (Hindmarsh et al., 2005) for computing the objective/constraint functions and their gradients.

 $<sup>^1\,</sup>$  available from: https://github.com/omega-icl/cronos

Any solution  $\overline{\mathbf{x}}(\cdot, \mathbf{p})$  to the problem (3) with T > 0 corresponds to a CSS of NIFTE. The stability of the corresponding limit cycle can be characterized by analyzing the so-called monodromy matrix  $\Phi_{\mathbf{p}}(T, 0)$ , which is the fundamental matrix solution to the variational equation (Grass et al., 2008; Teschl, 2012)

$$\forall \tau, t \in [0, T], \quad \frac{\partial \Phi_{\mathbf{p}}(t, \tau)}{\partial t} = \frac{\partial \mathbf{f}(t, \overline{\mathbf{x}}(t, \mathbf{p}), \mathbf{p})}{\partial \mathbf{x}} \Phi_{\mathbf{p}}(t, \tau) \quad (4a)$$

with 
$$\Phi_{\mathbf{p}}(\tau, \tau) = I$$
. (4b)

One (at least) of the eigenvalues of the monodromy matrix shall be equal to 1. The limit cycle  $\overline{\mathbf{x}}(\cdot, \mathbf{p})$  is asymptotically stable (attractor) if all of the remaining eigenvalues lie within the open unit disk,

$$\max |\lambda_i(\Phi_{\mathbf{p}}(T,0))| < 1.$$
(5)

If any of these eigenvalues has a modulus greater than 1, the limit cycle is otherwise unstable (repeller).

### 3. RESULTS AND DISCUSSIONS

### 3.1 CSS Multiplicity and Stability in NIFTE

By applying the optimization-based approach in Sec. 2.2, we find that a given engine may exhibit multiple CSS for certain values of the model parameters, an observation that had never been reported before. Moreover, these CSS may be either stable or unstable.

An illustration of this behavior is presented next with the simulation of an engine having all its RLC parameters at their nominal values in Table 1, except for the value of parameter  $L_f = 4.74 \times 10^5$  kg/m<sup>4</sup>s (10 times smaller than the nominal value). Three solutions to the optimization problem (3) are detected, two of which are non-trivial CSS with different periods (CSS1 with T = 1.30s and CSS2 with T = 0.88s), and the third one is an equilibrium point at the origin (EQ1). The eigenvalues of the monodromy matrices in the cases of CSS1 and CSS2, as well as the eigenvalues of the Jacobian matrix in the case of EQ1, are reported in Table 2.

Table 2. Stability of CSS solutions detected for an engine with nominal parameter values, except  $L_f = 4.74 \times 10^5 \text{ kg/m}^4 \text{s}.$ 

CSS1	CSS2	EQ1
-0.4704 + 0.2959i	-0.3848 + 1.2078i	8.8167+42.9535i
-0.4704 - 0.2959i	-0.3848 - 1.2078i	8.8167 - 42.9535i
0.1028	0.0348	-0.9957 + 29.2728i
0.3093	0.7958	-0.9957 - 29.2728i
1.0000	1.0000	-24.8325

Since all the eigenvalues of the monodromy matrix for CSS1 lie within the unit disk, this CSS describes a stable limit cycle. By contrast, the monodromy matrix for CSS2 has two of its eigenvalues outside the unit disk, and so CSS2 describes an unstable limit cycle. Lastly, the Jacobian matrix associated with CSS3 has eigenvalues with positive real parts, which indicates that the origin is an unstable equilibrium point.

The pressure-volume (PV) phase diagrams shown in Fig. 3 are obtained by simulating the engine over an extensive time horizon, as a result of initializing the simulation in the vicinity of CSS1, CSS2, and EQ1. It is clear that CSS1 in Fig. 3(a) acts as an attractor for the trajectories. By

contrast, the trajectory initialized near CSS2 eventually escape away from that limit cycle and are attracted by CSS1, thereby showing two clusters of trajectories in Fig. 3(b). Finally, the trajectory initialized near EQ1 at the origin is initially repelled and then attracted by CSS1 in Fig. 3(c). All of these observations are in good agreement with the numerical stability results in Table 2.



Fig. 3. PV phase diagrams for the load of an engine with nominal parameter values, except  $L_f = 4.74 \times 10^5 \text{ kg/m}^4$ s. The red dots indicate the respective starting points of the PV trajectories.

A comparison of the engine performance when operated at CSS1 and CSS2 is presented in Table 3. It is noteworthy that operating at the unstable CSS2 leads to significantly higher exergetic efficiencies compared to the stable CSS1 in this configuration. Nevertheless, the useful work outputs at CSS1 and CSS2 are comparable. This is because CSS2 has a shorter period and amplitude, and so significantly less exergy  $B_{th}$ , as given by

# $B_{th} := \overline{P_{th}(t)U_{th}(t)} \,,$

is supplied to the engine during a single cycle at CSS2.

Table 3. Performance comparison at CSS1 and CSS2 of an engine with nominal parameter values, except  $L_f = 4.74 \times 10^5 \text{ kg/m}^4 \text{s.}$ 

	period (s)	$\eta_{dev}$ (%)	$\eta_{sys}$ (%)	$W_l$ (J/cycle)	$B_{th}$ (J/cycle)
CSS1 (stable)	1.30	3.07	3.39	5.16	168.2
CSS2 (unstable)	0.88	24.4	24.9	5.88	23.9

### 3.2 CSS Parametric Analysis

Previous studies have shown that the performance of NIFTE is most sensitive to the parameter  $R_f$ , representing the feedback resistance due to fluid drag (Markides and Smith, 2011; Markides et al., 2013). The results in Fig. 4(a) are for an engine with  $R_f$  varying in a wide range around its nominal value of  $2.13 \times 10^6$  kg/m<sup>4</sup>s, all of the other parameters being kept at their nominal values in Table 1. Only one stable CSS is detected across the whole range of  $R_f$  (apart from the unstable origin equilibrium point), which is in agreement with the results in Markides et al. (2013).

The plot in Fig. 4(b) shows the relationships between the work output  $W_l$  over a single CSS oscillation and exergetic efficiencies  $\eta_{dev}$  and  $\eta_{sys}$  as  $R_f$  is varied. While the relationship between  $W_l$  and  $\eta_{dev}$  is essentially monotonic, the other relationship  $W_l$  and  $\eta_{sys}$  is more complex. This indicates a potential trade-off between efficiency and work output, which should be taken into consideration for the practical design of NIFTE.

The modification of  $R_f$  may potentially lead to a change in the value of  $L_f$ , representing the inductance in the feedback tube (due to fluid inertia), as they are both dependent on the geometry of the feedback tube. The results in Fig. 5(a) are for an engine with  $L_f$  varying in a wide range around its nominal value of  $4.74 \times 10^5$  kg/m<sup>4</sup>s, all of the other parameters being kept at their nominal values in Table 1. In agreement with the earlier discussion in Sec. 3.1, multiple CSS are obtained for certain values of  $L_f$ . The stability of these CSS is represented using solid lines for a stable CSS and a dotted line for an unstable CSS.

We find that CSS2 is stable for the smallest  $L_f$  values, before becoming unstable as  $L_f$  increases, and finally disappearing around  $L_f^+ \approx 0.12$ . Also, the exergetic efficiency of CSS2 increases monotonically with  $L_f$ . CSS1 appears for  $L_f^+$  greater than 0.033. This CSS is initially unstable, before becoming stable as  $L_f$  keeps increasing. The variations in both exergetic efficiencies for CSS1 turn out to be rather complex as  $L_f$  varies. It is noteworthy that there exists a small range of values for  $L_f$ , wherein both CSS1



Fig. 4. (a) Effect of the feedback resistance  $R_f$  on the device and system exergetic efficiencies,  $\eta_{dev}$  and  $\eta_{sys}$  for the NIFTE engine.  $R_f^+$  refers to  $R_f$  normalized by its nominal value in Table 1. (b) Useful work output from the load over a single cycle of oscillation against the device and system exergetic efficiencies for the NIFTE engine as  $R_f$  is varied.

and CSS2 are stable. This presents a clear opportunity for improving the performance of NIFTE, without the need for controlling the engine at an unstable limit cycle. However, it is clear from Fig. 5(b) that operating at CSS1 can also generate more useful work than CSS2 for some certain values of  $L_f$ . Therefore, whether the primary target is to maximize the efficiency or to produce more work will drive the design of the engine.

### 4. CONCLUSIONS

This paper has presented an analysis of NIFTE under cyclic steady-state conditions, by applying an optimization-based approach to detect the CSS and subsequently determining their stability. It has been found that NIFTE can exhibit multiple CSS for certain values of the design parameters, which may be either stable or unstable, a result that had not been reported before. This CSS multiplicity could have major implications for improving the performance of NIFTE, both in terms of design and control. As part of future work, we will investigate the application of systematic optimization methods based on



Fig. 5. (a) Effect of the feedback inductance  $L_f$  on the device and system exergetic efficiencies,  $\eta_{dev}$  and  $\eta_{sys}$  for the NIFTE engine.  $L_f^+$  refers to  $L_f$  normalized by its nominal value in Table 1. (b) Useful work output from the load over a single cycle of oscillation against the device and system exergetic efficiencies for the NIFTE engine as  $L_f$  is varied. #1 and #2 refer to CSS1 and CSS2, respectively. Unstable CSS are marked with dashed lines, while stable CSS are marked with solid lines.

integrated design and control to NIFTE, which presents major numerical challenges.

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#### NOMENCLATURE

- capacitance due to adiabatic compressibility,  $\rm m^4s^2kg^{-1}$  $C_{ad}$ capacitance in the displacer cylinder,  $\rm m^4 s^2 kg^{-1}$  $C_d$  $C_p$ capacitance in the power cylinder, m<sup>4</sup>s<sup>2</sup>kg<sup>-1</sup> -4 inductance in the displacer cylinder,  ${\rm kg\,m^-}$  $L_d$ inductance in the feedback tube, kg m  $L_f$ inductance in the load,  $\rm kg\,m^{-4}$  $L_l$  $L_p$ inductance in the power cylinder,  ${\rm kg\,m^{-4}}$  $P_{ad}$ pressure in the adiabatic region relative to atmospheric pressure, Pa  $P_d$ hydrostatic pressure in the displacer cylinder, Pa  $P_0$ reference pressure for scaling, Pa  $\hat{P}$ dimensionless pressure  $R_{f}$ resistance in the feedback tube,  $\rm kg\,m^{-4}s^{-1}$  $R_l$ resistance in the load,  $kg m^{-4} s^{-1}$ thermal resistance,  $\rm kg\,m^{-4}s^{-1}$  $R_{th}$ time. s  $U_f$ volumetric flow rate in the feedback tube,  $m^3s^{-1}$  $U_p$ volumetric flowrate in the power cylinder, m<sup>3</sup>s<sup>-1</sup>
- reference flowrate for scaling, m<sup>3</sup>s<sup>-1</sup>  $U_0$
- $\hat{U}$ dimensionless volumetric flowrate
- Kdimensionless heat transfer gain
- Λ dimensionless heat transfer slope

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