PI-control design of continuum models of production systems governed by scalar hyperbolic partial differential equation

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Abstract: A production system which produces plenty of items in many steps can be modelled as a continuous flow problem governed by a nonlinear and nonlocal hyperbolic partial differential equation. One way to adjust the output of such process is by manipulating the start rate. This paper considers the control and regulation by proportional-integral (PI) controllers for the continuum production systems. In the considered system, the input and output are located at the boundary of production system. In particular, the closed-loop stability of the linearized continuum production model with the designed PI-controller is proved using spectral analysis and Lyapunov theory. Numerical results demonstrate successful tracking for step inputs in the demand rate.

Keywords: Optimal state estimation, Inequality constraints, Linear time-invariant system.

1. INTRODUCTION

Recent years, several continuum models were introduced to simulate the average behavior of production systems at an aggregate level, see Armbruster et al. (2006), La Marca et al. (2010). The continuum model is governed by scalar hyperbolic partial differential equations and the model description is appropriate, for this work a semiconductor factory producing a large number of items in many steps. The mathematical variable used to describe the production flow is a density variable $\rho(x,t)$ denoting the production density at stage x at a time t. In this work, without loss of generality, we scale $x \in [0,1]$, where x = 0denotes the beginning of the production flow line and x = 1 denotes the end. It should be noted that the velocity of the production flow along entire system is a constant. This explains that in a real world factory, all parts move through the factory with the same speed. In a serial production system, velocity through the factory is dependent on all items and machines downstream.

The problem of control systems described by hyperbolic partial differential equations has received a considerable amount of attentions Aksikas et al. (2008), Bastin and Coron (2016), Xu and Dubljevic (2016). For a complete and detailed review, readers should refer Luo et al. (2012). Numerous, available results have been provided to guarantee asymptotical and exponential stability of the closedloop hyperbolic systems Coron et al. (2007), Dos Santos et al. (2008), Coron and Bastin (2015) and in addition to realize the regulation Xu et al. (2017), Paunonen (2017). The backstepping methods have been explored for the regulation of infinite-dimensional systems in Deutscher (2015), Deutscher (2017), Xu and Dubljevic (2017b). In this work, our objective is to design a PI-controller to ensure the stability of the equilibrium of the nonlinear close-loop production system and the output regulation to a desired set-point. The idea of using output feedback control for infinite-dimensional systems is motivated by Pohjolainen (1982), Deutscher (2011), Xu and Dubljevic (2017a). In Blandin et al. (2017), an internal state feedback controller has been designed to stabilize the entropy solutions around a constant equilibrium in the L^1 and L^{∞} . A nonlocal stabilization boundary controller has been developed in Perrollaz (2013) to obtain asymptotic stability of the constant equilibrium in the L^2 . In this paper, a 1-D scalar hyperbolic PDE is considered and we are interested in a boundary PI-controller design to asymptotically stabilize the system around its equilibrium profile. Inspired by Coron et al. (2007) and Trinh et al. (2017), we construct a new Lyapunov functional to prove exponential stabilization of the closed-loop system with the designed PI-controller.

The paper is organized as follows: Section 2 introduces the statement of the considered model and problem. Then, Lypunov direct method is provided and applied to the linearized model. Section 3 gives the discussion of the stability from the point of frequency domain. Section 4 implements the PI-controller through numerical simulations. Finally, our conclusions are given in Section 5.

2. STATEMENT OF THE PROBLEM AND RESULT

In this work, the time evaluation of the product density governed by the following 1-D hyperbolic PDE is considered:

$$\partial_t \rho(x,t) + \lambda(W) \partial_x \rho(x,t) = 0$$

$$\lambda(W) \rho(0,t) = u(t)$$
(1)

$$\lambda(W) \rho(1,t) = y(t)$$

where $\rho(x,t)$ is a density variable describing the density of products at stage x of the production at a time t, and $\lambda(W)$ is a velocity function that depends on the density $\rho(x,t)$ only, while $\lambda(W)$ is given by:

$$\lambda(W) = \frac{1}{1+W}, W = \int_0^1 \rho(x,t) dx$$

Note that the velocity function λ is bounded, positive and monotonically decreasing. Here, we scale $x \in [0, 1]$ with x = 0 denoting the beginning of the production line and x = 1 the end.

Obviously, if the influx u(t) and the initial condition $\rho_0(x)$ are nonnegative, then the production density is and will always remain nonnegative. In addition, the system satisfies the following properties:

- The system velocity $\lambda(\rho)$ is positive;
- $\lambda(\rho)$ is bounded; $0 < \lambda < v_{\max}$;
- $\lambda(\rho)$ has no spacial dependence because of integration.

u(t) and y(t) denote the rates of products entering and exiting the fab at x = 0 and x = 1, i.e., in-flux and out-flux.

Our control objective is to design a dynamic output feedback controller such that asymptotic stabilization of the closed-loop system is achieved and it is ensured that the outflux y(t) converges to a desired set-point $y_r \in \mathbb{R}$, as $t \to +\infty$.

In this work, the considered output feedback control law has an integral structure and hence we write the control law as follows:

$$u(t) = k_P \left(y(t) - \bar{\rho}\lambda(\bar{\rho}) \right) - k_I \zeta(t) + \lambda(\bar{\rho})\bar{\rho}$$

$$\dot{\zeta}(t) = \lambda(W)\rho(1,t) - \lambda(\bar{\rho})\bar{\rho}$$
(2)

in which $k_I \in \mathbb{R}$ is a tunning parameter and $\bar{\rho} \in \mathbb{R}$ is the equilibrium of interest at which stabilization is desired as time $t \to +\infty$.

2.1 Stabilization to $\bar{\rho}$ for the linearized system

Since it is rather complex to study the control problem for the nonlinear system (1). In this work, we focus on the development of an integral boundary control law for the linearized model at $\bar{\rho} \in \mathbb{R}$.

Let us define the following variables:

$$\rho(x,t) = \tilde{\rho}(x,t) + \bar{\rho}, \xi(t) = \frac{\zeta(t)}{\lambda(\bar{\rho})}$$

and approximate $\lambda(W)$ using

$$\Lambda(W) = \lambda(\bar{\rho}) \tag{3}$$

Then, the linearized model at $\bar{\rho}$ is given by:

$$\partial_t \tilde{\rho}(x,t) + \lambda(\bar{\rho}) \partial_x \tilde{\rho}(x,t) = 0$$

$$\tilde{\rho}(0,t) = k_P \tilde{\rho}(1,t) - k_I \xi(t)$$
(4)

$$\dot{\xi}(t) = \tilde{\rho}(1,t)$$

During the linearization, $\lambda(W)$ in the process was approximated by $\lambda(\bar{\rho})$ to remove the nonlinearity. Similar linearization can be found in Coron and Wang (2013).

Lyapunov candidate V is defined as:

$$V(t) = \int_0^1 \left[\tilde{\rho}^2(x) e^{-\mu x} + q_1 \xi \tilde{\rho}(x) e^{-\frac{\mu}{2}x} \right] dx + q_2 \xi^2$$

where $\mu > 0$, $q_i > 0 \forall i = 1, 2$ are chosen later. Consider the function $\Gamma : [0, 2] \to \mathbb{R}$ such that $\Gamma(z) = \sqrt{z(2-z)}e^{-z/2}$. It is easy to find the maximum value of $\Gamma(z)$ in [0, 2] is located at $z = 2 - \sqrt{2}$.

Theorem 1. Let $k_I^* = \frac{\lambda(\bar{\rho})}{2}\Gamma(2-\sqrt{2})$. Chose $k_I \in (0, k_I^*)$, $k_P \in [0, \frac{\sqrt{2}}{2}e^{-\mu/2}]$ and $\mu \in (0, 2-\sqrt{2})$. Let $q_1 = 2k_I$ and $q_2 = k_I\lambda(\bar{\rho})\left(e^{-\frac{\mu}{2}} + 3k_P\right)$. Then, there exists positive constant $\phi > 0$ such that

$$\dot{V}(t) \le -\phi V(t) - \lambda \left(\bar{\rho}\right) \left[\frac{e^{-\mu}}{2} - k_P^2\right] \tilde{\rho}^2(1, t) \qquad (5)$$

Proof: Rewrite V in the following:

$$V(t) = \int_0^1 \left[\tilde{\rho}(x) e^{-\mu x/2} \xi \right] Q \left[\frac{\tilde{\rho}(x) e^{-\mu x/2}}{\xi} \right] dx$$
re

where

$$Q = \begin{bmatrix} 1 & \frac{q_1}{2} \\ \frac{q_1}{2} & q_2 \end{bmatrix}$$

If $det(Q) \ge 0$, then V(t) is positive definite for all $t \ge 0$. Compute det(Q):

$$\det(Q) = \left[\frac{\lambda\left(\bar{\rho}\right)\Gamma(\mu)}{2} - k_I + \frac{\lambda\left(\bar{\rho}\right)}{2}e^{-\frac{\mu}{2}}\left(2 - \sqrt{\mu(2-\mu)}\right) + 3\lambda\left(\bar{\rho}\right)k_P\right]$$

Since $k_I < \frac{\lambda(\bar{\rho})\Gamma(\mu)}{2}$, $\mu < 2$ and $k_P > 0$, it is easy to conclude that $\det(Q) > 0$. Obviously, with configuration of k_I and k_P , we easily conclude that $q_1 > 0$ and $q_2 > 0$ and V(t) is positive.

We assume that the initial conditions are smooth enough so that the solution of (4) is continuously differentiable with respect to time t and space x. Then, differentiating V along the the state trajectories and using integration by parts gives:

$$\begin{split} \dot{V}(t) &= \int_{0}^{1} 2\tilde{\rho}(x)\partial_{t}\tilde{\rho}(x)e^{-\mu x}dx + 2q_{2}\xi\dot{\xi} \\ &+ q_{1}\xi\int_{0}^{1}\partial_{t}\tilde{\rho}(x)e^{-\frac{\mu}{2}x}dx + q_{1}\dot{\xi}\int_{0}^{1}\tilde{\rho}(x)e^{-\frac{\mu}{2}x}dx \\ &= -\lambda\left(\bar{\rho}\right)\tilde{\rho}^{2}(1)e^{-\mu} + \lambda\left(\bar{\rho}\right)\tilde{\rho}^{2}(0) - \lambda\left(\bar{\rho}\right)\mu\int_{0}^{1}\tilde{\rho}^{2}(x)e^{-\mu x}dx \\ &- q_{1}\lambda\left(\bar{\rho}\right)\xi\tilde{\rho}(1)e^{-\frac{\mu}{2}} + \lambda\left(\bar{\rho}\right)q_{1}\xi\tilde{\rho}(0) \\ &-\lambda\left(\bar{\rho}\right)q_{1}\xi\frac{\mu}{2}\int_{0}^{1}\tilde{\rho}(x)e^{-\frac{\mu}{2}x}dx \\ &+ q_{1}\tilde{\rho}(1)\int_{0}^{1}\tilde{\rho}(x)e^{-\frac{\mu}{2}x}dx + 2q_{2}\xi\tilde{\rho}(1) \end{split}$$

$$= -\lambda \left(\bar{\rho}\right) \left[e^{-\mu} - k_P^2\right] \tilde{\rho}^2(1) - \lambda \left(\bar{\rho}\right) \mu \int_0^1 \tilde{\rho}^2(x) e^{-\mu x} dx \\ -\lambda \left(\bar{\rho}\right) k_I^2 \xi^2 - \lambda \left(\bar{\rho}\right) \mu k_I \xi \int_0^1 \tilde{\rho}(x) e^{-\frac{\mu}{2}x} dx \\ +2k_I \tilde{\rho}(1) \int_0^1 \tilde{\rho}(x) e^{-\frac{\mu}{2}x} dx \\ +2 \left[q_2 - k_I \lambda \left(\bar{\rho}\right) e^{-\frac{\mu}{2}} - k_I \lambda \left(\bar{\rho}\right) 3k_P\right] \xi \tilde{\rho}(1)$$

By using the Young and Cauchy-Schwarz inequalities, we have

$$2k_{I}\tilde{\rho}(1)\int_{0}^{1}\tilde{\rho}(x)e^{-\frac{\mu}{2}x}dx$$

$$\leq \left(\frac{\lambda(\bar{\rho})e^{-\mu}}{2}\right)\tilde{\rho}^{2}(1,t) + \left(\frac{2k_{I}^{2}e^{\mu}}{\lambda(\bar{\rho})}\right)\int_{0}^{1}e^{-\mu x}\tilde{\rho}^{2}(x,t)dx$$

$$\left|\lambda(\bar{\rho})\mu k_{I}\xi\int_{0}^{1}\tilde{\rho}(x)e^{-\frac{\mu}{2}x}dx\right|$$

$$\leq \left(\frac{\lambda(\bar{\rho})k_{I}^{2}}{2}\right)\xi^{2}(t) + \left(\frac{\lambda(\bar{\rho})\mu^{2}}{2}\right)\int_{0}^{1}e^{-\mu x}\tilde{\rho}^{2}(x,t)dx$$

Then, by substituting the provided q_1 and q_2 into the above inequalities we obtain the following:

$$\begin{split} \dot{V}(t) &\leq -\lambda \left(\bar{\rho}\right) \left[\frac{e^{-\mu}}{2} - k_P^2\right] \tilde{\rho}^2(1) \\ &-\lambda \left(\bar{\rho}\right) k_I^2 \xi^2 + \frac{\lambda \left(\bar{\rho}\right)}{2} k_I^2 \xi^2 \\ &- \left[\lambda \left(\bar{\rho}\right) \mu - \frac{\lambda \left(\bar{\rho}\right) \mu^2}{2} - \frac{2k_I^2 e^{\mu}}{\lambda \left(\bar{\rho}\right)}\right] \int_0^1 \tilde{\rho}^2(x) e^{-\mu x} dx \end{split}$$

We rewrite the above inequality:

$$\dot{V}(t) \leq -\lambda \left(\bar{\rho}\right) \left[\frac{e^{-\mu}}{2} - k_P^2\right] \tilde{\rho}^2(1) - \frac{\lambda \left(\bar{\rho}\right)}{2} k_I^2 \xi^2 -\frac{2e^{\mu}}{\lambda \left(\bar{\rho}\right)} \left(\frac{\lambda \left(\bar{\rho}\right)}{2} \Gamma(\mu) + k_I\right) \left(\frac{\lambda \left(\bar{\rho}\right)}{2} \Gamma(\mu) - k_I\right) J(t)$$

where

$$J(t) = \frac{2e^{\mu}}{\lambda(\bar{\rho})} \int_0^1 \bar{\rho}^2(x) e^{-\mu x} dx$$

With chosen μ , we have $k_I < \frac{\lambda(\bar{\rho})\Gamma(\mu)}{2}$. Then, there exits a positive real number M such that

$$\begin{split} \dot{V}(t) &\leq -M \left[\xi^2(t) + \int_0^1 \tilde{\rho}^2(x) e^{-\mu x} dx \right] \\ &-\lambda \left(\bar{\rho} \right) \left[\frac{e^{-\mu}}{2} - k_P^2 \right] \tilde{\rho}^2(1,t) \end{split}$$
plete the proof.

This complete the proof.

3. SPECTRAL ANALYSIS

3.1 P-Controller

Suppose that $k_I = 0$ and that we are interested simply in speeding up the dynamics of the system:

$$\partial_t \tilde{\rho}(x,t) + \lambda(\bar{\rho}) \partial_x \tilde{\rho}(x,t) = 0$$

$$\tilde{\rho}(0,t) = k_P \tilde{\rho}(1,t)$$
(6)

We set the new control input: $\tilde{\rho}(0,t) = k_P \tilde{\rho}(1,t) + v(t)$. Carrying out the Laplace transformation gives:

 $s\hat{\rho}(x,s) + \lambda(\bar{\rho})\partial_x\hat{\rho}(x,s) = 0$ $\hat{\rho}(0,s) = k_P \hat{\rho}(1,\hat{s})$

and the transfer function is obtained:

$$G(s) = \frac{\lambda e^{-\lambda^{-1}s}}{1 - k_P e^{-\lambda^{-1}s}}$$

The corresponding characteristic equation is written as:

$$1 - k_P e^{-\lambda^{-1}s} = 0 (7)$$

Let
$$\mu = \lambda^{-1}s$$
 and then the equation (7) becomes:

$$1 - k_P e^{-\mu} = 0 \tag{8}$$

Let set $\mu = \sigma + i\eta$ with $\sigma, \eta \in \mathbb{R}$. Then (8) is written as follows by separating the real part and the imaginary part:

$$e^{\sigma}\cos\eta - k_P = 0 \tag{9}$$

$$e^{\sigma} \sin \eta = 0 \tag{10}$$

Obviously, (10) implies $\eta = n\pi$ with $n \in \mathbb{N}$. Then, (9) becomes:

$$\begin{cases} e^{\sigma} + k_P = 0, \eta = (2n+1)\pi\\ e^{\sigma} - k_P = 0, \eta = 2n\pi \end{cases}$$

It is easy to conclude that for the case $k_P \in (-1, 1)$, the equation has no solution $\sigma \geq 0$, i.e., the closed-loop system is exponentially stable.

3.2 PI-Controller

Suppose now that a suitable $k_P \in (-1, 0)$ has been found and let us now apply the controller:

$$\tilde{\rho}(0,t) = k_P \tilde{\rho}(1,t) - k_I \int_0^t y(\tau) - y_r(\tau) d\tau \qquad (11)$$

to the system (4).

Proposition 1. let S(t) for $t \ge 0$ be the C_0 -semigroup on $L^{(0,1)}$ the corresponds to the solution of (4) with (11). Let A be the infinitesimal generator of the semigroup $S(t)(t \ge 0)$ and let $\sigma(A)$ be the spectrum of A. The stability margin is defined as

$$\omega(A) := \inf \left\{ \omega \in R | \exists M = M(\omega) : ||S(t)|| \le M e^{\omega t} \ge 0 \right\}$$

Then, the following equation holds:

$$\omega(A) = s(A)$$

where $s(A) := \sup \{ \Re(\mu) | \mu \in \sigma(A) \}$ and $\Re(\mu)$ denotes the real part of μ .

To formulate the transfer function, we set v(t) as the new control input with y(t) as the output:

$$\tilde{\rho}(0,t) = k_P \tilde{\rho}(1,t) - k_I \xi(t) + v(t)$$

By taking Laplace transform, we have:

$$\begin{aligned} s\hat{\rho} + \lambda\hat{\rho}_x &= 0\\ s\hat{\xi} &= \hat{\rho}(1,s)\\ \hat{\rho}(0,s) &= k_P\hat{\rho}(1,s) - k_I\hat{\xi}(s) + \hat{v}(s)\\ \hat{y}(s) &= \lambda\hat{\rho}(1,s) \end{aligned}$$

Finally, we get the transfer function as follows:

$$\frac{\hat{y}(s)}{\hat{v}(s)} = \frac{\lambda}{e^{\lambda^{-1}s} - k_P + \frac{k_I}{s}}$$

We now analyze the nontrivial solution of the following equation:

$$e^{\lambda^{-1}s} - k_P + \frac{k_I}{s} = 0$$

We set

 $\mu = s\lambda^{-1}(\bar{\rho}), \alpha = k_I\lambda^{-1}(\bar{\rho})$

Then the characteristic equation now becomes:

$$1 - k_P e^{-\mu} + \alpha \frac{e^{-\mu}}{\mu} = 0, \mu \in \mathbb{C} \setminus \{0\}$$

It is equivalent to study the zeros of the following continuous and holomorphic function:

$$f_{\alpha,k_P}(\mu) = 1 - k_P e^{-\mu} + \alpha \frac{e^{-\mu}}{\mu}$$

Here, we just define the right half plan

$$\Omega = \{ \mu \in \mathbb{C} \setminus \{0\} | \Re(\mu) \ge 0 \}$$

Tanking the derivative of f gives:

$$\frac{df_{\alpha,k_P}(\mu)}{d\mu} = 1 + k_P e^{-\mu} - \alpha \frac{e^{-\mu}}{\mu} \left(1 + \frac{1}{\mu}\right)$$

Suppose that $\mu^* \in \Omega$ is one of the solution of $f_{\alpha,k_P}(\mu) = 0$, then we have:

$$f_{\alpha,k_P}(\mu^*) = 0 \Leftrightarrow 1 - k_P e^{-\mu^*} = -\alpha \frac{e^{-\mu^*}}{\mu^*}$$

Furthermore,

$$\frac{df_{\alpha,k_P}(\mu^*)}{d\mu} = 1 + k_P e^{-\mu^*} + \left(1 - k_P e^{-\mu^*}\right) \left(1 + \frac{1}{\mu^*}\right) > 0$$

Therefore, $0 = f_{\alpha,k_P}(\mu)$ is a regular value of f_{α,k_P} . Then, generic degree theory can be applied.

In the following, we consider two cases:

- If $\alpha < 0$, notice that $\lim_{\mu \to 0^+} f_{\alpha,k_P}(\mu) \to -\infty$ and that $\lim_{\mu \to +\infty} f_{\alpha,k_P}(\mu) \to 1$ with $\mu \in \mathbb{R}$. Due to the continuity of $f_{\alpha,k_P}(\mu)$, $f_{\alpha,k_P}(\mu)$ has at least one zero in $(0, +\infty)$.
- If $\alpha > 0$, we apply degree theory for holomorphic functions to show that $s(A) \leq 0$, namely, f_{α,k_P} has no zero in the right half plane $\bar{\Omega} := \{\mu \in \mathbb{C} \setminus \{0\} | \Re(\mu) \geq 0\}$. In fact, f_{α,k_P} behaves like $1 k_P e^{-\mu}$ as $|\mu| \to +\infty$. Let

$$H(\theta, \alpha, k_P, \mu) := f_{\theta\alpha, k_P}(\mu) = 1 - k_P e^{-\mu} + \theta \alpha \frac{e^{-\mu}}{\mu} \quad (12)$$

where $\theta \in [0, 1]$. Consequently, the set Ω can be rewritten as:

 $\Omega = \Omega_{R1} \cup \Omega_{R2}$

and one has:

$$\Omega_{R1} = \{ \mu \in \mathbb{C} \setminus \{0\} | \Re(\mu) \ge 0 \text{ and } |\mu| \ge R \}$$

$$\Omega_{R2} := \{ \mu \in \mathbb{C} \setminus \{0\} | \Re(\mu) > 0 \text{ and } |\mu| < R \}$$

Then, in particular, $H(0, \alpha, k_P, \mu) = f_{0,k_P}(\mu) = 1 - k_P e^{-\mu}$ and $H(1, \alpha, k_P, \mu) = f_{\alpha,k_P}(\mu)$. It is easy to see that

$$|f_{\theta\alpha,k_P}(\mu)| \ge 1 - |k_P| - \frac{|\theta\alpha|}{|\mu|}$$

Obviously, for R > 0 sufficiently large, we have

$$H\left(\theta, \alpha, k_P, \mu\right) = f_{\theta\alpha, k_P}(\mu) \neq 0, \forall \mu \in \Omega_{R1}$$
(13)

Now, we analyze the zeros of $f_{\theta\alpha,k_P}(\mu)$ in the set Ω_{R2} . For any R > 0, $H(0,\alpha,k_P,\mu) = f_{0,k_P}(\mu)$ has no zero in Ω_{R2} , since

$$|f_{0,k_P}(\mu)| \ge 1 - |k_P e^{-\mu}| \ge 1 - |k_P| > 0$$



Fig. 1. The graphic expression of Ω_{R1} , Ω_{R2} and $\partial \Omega_{R2}$.

For a holomorphic function $f0, k_P(\mu)$, the regular value $0 \notin f0, k_P(\partial \Omega_{R2})$ the corresponding degree on Ω_{R2} is equal to the number of zeros of $f0, k_P(\mu) = 0$.

Next, we claim that for R > 0 sufficiently large,

$$H\left(\theta, \alpha, k_P, \mu\right) = f_{\theta\alpha, k_P}(\mu) \neq 0, \forall \mu \in \partial\Omega_{R2} \quad (14)$$

To prove (14), since we already have (13), it is sufficient to prove that $f_{\theta\alpha,k_P}(\mu)$ does not vanish on the imaginary axis.

Suppose that $\mu = ib$ $(b \in \mathbb{R} \setminus \{0\})$ is a zero of $f_{\theta\alpha,k_P}(\mu)$. By substituting μ into $f_{\theta\alpha,k_P}(\mu) = 0$, we have

$$1 - k_P \left(\cos(b) - i\sin(b)\right) + \theta \alpha \frac{\left(\cos(b) - i\sin(b)\right)}{ib} = 0$$

which is equivalent to the following equation:

$$b - bk_P \cos(b) - \theta \alpha \sin(b) = 0 \tag{15}$$

$$\theta \alpha \cos(b) - bk_P \sin(b) = 0 \tag{16}$$

Multiplying (15) with $\cos(b)$ and (16) with $\sin(b)$, one gets that

$$b\left(\cos(b) - k_P\right) = 0 \tag{17}$$

which implies that $\cos(b) = k_P$. Then, from (16), we have

$$\theta \alpha = b \sin(b) \tag{18}$$

If let $\alpha \in (0, \frac{\pi}{2})$, then b has to stay in $(0, \frac{\pi}{2})$. However, $k_P = \cos(b) \in (0, 1)$ which contradicts $k_P \in (-1, 0)$.

Lemma 1. Let $\theta \to f_{\theta}, \theta \in [0, 1]$ be a continuous path in $C^1(\overline{\Omega})$ with $p \notin f_{\theta}(\partial \Omega)$ for all $\theta \in [0, 1]$ and let pbe a regular value for both f_0 and f_1 . Then,

$$\deg\left(f_0,\Omega,p\right) = \deg\left(f_1,\Omega,p\right)$$

As a result, according to Lemma 1, one has the following:

$$0 \notin f_{\theta\alpha,k_P} \left(\partial \Omega_{R2} \right)$$

 $0 \notin f_{0,k_P}\left(\partial \Omega_{R2}\right)$

 $deg (f_{\alpha,k_P}(\mu), \Omega_{R2}, 0)$ $= deg (H (1, \alpha, k_P, \mu), \Omega_{R2}, 0)$ $= deg (H (1, \alpha, k_P, \mu), \Omega_{R2}, 0)$ $= deg (f_{0,k_P}(\mu), \Omega_{R2}, 0) = 0$ Moreover, one would have the following results:

$$0 \notin f_{\theta\alpha,k_P} (\partial\Omega)$$

$$0 \notin f_{0,k_P} (\partial\Omega)$$

$$\deg (f_{\alpha,k_P}(\mu), \Omega, 0)$$

$$= \deg (H (1, \alpha, k_P, \mu), \Omega, 0)$$

$$= \deg (H (1, \alpha, k_P, \mu), \Omega, 0)$$

$$= \deg (f_{0,k_P}(\mu), \Omega, 0) = 0$$

Therefore, f_{α,k_P} does not vanish in Ω further in the right half plane $\overline{\Omega}$, namely, $s(A) \leq 0$.

Then, we show that s(A) < 0. For any $\alpha \in (0, \frac{\pi}{2})$ and $k_P \in (-1, 0)$, there exits $\gamma > 0$ such that

 $1 - k_P e^{-\mu} \ge 1 - |k_P| \left| e^{-\mu} \right| = 1 - |k_P| e^{-\Re(\mu)} > 0$ for all $\mu \in \{\mu \in C \setminus \{0\} | \Re(\mu) \ge -\gamma\}$. Because $f_{\alpha, k_P}(\mu) \approx 1 - k_P e^{-\mu}$

as $|\mu|$ tends to $+\infty$, there exists R > 0 so that $f_{\alpha,k_P}(\mu) \neq 0$ for all

 $\mu \in \{\mu \in C \setminus \{0\} | \Re(\mu) \ge -\gamma \text{ and } |\mu| > R\}$ If $s(A) < -\gamma$, we are done.

Now, we consider $-\gamma \leq s(A) \leq 0$. Then, because $\mu \to f_{\alpha,k_P}(\mu)$ is continuous. s(A) must be achieved by some

 $\mu \in \{\mu \in C \setminus \{0\} | f_{\alpha,k_P}(\mu) = 0, 0 \ge \operatorname{Re}(\mu) \ge -\gamma, |\mu| \le R\}$ Then, since $f_{\alpha,k_P}(\mu) = 0$ has no solution on the imaginary axis, we conclude that s(A) < 0.

4. NUMERICAL SIMULATIONS

In this section, we discuss several different experiments for the demand tracking problem. The proposed PI-controller will be directly applied to the original nonlinear hyperbolic PDE model. As a prototypical experiment, we consider a constant demand that increases by a one step jump around the system steady state in the time interval. In particular, we have a constant initial density profile $\rho_0(x) = 1$, a $v_{max} = 4$, a constant influx u(t) = 2 and a demand function that jumps from 2 to 2.5 at time t = 5. In the numerical simulation, the integral gain and the proportional gain are chosen as $k_I = \frac{\lambda(\bar{\rho})}{4}\Gamma(\mu)$ and $\frac{1}{6}e^{-\mu/2}$ with $\mu = 2 - \sqrt{2}$. In addition, in order to improve the performance of the PI-controller, two tuning parameters are added as θ and ψ and then we have $K_I = \theta k_I$ and $K_P = \psi k_P$. In the following, the performance of PIcontroller is given with different tunning parameter values.

Figure 4 shows asymptotic stability of the nonlinear closed-loop system and demonstrate the evolution of the state $\rho(x, t)$. Moreover, the regulation of the outflux y(t) to the desired demand y_r and the evolution of influx u(t) are shown in Figure 3 and 6 with different tunning numbers. Correspondingly, the tracking error is shown in Figure 2 and 5. As clearly indicated via the simulations, the output converges by PI-action to the desired demand as $t \to +\infty$ exponentially.

Moreover, if we observe Figure 2 and 5, it is not hard to conclude that the controller with tunning parameters $\theta = 0.7$ and $\psi = 0.7$ has a better reactivity since the corresponding proportional gain is increased.



Fig. 2. The evolution of error $||y(t) - y_r||$ with $\theta = 0.5$ and $\psi = 0.5$.



Fig. 3. The evolution of outflux y(t), influx u(t) and demand y_r and here $\theta = 0.5$ and $\psi = 0.5$.



Fig. 4. The profile and evolution of the production density $\rho(x,t)$ with $\theta = 0.5$ and $\psi = 0.5$.

5. CONCLUSION

In this work, we have considered the design of regulation PI-controller for the continuum production system described by a scalar hyperbolic PDE. Due to the nonlinearity of the considered system, the proposed PI-controller is constructed based on the linearized model around the desired production density $\bar{\rho}$. For the linearized system, we have been able to find the sufficient condition for the proportional gain and the integral gain to obtain exponential stability of the closed-loop system in L^2 norm. Numerical simulations for the nonlinear close-loop system have been carried out to show the performance of the PI-controller. In the future, our work will be focused on the design of



Fig. 5. The evolution of error $||y(t) - y_r||$ with $\theta = 0.7$ and $\psi = 0.7$.



Fig. 6. The evolution of outflux y(t), influx u(t) and demand y_r and here $\theta = 0.7$ and $\psi = 0.7$.

PI-controller for the original nonlinear production system to guarantee the local exponential stability of the closedloop system in H^2 norm. Moreover, the robustness of the controller will be also considered in the future design.

REFERENCES

- Aksikas, I., Winkin, J.J., and Dochain, D. (2008). Optimal LQ-feedback control for a class of first-order hyperbolic distributed parameter systems. *ESAIM: Control, Optimisation and Calculus of Variations*, 14(04), 897–908.
- Armbruster, D., Marthaler, D.E., Ringhofer, C., Kempf, K., and Jo, T.C. (2006). A continuum model for a reentrant factory. *Operations research*, 54(5), 933–950.
- Bastin, G. and Coron, J.M. (2016). *Stability and bound*ary stabilization of 1-d hyperbolic systems, volume 88. Springer.
- Blandin, S., Litrico, X., Delle Monache, M.L., Piccoli, B., and Bayen, A. (2017). Regularity and lyapunov stabilization of weak entropy solutions to scalar conservation laws. *IEEE Transactions on Automatic Control*, 62(4), 1620–1635.
- Coron, J.M., d'Andrea Novel, B., and Bastin, G. (2007). A strict lyapunov function for boundary control of hyperbolic systems of conservation laws. *Automatic Control, IEEE Transactions on*, 52(1), 2–11.
- Coron, J.M. and Bastin, G. (2015). Dissipative boundary conditions for one-dimensional quasi-linear hyperbolic systems: Lyapunov stability for the c[^]1-norm. SIAM Journal on Control and Optimization, 53(3), 1464–1483.

- Coron, J.M. and Wang, Z. (2013). Output feedback stabilization for a scalar conservation law with a nonlocal velocity. SIAM Journal on Mathematical Analysis, 45(5), 2646–2665.
- Deutscher, J. (2011). Output regulation for linear distributed-parameter systems using finite-dimensional dual observers. Automatica, 47(11), 2468–2473.
- Deutscher, J. (2015). A backstepping approach to the output regulation of boundary controlled parabolic pdes. *Automatica*, 57, 56–64.
- Deutscher, J. (2017). Backstepping design of robust state feedback regulators for linear 2×2 hyperbolic systems. *IEEE Transactions on Automatic Control*, 62(10), 5240-5247.
- Dos Santos, V., Bastin, G., Coron, J.M., and dAndréa Novel, B. (2008). Boundary control with integral action for hyperbolic systems of conservation laws: Stability and experiments. *Automatica*, 44(5), 1310–1318.
- La Marca, M., Armbruster, D., Herty, M., and Ringhofer, C. (2010). Control of continuum models of production systems. *IEEE Transactions on Automatic Control*, 55(11), 2511–2526.
- Luo, Z.H., Guo, B.Z., and Morgül, O. (2012). Stability and stabilization of infinite dimensional systems with applications. Springer Science & Business Media.
- Paunonen, L. (2017). Robust controllers for regular linear systems with infinite-dimensional exosystems. SIAM Journal on Control and Optimization, 55(3), 1567–1597.
- Perrollaz, V. (2013). Asymptotic stabilization of entropy solutions to scalar conservation laws through a stationary feedback law. In Annales de l'Institut Henri Poincare (C) Non Linear Analysis, volume 30, 879–915. Elsevier.
- Pohjolainen, S. (1982). Robust multivariable PI-controller for infinite dimensional systems. Automatic Control, IEEE Transactions on, 27(1), 17–30.
- Trinh, N.T., Andrieu, V., and Xu, C.Z. (2017). Design of integral controllers for nonlinear systems governed by scalar hyperbolic partial differential equations. *IEEE Transactions on Automatic Control.*
- Xu, X. and Dubljevic, S. (2016). Output regulation problem for a class of regular hyperbolic systems. *International Journal of Control*, 89(1), 113–127.
- Xu, X. and Dubljevic, S. (2017a). Output and error feedback regulator designs for linear infinite-dimensional systems. *Automatica*, 83, 170–178.
- Xu, X. and Dubljevic, S. (2017b). Output regulation for a class of linear boundary controlled first-order hyperbolic pide systems. *Automatica*, 85, 43–52.
- Xu, X., Pohjolainen, S., and Dubljevic, S. (2017). Finitedimensional regulators for a class of regular hyperbolic pde systems. *International Journal of Control*, 1–18.