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Semi-Global Asymptotic Control by Sampled-Data Output Feedback ⋆

Wei Lin and **Wei Wei** *

* *Dongguan University of Technology, Guangdong, China, and Dept. of Dept. of Electrical Engineering and Computer Science, Case Western Reserve University, Cleveland, Ohio, USA*

Abstract: This paper shows that for a class of nonlinear systems with a lower-triangular structure, the problem of semi-global asymptotic stabilization is solvable by sampled-data output feedback, without requiring restrictive conditions on the nonlinearities and unmeasurable states of the system, such as linear growth, output-dependent growth or homogeneous growth conditions as commonly assumed in the case of global output feedback stabilization. The main contribution is to point out that semi-global asymptotic rather than practical stabilizability of certain classes of nonlinear systems is still possible by sampled-data output feedback if a sampling time is small enough. A design method is also given for the construction of semi-globally stabilizing, sampled-data output feedback controllers.

1. INTRODUCTION

This paper studies the problem of sampled-data output feedback control for the nonlinear system

$$\begin{aligned}\dot{x}_1 &= x_2 + f_1(x_1)\\ \dot{x}_2 &= x_3 + f_2(x_1, x_2)\\ &\vdots\\ \dot{x}_{n-1} &= x_n + f_{n-1}(x_1, x_2, \cdots, x_{n-1})\\ \dot{x}_n &= u + f_n(x_1, x_2, \cdots, x_n), \qquad y = x_1, \quad (1.1)\end{aligned}$$

where $x \in \mathbb{R}^{\mathrm{n}}, \mathrm{u} \in \mathbb{R}$ and $y \in \mathbb{R}$ are the system state, input and output, respectively. The mappings $f_i : \mathbb{R}^{\mathrm{i}} \to \mathbb{R},\ \mathrm{i} = 1, 2, \cdots, \mathrm{n}$ are C^1 with $f_i(0, \cdots, 0) = 0$.

In the recent years, sampled-data control of nonlinear systems has received increasing attention because most of the controllers nowadays are implemented digitally, using sampler and zero-order holder. When the system state is available for feedback design, both discrete-time and sampled-data control of continuous-time nonlinear systems were studied, for instance, in Byrnes and Lin (1994); Lin (1996); Karafyllis and Kravaris (2009); Monaco and Normand-Cyrot (2011); Monaco et al. (2011); Tsinias (2012); Theodosis and Tsinias (2015) and references therein. One approach is to design sampled-data controllers by a discrete-time approximation of continuous-time nonlinear systems. Due to the approximation errors, such control strategies usually lead to local or regional stabilization results. The other method is to design sampled-data controllers using the emulation technique or by discretizing continuous-time controllers. With an appropriate choice of a sampling frequency, the emulation controllers can achieve regional, semi-global or global stability for continuous-time nonlinear plants Monaco and Normand-Cyrot (2011); Monaco et al. (2011); Monaco and Normand-Cyrot (2011).

When the system states are unmeasurable but only the system output is available, sampled-data output instead of state feedback must be utilized to control continuous-time nonlinear systems. The digital realization of dynamic output compensators demands for the development of effective sampled-data output feedback control approaches. In view of the fact that even in the continuous-time case, global stabilization of nonlinear systems by output feedback is usually impossible unless some restrictive growth conditions are imposed on the system nonlinearities and unmeasurable states, it is not surprising that global asymptotic stabilization of nonlinear systems by sampled-data output feedback is a challenging problem when dynamic output compensators are implemented digitally. In the existing literature, only few papers were devoted to sampled-data output feedback control of nonlinear systems Dabroom and Khalil (2001); Khalil (2004); Qian and Du (2012); Lin et al. (2016); Lin and Wei (2018). In Shim and Teel (2003), the problem of *semi-global practical* stabilization was studied for nonlinear systems by sampled-data output feedback, under asymptotic controllability and observability hypotheses.

Recognizing the difficulty of global output feedback control and the need of certain restrictive conditions for global stabilization of nonlinear systems, we focus in this paper on the problem of *semi-global* instead of global asymptotic stabilization via output feedback. For a SISO nonlinear control system, it was shown in Teel and Praly (1994) that global stabilizability by smooth state feedback and uniform observability imply semi-global asymptotic stabilizability by output feedback, although they cannot guarantee global asymptotic stabilizability by output feedback. In the work Isidori (2000), semi-global output feedback control schemes were presented for nonminimum-phase systems. Recently, it has been proved in Yang and Lin

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(2014) that for a significant class of nonlinear systems that may be neither smoothly stabilizable nor uniformly observable, semi-global asymptotic stabilization is still possible by non-smooth rather than smooth output feedback.

Motivated by the aforementioned semi-global output feedback control results, the paper Lin and Qian (2001) and the recent work Qian and Du (2012); Lin et al. (2016); Lin and Wei (2018) on global stabilization by sampled-data output feedback for certain classes of nonlinear systems, we shall address in this paper the problem of *semi-global asymptotic* stabilization of the nonlinear system (1.1) by sampled-data output feedback. The semi-global sampled-data control problem can be formulated as follows.

Definition 1.1. The origin of the nonlinear system (1.1) is said to be semi-globally asymptotically stabilizable by sampled-data output feedback if, for a given bound $r > 0$ and a maximal sampling period T^*, there exists a C^1 dynamic output compensator, which depends on r and $0 < T < T^*$, of the form

$$\eta(t_{k+1}) = N(\eta(t_k), y(t_k)), \qquad \eta \in \mathbb{R}^q, \tag{1.2}$$
$$u(t) = u(t_k) = \sigma(\eta(t_k), y(t_k)), \quad \forall t \in [t_k, t_{k+1}),$$

with $t_k = kT, \ k = 0, 1 \cdots$, being the sampling points, $N(0,0) = 0$ and $\sigma(0,0) = 0$, suc that

- Semi-global attractivity: all the trajectories of the closed-loop hybrid system (1.1)-(1.2) starting from the compact set $\Gamma_x \times \Gamma_\eta \stackrel{\Delta}{=} [-r, r]^n \times [-r, r]^q \subset \mathbb{R}^n \times \mathbb{R}^q$ converge to the origin $(x, \eta) = (0, 0)$;
- Local asymptotic stability: the closed-loop hybrid system (1.1)-(1.2) is locally asymptotically stable at the equilibrium $(x, \eta) = (0, 0)$.

Clearly, the sampled-data controller (1.2) uses only the system output at the sampling point t_k to control the continuous-time nonlinear system (1.1). Different from the previous work in Shim and Teel (2003), where the problem of semi-global *practical* stabilization by sampled-data output feedback was studied for a general nonlinear system under asymptotic controllability and observability, our work proves that for a class of nonlinear systems in a lower-triangular form, it is possible to achieve *semi-global asymptotic* instead of practical stabilization by sampled-data output feedback. This is the main contribution of this work when compared with the semi-global practical stabilization result in Shim and Teel (2003).

The paper is organized as follows: Section 2 contains notation and key lemmas. In Section 3, we briefly review how a continuous-time, semi-globally stabilizing output feedback controller can be designed for the system (1.1). Based on the continuous-time compensator thus constructed, we then develop a sampled-data output feedback control scheme by the emulation technique. Notably, the semi-global design leads to a different sampled-data output feedback controller from the global case Qian and Du (2012); Lin et al. (2016) and enables us to establish in Section 4 the *semi-global asymptotic* stabilization result for the nonlinear system (1.1), without imposing restrictive growth conditions commonly required in the global stabilization case Qian and Du (2012); Lin et al. (2016); Lin and Wei (2018). Conclusions are drawn in Section 5.

2. NOTATION AND KEY LEMMAS

In this section, we introduce several lemmas that are essential for the analysis and synthesis of semi-global asymptotic stability of nonlinear systems.

Recall that a saturation function with the threshold $M > 0$ is defined as

$$\text{sat}_M(z) = \begin{cases} -M & \text{if} \quad z < -M \\ z & \text{if} \quad |z| \leq M \\ M & \text{if} \quad z > M. \end{cases}$$

Obviously, the saturation function thus defined is continuous $\forall z \in \mathbb{R}$ and bounded by M. Moreover, it has the following property.

Lemma 2.1. Let b be a real number in $[-M, M]$. Then,

$$|b - \text{sat}_M(z)| \leq \min\{|b - z|, 2M\}, \quad \forall z \in \mathbb{R}.$$

The next lemma characterizes an important property of smooth functions on a compact set.

Lemma 2.2. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a C^1 mapping and $\Gamma = [-N, N]^n$ a compact set in $\mathbb{R}^n$, where $N > 0$ being a real number. Then, there exists a constant $K \geq 1$ depending on N, such that

$$||f(a_1, \cdots, a_n) - f(b_1, \cdots, b_n)|| \leq K[|a_1 - b_1| + \cdots + |a_n - b_n|]$$
$$\forall (a_1, \cdots, a_n) \in \Gamma, \ \forall (b_1, \cdots, b_n) \in \Gamma.$$

Lemma 2.3. For any $K > 0$,

$$\frac{z^2}{1 + Kz^2} \geq \frac{1}{1 + K} \min\{z^2, 1\}, \quad \forall z \in \mathbb{R}.$$

Lemma 2.4. If $z \geq 0$ is a real number, then

$$\ln(1 + z) \leq z.$$

Lemma 2.5. Let χ be a compact set in $\mathbb{R}$ containing the origin. Then, there exists a constant $K > 1$ such that

$$z^2 \leq K \ln(1 + z^2), \qquad \forall z \in \chi.$$

The proofs of Lemmas 2.1-2.5 are straightforward. Lemma 2.2 is a direct consequence of the mean value theorem. Lemma 2.3 is trivial by simply discussing the cases when $z^2 \geq 1$ and $z^2 \leq 1$. Lemma 2.4 is obvious as it is equivalent to $e^z \geq 1 + z$. Lemma 2.5 is true as the non-negative function $\frac{z^2}{\ln(1+z^2)}$ is well-defined and continuous on any compact set in R, and hence there is a maximal value on the compact set.

3. DESIGN OF SAMPLED-DATA OUTPUT FEEDBACK CONTROLLERS

To better understand the design of a sampled-data output feedback controller, we briefly review how a continuous-time, semi-globally stabilizing output feedback controller is designed for the nonlinear system (1.1).

3.1 Continuous-Time Semi-Global Output Feedback Design

The following result is a consequence of the paper Teel and Praly (1994).

Proposition 3.1. The nonlinear system (1.1) is semi-globally asymptotically stabilizable by output feedback.

In Teel and Praly (1994), a dynamic output feedback compensator was designed by the idea of dynamic extension. The dynamic extension method resulted in a $2n$-dimensional output feedback compensator. Following the idea of Yang and Lin (2014, 2006), we present in this subsection an n-dimensional nonlinear observer and output feedback controller for the system (1.1).

We begin by using the method of adding an integrator to get a globally stabilizing state feedback control law

$$u^*(x) = -\xi_n \beta_n(x_1, x_2, \cdots, x_n), \tag{3.1}$$

such that the closed-loop system (1.1)-(3.1) satisfies

$$\dot{V}_c(x) \le -(\xi_1^2 + \cdots + \xi_n^2) + (u - u^*(x))^2, \tag{3.2}$$

where $V_c(x) = \frac{1}{2}(\xi_1^2 + \cdots + \xi_n^2)$ is a Lyapunov function and

$$\begin{aligned} \xi_1 &= x_1 - x_1^*, & x_1^* &= 0, \\ \xi_2 &= x_2 - x_2^*, & x_2^* &= -\xi_1 \beta_1(x_1), \\ &\vdots & &\vdots \\ \xi_n &= x_n - x_n^*, & x_n^* &= -\xi_{n-1}\beta_{n-1}(x_1, \cdots, x_{n-1}), \end{aligned} \tag{3.3}$$

with $\beta_i(\cdot) > 0,\ i = 1, \cdots, n$, being smooth functions.

Because $V_c(\cdot)$ is positive definite and proper, one can introduce the level set

$$\Omega_x = \{x \in R^n \mid V_c(x) \le 2r_0\},$$

where $r_0 > 0$ is a constant such that

$$B_r = [-r, r]^n \ \subseteq\ \{x \in R^n \mid V_c(x) \le r_0\}.$$

Then, define $M = \max_{x \in \Omega_x} \|x\|_\infty$ and $B_M = [-M, M]^n$.

From the Lyapunov inequality (3.2), it is clear that the smooth state feedback controller (3.1) would globally asymptotically stabilize the nonlinear system (1.1) if the state x were measurable. When the state of (1.1) is not available for feedback design and only measurable signal is $y = x_1$, output feedback controllers must be designed.

To estimate the state x of system (1.1), we propose the following nonlinear observer with saturation of the estimated state

$$\begin{aligned} \dot{\hat{x}}_1 &= \hat{x}_2 + \hat{f}_1(\hat{x}_1) + La_1(x_1 - \hat{x}_1) \\ \dot{\hat{x}}_2 &= \hat{x}_3 + \hat{f}_2(\hat{x}_1, \hat{x}_2) + L^2 a_2(x_1 - \hat{x}_1) \\ &\vdots \\ \dot{\hat{x}}_n &= u + \hat{f}_n(\hat{x}_1, \hat{x}_2, \cdots, \hat{x}_n) + L^n a_n(x_1 - \hat{x}_1) \end{aligned} \tag{3.4}$$

where $a_1, a_2, \cdots, a_n$ are the coefficients of the Hurwitz polynomial $p(s) = s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n$, $L \ge 1$ is a gain constant to be determined later on and

$$\hat{f}_i(\cdot) := f_i(\text{sat}_M(\hat{x}_1), \cdots, \text{sat}_M(\hat{x}_i)), \quad i = 1, \cdots, n. \tag{3.5}$$

By the certainty equivalence principle, replacing the state x in the controller (3.1) by its estimate $\hat{x}$ from the observer (3.4)-(3.5), we obtain

$$u = u(\hat{x}) := u^*(\text{sat}_M(\hat{x}_1), \cdots, \text{sat}_M(\hat{x}_n)). \tag{3.6}$$

For the convenience of the semi-global stability analysis, introduce the estimate error $e = [e_1, e_2, \cdots, e_n]^T$, where $e_i = L^{n-i}(x_i - \hat{x}_i),\ i = 1, \cdots, n$. Then, it is deduced from (3.4) and (1.1) that the error dynamics satisfy

$$\dot{e} = LAe + F(\cdot) \tag{3.7}$$

where

$$A = \begin{bmatrix} -a_1 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_{n-1} & 0 & 0 & \cdots & 1 \\ -a_n & 0 & 0 & \cdots & 0 \end{bmatrix}, F(\cdot) = \begin{bmatrix} L^{n-1}(f_1(\cdot) - \hat{f}_1(\cdot)) \\ \vdots \\ L(f_{n-1}(\cdot) - \hat{f}_{n-1}(\cdot)) \\ f_n(\cdot) - \hat{f}_n(\cdot) \end{bmatrix}$$

Following an argument similar to the one used in Yang and Lin (2014, 2006), we consider the Lyapunov function

$$V(x, \hat{x}) = \frac{V_c(x)}{2} + \frac{r_0}{2}\frac{\ln(1 + V_e(e))}{\ln(1 + \mu(L))} \tag{3.8}$$

for the closed-loop system (1.1)-(3.7), where $V_e(e) = e^T P e$, the matrix $P = P^T > 0$ satisfies $A^T P + PA = -I$ and $\mu(L) = 4nr^2 L^{2n}\|P\|$. Notably, the Lyapunov function (3.8) is quite different from the ones in Teel and Praly (1994), making the estimation of domain attractions and the corresponding semi-global stability analysis much easier and relatively simpler.

Associated with $V(x, \hat{x})$, define the corresponding level set

$$\Omega = \{(x, \hat{x}) \in \mathbb{R}^n \times \mathbb{R}^n \mid V(x, \hat{x}) \le r_0\} \tag{3.9}$$

Then, a delicate analysis and subtle estimation, similar to the one performed in Yang and Lin (2014, 2006), leads to the conclusion that there is a high gain $L^* > 0$, such that

$$\dot{V}\big|_\Omega \le -\frac{1}{2}\|\xi\|^2 - \frac{\|e\|^2}{1 + \|P\|\|e\|^2}, \quad \forall L \ge L^*, \tag{3.10}$$

where $\xi = [\xi_1, \xi_2, \cdots, \xi_n]^T$ given in (3.3).

The Lyapunov inequality (3.10) implies that the nonlinear system (1.1) is semi-globally asymptotically stabilizable by the n-dimensional dynamic output compensator (3.4)-(3.5) and (3.6), with the domain of attraction $B_r \times B_r$ contained in the level set Ω.

3.2 Sampled-Data Output Feedback Controllers

Based on the continuous-time controller (3.4)-(3.6), a semi-globally asymptotically stabilizing, sampled-data output feedback controller can be designed for the nonlinear system (1.1). The basic idea is to discretize the continuous-time compensator (3.4)-(3.5)-(3.6) using the emulated versions of the continuous-time solutions of the controller through zero-order hold (ZOH).

Specifically, a sampled-date observer can be built, on one hand, by discretizing the continuous-time observer (3.4)-(3.5), particularly, by replacing the output $y(t)$ with the constant $y(t_k) = y(kT)$ and all the nonlinear terms in the observer (3.4)-(3.5) via the corresponding constant values measured at $t_k = kT$ during the sampling interval $[t_k, t_{k+1})$. On the other hand, a sampled-data controller can be simply obtained through zero-order hold (ZOH), i.e., by holding $u(t)$ in (3.6) as a constant $u(t_k) = u(kT)$ over the sampling interval $[t_k, t_{k+1})$. In this way, a sampled-data output feedback compensator of the form (1.2) is obtained by integrating both emulated versions of the observer (3.4)-(3.5) and controller (3.6) over each sampling period $[t_k, t_{k+1})$ for $k = 1, 2, \cdots$. The described design leads to

$$
\begin{aligned}
\dot{\hat{x}}_1 &= \hat{x}_2 + \hat{f}_1(\hat{x}_1(t_k)) + La_1(x_1(t_k) - \hat{x}_1)\\
\dot{\hat{x}}_2 &= \hat{x}_3 + \hat{f}_2(\hat{x}_1(t_k), \hat{x}_2(t_k)) + L^2 a_2(x_1(t_k) - \hat{x}_1)\\
&\vdots \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (3.11)\\
\dot{\hat{x}}_n &= u + \hat{f}_n(\hat{x}_1(t_k), \cdots, \hat{x}_n(t_k)) + L^n a_n(x_1(t_k) - \hat{x}_1)\\
u &= u(t_k) = u^*(\mathrm{sat}_M(\hat{x}_1(t_k)), \cdots, \mathrm{sat}_M(\hat{x}_n(t_k))), t \in [t_k, t_{k+1})
\end{aligned}
$$

which can also be represented as

$$\dot{\hat{x}} = \hat{A}\hat{x} + \hat{B}u(t) + \hat{H}y(t_k) + \Phi(\hat{x}(t_k)), \qquad (3.12)$$

$$u(t) = u(t_k) := u^*(\mathrm{Sat}(\hat{x}(t_k))), \quad t \in [t_k, t_{k+1}) \quad (3.13)$$

where $\hat{x} = [\hat{x}_1 \cdots \hat{x}_n]^T$, $\mathrm{Sat}(\hat{x}) = [\mathrm{sat}_M(\hat{x}_1) \cdots \mathrm{sat}_M(\hat{x}_n)]^T$ and

$$\hat{A} = \begin{bmatrix} -La_1 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -L^{n-1}a_{n-1} & 0 & 0 & \cdots & 1 \\ -L^n a_n & 0 & 0 & \cdots & 0 \end{bmatrix}, \quad \hat{B} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

$$\hat{H} = \begin{bmatrix} La_1 \\ L^2 a_2 \\ \vdots \\ L^n a_n \end{bmatrix}, \quad \Phi(\hat{x}(t_k)) = \begin{bmatrix} \hat{f}_1(\hat{x}_1(t_k)) \\ \hat{f}_2(\hat{x}_1(t_k), \hat{x}_2(t_k)) \\ \vdots \\ \hat{f}_n(\hat{x}(t_k)) \end{bmatrix}$$

with the coefficient a_i and the function $\hat{f}_i(\cdot)$, $i = 1, 2, \cdots, n$ being defined in (3.4)-(3.5).

Integrating the compensator (3.12)-(3.13) on the sampling period $[t_k, t_{k+1})$, $t_k = kT, k = 0, 1, \cdots$, yields

$$
\begin{aligned}
\hat{x}(t_{k+1}) &= e^{\hat{A}T}\hat{x}(t_k) + \int_0^T e^{\hat{A}(T-s)} ds [\hat{B}u(t) + \hat{H}y(t_k) + \Phi(\hat{x}(t_k))]\\
&= e^{\hat{A}T}\hat{x}(t_k) + \int_0^T e^{\hat{A}s} ds \ \hat{B}u^*(\mathrm{Sat}(\hat{x}(t_k)))\\
&\quad + \int_0^T e^{\hat{A}s} ds \ \hat{H}y(t_k) + \int_0^T e^{\hat{A}s} ds \ \Phi(\hat{x}(t_k))\\
&:= N(\hat{x}(t_k), y(t_k)) \qquad (3.14)
\end{aligned}
$$

$$u(t) = u(t_k) = u^*(\mathrm{Sat}(\hat{x}(t_k))) \qquad (3.15)$$

which is in the form (1.2). In other words, the discrete-time nonlinear observer (3.14) and controller (3.13) have provided a sampled-data output feedback controller for the nonlinear system (1.1).

4. SEMI-GLOBAL ASYMPTOTIC STABILIZATION

In this section, we prove that the proposed sampled-data output feedback controller (3.14)-(3.15) semi-globally asymptotically stabilize the nonlinear system (1.1) as long as the sampling period T is small enough. Because the discrete-time nonlinear observer (3.14) and its continuous counterpart (3.12) produce exactly the same estimation $\hat{x}(t_k)$, $t_k = kT$, $k = 0, 1, \cdots$, while the discrete-time controller (3.15) is identical to (3.13) on $[t_k, t_{k+1})$, we shall use the equivalent output feedback controller (3.12)-(3.13) or (3.11) for the proof of semi-global asymptotic stability.

Similar to Section 3, introduce the estimate error $e = [e_1, \cdots, e_n]^T$ with $e_i = L^{n-i}(x_i - \hat{x}_i)$, $i = 1, \cdots, n$. Clearly, the error dynamics based on (1.1) and (3.11) is given by

$$\dot{e} = LAe + F(\cdot) + \bar{F}(\cdot) \qquad (4.1)$$

where A and $F(\cdot)$ are the same as in (3.7) and

$$\bar{F}(\cdot) = \begin{bmatrix} L^{n-1}[\hat{f}_1(\hat{x}_1) - \hat{f}_1(\hat{x}_1(t_k))] - L^n a_1[x_1(t_k) - x_1] \\ L^{n-2}[\hat{f}_2(\hat{x}_1, \hat{x}_2) - \hat{f}_2(\hat{x}_1(t_k), \hat{x}_2(t_k))] \\ \qquad -L^n a_2[x_1(t_k) - x_1] \\ \vdots \\ \hat{f}_n(\hat{x}) - \hat{f}_n(\hat{x}(t_k)) - L^n a_n[x_1(t_k) - x_1] \end{bmatrix}$$

which contains the differences between the functions at t and the corresponding values at the sampling time t_k.

By Lemmas 2.1 - 2.2, $L \geq 1$ and the fact that $|x_i - \hat{x}_i| = |e_i|/L^{n-i} \leq |e_i|$, we have $\forall (x, \hat{x}) \in B_M \times \mathbb{R}^n$,

$$
\begin{aligned}
&|L^{n-i}(f_i(\cdot) - \hat{f}_i(\cdot))|\big|_{B_M \times \mathbb{R}^n}\\
&\leq L^{n-i}K(|x_1 - \mathrm{sat}_M(\hat{x}_1)| + \cdots + |x_i - \mathrm{sat}_M(\hat{x}_i)|)\\
&\leq K(|e_1| + \cdots + |e_i|) \leq K_1\|e\|, \qquad i = 1, \cdots, n \quad (4.2)
\end{aligned}
$$

where $K_1 > 0$ is a constant independent of the gain L.

Similarly, it is deduced from Lemmas 2.1 - 2.2 that

$$
\begin{aligned}
&|L^{n-i}[f_i(\hat{x}_1, \cdots, \hat{x}_i) - \hat{f}_i(\hat{x}_1(t_k), \cdots, \hat{x}_i(t_k))]|\\
&\quad \leq L^{n-i}K(|\mathrm{sat}_M(\hat{x}_1) - \mathrm{sat}_M(\hat{x}_1(t_k))| + \cdots \\
&\qquad + |\mathrm{sat}_M(\hat{x}_i) - \mathrm{sat}_M(\hat{x}_i(t_k))|) \qquad (4.3)\\
&\quad \leq L^{n-1}nK\|\mathrm{Sat}(\hat{x}) - \mathrm{Sat}(\hat{x}(t_k))\|, \quad i = 1, \cdots, n
\end{aligned}
$$

and

$$|L^n a_i(x_1(t_k) - x_1)| \leq L^n \bar{K}_2|x_1(t_k) - x_1| \qquad (4.4)$$

where $\bar{K}_2 = \max_{1 \leq i \leq n}\{|a_i|\}$.

For the error dynamics (4.1), choose the Lyapunov function $V_e(e) = e^T Pe$. From (4.2), (4.3) and (4.4), it follows that (by the completion of square)

$$
\begin{aligned}
\dot{V}_e\big|_{B_M \times \mathbb{R}^n} &\leq Le^T(A^T P + PA)e + 2e^T P\big(F(\cdot) + \bar{F}(\cdot)\big)\\
&\leq -(L - \bar{K}_1)\|e\|^2 + C_1(L)\Big(|x_1(t_k) - x_1|^2\\
&\qquad + \|\mathrm{Sat}(\hat{x}) - \mathrm{Sat}(\hat{x}(t_k))\|^2\Big) \qquad (4.5)
\end{aligned}
$$

where $C_1(L) = L^{2n}n\bar{K}_2^2 + L^{2n-2}n^3K^2$ and $\bar{K}_1 = 2\sqrt{n}K_1\|P\| + 2\|P\|^2$.

On the other hand, using Lemma 2.2 yields

$$
\begin{aligned}
&|u^*(x) - u^*(\mathrm{Sat}(\hat{x}(t_k)))|\big|_{B_M \times \mathbb{R}^n} \leq |u^*(x) - u^*(\mathrm{Sat}(\hat{x}))|\big|_{B_M \times \mathbb{R}^n}\\
&\qquad + |u^*(\mathrm{Sat}(\hat{x})) - u^*(\mathrm{Sat}(\hat{x}(t_k)))|\big|_{B_M \times \mathbb{R}^n}\\
&\leq \bar{K}(\min\{\|e\|, 1\} + \|\mathrm{Sat}(\hat{x}) - \mathrm{Sat}(\hat{x}(t_k))\|) \qquad (4.6)
\end{aligned}
$$

where $\bar{K} > 0$ is a constant independent of L.

The inequality (4.6), together with (3.2) and $u = u^*(\mathrm{Sat}(\hat{x}(t_k)))$, results in

$$
\begin{aligned}
\dot{V}_c(x)\big|_{B_M \times \mathbb{R}^n} &\leq -\|\xi\|^2 + 2\bar{K}^2\Big(\min\{\|e\|^2, 1\}\\
&\qquad + \|\mathrm{Sat}(\hat{x}) - \mathrm{Sat}(\hat{x}(t_k))\|^2\Big) \quad (4.7)
\end{aligned}
$$

Since $e_i = L^{n-i}(x_i - \hat{x}_i)$ and $L \geq 1$, we have

$V_e(e) \le \|P\|\|e\|^2 \le L^{2n}\|P\|[(x_1-\hat{x}_1)^2+\cdots+(x_n-\hat{x}_n)^2]$
Thus, $V_e(e)\big|_{B_r\times B_r} \le 4nr^2L^{2n}\|P\|, \ \forall (x,\hat{x}) \in B_r\times B_r$. Define

$$\mu(L) = 4nr^2L^{2n}\|P\| \ge \max_{(x,\hat{x})\in B_r\times B_r} V_e(e) > 0. \quad (4.8)$$

Now, we use the Lyapunov function (3.8) to conduct a stability analysis for the hybrid closed-loop system (1.1)-(3.11) or (1.1)-(4.1) on the compact set Ω defined by (3.9).

Observe that semi-global asymptotic stability of the hybrid closed-loop system (1.1)-(4.1) is guaranteed if there is a constant $\rho\in(0,1)$, such that

$$V(x,\hat{x})|_{t=t_{k+1}} \le \rho V(x,\hat{x})|_{t=t_k}, \qquad \forall (x,\hat{x})\in\Omega$$

for $k=0,1,2,\cdots$. With this idea in mind, in what follows we shall prove that such a $\rho<1$ indeed exists if the sampling time T is small enough.

Firstly, it is not difficult to verify the following properties:

i) For every $L\ge 1$, $V(\cdot)$ is positive definite and proper and Ω is a compact set in $\mathbb{R}^n\times\mathbb{R}^n$. If $L=L^*$ is fixed, $V(\cdot)$ and Ω are also fixed.

ii) $\forall L\ge 1, \quad \Omega\supset B_r\times B_r$.
This fact follows directly from (3.8), $V_c\le r_0$ and

$$\frac{r_0}{2}\frac{\ln(1+V_e(e))}{\ln(1+\mu(L))} \le \frac{r_0}{2}, \quad \forall (x,\hat{x})\in B_r\times B_r.$$

iii) $\forall L\ge 1, \quad B_M\times\mathbb{R}^n\supset\Omega_x\times\mathbb{R}^n\supset\Omega$.
The first inclusion follows trivially from the definition of M or B_M. The second relation is because $V(x,\hat{x})\le r_0$ implies $V_c(x)\le 2r_0$.

In view of the properties ii) and iii), we deduce from (4.5)-(4.7) and Lemma 2.3 that

$$\begin{aligned}
\dot{V}\big|_\Omega &= \frac{1}{2}\dot{V}_c(x)\big|_\Omega + \frac{r_0}{2\ln(1+\mu(L))}\frac{\dot{V}_e}{1+V_e}\big|_\Omega \\
&\le -\frac{1}{2}\|\xi\|^2 + \bar{K}^2\min\{\|e\|^2,1\} \\
&\quad -\frac{r_0}{2\ln(1+\mu(L))}\frac{(L-\bar{K}_1)\|e\|^2}{1+V_e} + \bar{K}^2\|\mathrm{Sat}(\hat{x})-\mathrm{Sat}(\hat{x}(t_k))\|^2 \\
&\quad +\frac{r_0C_1(L)(|x_1(t_k)-x_1|^2+\|\mathrm{Sat}(\hat{x})-\mathrm{Sat}(\hat{x}(t_k))\|^2)}{2\ln(1+\mu(L))(1+V_e)} \\
&\le -\frac{1}{2}\|\xi\|^2 - \frac{\|e\|^2}{1+\|P\|\|e\|^2} \qquad (4.9)\\
&\quad \times\Big(\frac{r_0(L-\bar{K}_1)}{2\ln(1+\mu(L))} - \bar{K}^2(1+\|P\|)\Big) \\
&\quad +\bar{C}_1(L)(|x_1(t_k)-x_1|^2+\|\mathrm{Sat}(\hat{x})-\mathrm{Sat}(\hat{x}(t_k))\|^2)
\end{aligned}$$

where $\bar{C}_1(L) = \frac{r_0C_1(L)}{2\ln(1+\mu(L))(1+V_e)} + \bar{K}^2$.

By construction, $\mu(L)>0$ is a polynomial function of L with the order $2n$, while the constants $\bar{K}_1$ and $\bar{K}$ are independent of L. Consequently, there exists a constant $L^*\ge 1$ such that

$$\frac{r_0(L-\bar{K}_1)}{2\ln(1+\mu(L))} \ge \bar{K}^2(1+\|P\|)+1, \quad \forall L\ge L^*.$$

Hence, choosing $L=L^*$ yields

$$\begin{aligned}
\dot{V}\big|_\Omega &\le -\frac{1}{2}\|\xi\|^2 - \frac{\|e\|^2}{1+\|P\|\|e\|^2} \\
&\quad +\bar{C}_1(L^*)(|x_1(t_k)-x_1|^2+\|\mathrm{Sat}(\hat{x})-\mathrm{Sat}(\hat{x}(t_k))\|^2) \\
&\le -\frac{1}{2}\|\xi\|^2 - c_2\|e\|^2 + \bar{C}_1(L^*)\Big(|x_1(t_k)-x_1|^2\big|_\Omega \\
&\quad +\|\mathrm{Sat}(\hat{x})-\mathrm{Sat}(\hat{x}(t_k))\|^2\big|_\Omega\Big) \qquad (4.10)
\end{aligned}$$

where $c_2 = C_2(L^*) = \min_{(x,\hat{x})\in\Omega}\frac{1}{1+\|P\|\|e\|^2}$ are positive constants once L^* is fixed.

With the aid of (6.1) and (6.2) from Appendix, one can deduce from (4.10) that

$$\begin{aligned}
\dot{V}\big|_\Omega &\le -\frac{1}{2}\|\xi\|^2 - c_2\|e\|^2 + C_2(L)\big(\sup_{\tau\in[t_k,t)}\{\|\xi(\tau)\|^2\} \\
&\quad +\sup_{\tau\in[t_k,t)}\{\|e(\tau)\|^2\}\big)(t-t_k), \qquad t\in[t_k,t_{k+1})
\end{aligned}$$

where $C_2(L)>0$ is a constant when $L=L^*$ is fixed.

By Lemmas 2.4–2.5, a straightforward argument shows that there exist $h_1,h_2>0$ such that $\forall t\in[t_k,t_{k+1})$,

$$\begin{aligned}
\dot{V}\big|_\Omega &\le -2\times\frac{1}{4}\|\xi\|^2 - \frac{c_2}{h_2}h_2\|e\|^2 \\
&\quad +C_2(L)\Big[\sup_{\tau\in[t_k,t)}\{4V(\tau)\} + \sup_{\tau\in[t_k,t)}\{\frac{V(\tau)}{h_1}\}\Big](t-t_k) \\
&\le -\beta V(x,\hat{x}) + \gamma\max_{\tau\in[t_k,t_{k+1}]}\{V(\tau)\}(t-t_k) \qquad (4.11)
\end{aligned}$$

where $\beta=\min\{2,c_2/h_2\}, \gamma=C_2(L)\cdot\max\{4,1/h_1\}$.

Using (4.11) and an argument similar to the one in Qian and Du (2012); Lin et al. (2016), one can prove that

$$\max_{\forall\tau\in[t_k,t_{k+1}]} V(\theta(\tau))\big|_\Omega = V(\theta(t_k)), \quad \theta := [x^T,\hat{x}^T]^T \quad (4.12)$$

From (4.12) and (4.11), it follows that

$$\dot{V}\big|_\Omega \le -\beta V(\theta(t)) + T\gamma V(\theta(t_k)), \ \forall t\in[t_k,t_{k+1}) \quad (4.13)$$

Consequently, $\forall (x,\hat{x})\in\Omega$,

$$\frac{d}{dt}V(\theta(t)) + \beta V(\theta(t)) \le T\gamma V(\theta(t_k)), \quad \forall t\in[t_k,t_{k+1}).$$

Solving the differential inequality above yields

$$\begin{aligned}
V(\theta(t_{k+1})) &\le \big[e^{-\frac{1}{2}\beta T} + (1-e^{-\frac{1}{2}\beta T})\frac{T\gamma}{\beta}\big]V(\theta(t_k)) \\
&:= \rho V(\theta(t_k)), \qquad \forall (x,\hat{x})\in\Omega. \qquad (4.14)
\end{aligned}$$

By selecting the sampling period $T<\frac{\beta}{\gamma}$, one has $\frac{T\gamma}{\beta}<1$, which leads to $0<\rho<1$. Consequently, $V(\theta(t_k))$ converges to zero as k tends to $+\infty$, $\forall (x,\hat{x})\in\Omega$. This, in turn, implies that the nonlinear system (1.1) is semi-globally asymptotically stabilized by the sampled-data output feedback compensator (3.14)-(3.15), if the sampling period $T<\frac{\beta}{\gamma}$.

The proof above has led to the following theorem, which is main result of this paper.

Theorem 4.1. The nonlinear system (1.1) is semi-globally asymptotically stabilizable by sampled-data output feedback, in particular, by the sampled-data controller (3.14)-(3.15), if the sampling period T is small, for instance, $T<\beta/\gamma$.

Remark 4.2. The significance of Theorem 4.1 is the removal of the restrictive growth conditions that are required for GAS by digital output feedback. Of course, a trade-off is that only SGAS rather than GAS can be achieved. The two benchmark examples below, both of them are known not to be globally stabilizable by output feedback but can be SGAS by sampled-data output feedback:

$$\Sigma_1: \begin{array}{l}\dot{x}_1 = x_2\\ \dot{x}_2 = u_2 + x_2^p\\ y = x_1\end{array} \qquad \Sigma_2: \begin{array}{l}\dot{x}_1 = x_2\\ \dot{x}_2 = x_3\\ \dot{x}_3 = u_2 + x_3^2, \quad y = x_1,\end{array}$$

where $p > 2$ is a real number. For instance, a SGAS sampled-data output feedback controller for the system Σ_1 is, in view of Theorem 4.1, given by

$$\begin{aligned}\dot{\hat{x}}_1 &= \hat{x}_2 + L(y(t_k) - \hat{x}_1)\\ \dot{\hat{x}}_2 &= u(t) + [\mathrm{sat}_M(\hat{x}_2(t_k))]^p + L^2(y(t_k) - \hat{x}_1) \qquad (4.15)\\ u(t) &= -10\mathrm{sat}_M(\hat{x}_1(t_k)) - 10\mathrm{sat}_M(\hat{x}_2(t_k)) - [\mathrm{sat}_M(\hat{x}_2(t_k))]^p\end{aligned}$$

$\forall t \in [t_k, t_{k+1})$, with $t_k = kT,\ k = 0, 1, 2, \cdots$. Simulations of the hybrid closed-loop system by Σ_1 and (4.15) are conducted with $p = 3,\ M = 7, L = 40$ and the sampling period $T = 0.01$. Details are omitted for the sake of space.

5. CONCLUDING REMARKS

Although a lower-triangular system such as Σ_1 or Σ_2 is uniformly observable and globally stabilizable via smooth state feedback, it is usually not globally stabilizable by output feedback. In this paper, we have shown that semi-global asymptotic stabilization is achievable by sampled-data output feedback. Because a less demand control objective, namely semi-global instead of global asymptotic stabilization is sough, the result obtained in this work does not require any restrictive conditions such as linear growth, output-dependent growth or homogeneous growth conditions that are commonly assumed and somewhat necessary in the case of global output feedback control. Compared with the previous work Shim and Teel (2003), where a sampled-data output feedback control scheme was proposed for general nonlinear systems, achieving semi-global *practical* stabilization under the conditions of asymptotic controllability and observability, our contribution is to point out that for certain classes of nonlinear systems, it is still possible to achieve semi-global asymptotic rather than practical stabilization by sampled-data output feedback.

6. APPENDIX

The following estimations will be used in (4.10) for the proof of semi-global asymptotic stability. Details are omitted for the reason of space.

$$|x_1(t) - x_1(t_k)|^2\big|_\Omega \le D_1\Big(\sup_{\tau\in[t_k,t)}\{\|\xi(\tau)\|^2\} + \sup_{\tau\in[t_k,t)}\{\|e(\tau)\|^2\}\Big)(t - t_k), \quad \forall t \in [t_k, t_{k+1}) \quad (6.1)$$

where $D_1 > 0$ is a constant independent of L.

Similarly, it can be proved that there is a constant $D_2 > 0$, such that $\forall t \in [t_k, t_{k+1})$,

$$\|\mathrm{Sat}(\hat{x}) - \mathrm{Sat}(\hat{x}(t_k))\|^2\big|_\Omega \le D_2\Big(\sup_{\tau\in[t_k,t)}\{\|\xi(\tau)\|^2\} + \sup_{\tau\in[t_k,t)}\{\|e(\tau)\|^2\}\Big)(t - t_k) \quad (6.2)$$

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