Preprints, 10th IFAC International Symposium on
Advanced Control of Chemical Processes
Shenyang, Liaoning, China, July 25-27, 2018

A Synergistic Approach to Robust Output Feedback Control: Tube-based Multi-stage NMPC

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Abstract: Plant-model mismatch and estimation errors are critical issues in the practical implementation of Nonlinear Model Predictive Control (NMPC). To address these challenges, we formulate a robust output feedback NMPC scheme that is real-time implementable and provides robust constraint satisfaction in the presence of parametric and additive uncertainties, and estimation errors. The robustness is achieved by combining the tube-based and the multi-stage NMPC approaches. Two controllers are used in the proposed framework: a primary controller with tightened constraints that optimizes a given objective and an ancillary controller that tracks the trajectories provided by the primary controller. Unlike standard tube-based NMPC, the primary controller predicts different state trajectories for different realizations of the most important uncertainties using the multi-stage NMPC framework. The synergy between the two approaches leads to a better trade-off between optimality and complexity. The advantages of the proposed approach are demonstrated for an industrial-scale fed-batch polymerization reactor example.

Keywords: Model Predictive Control, Robust Control, Nonlinear Output Feedback, Process Control.

1. INTRODUCTION

Nonlinear Model Predictive Control (NMPC) schemes must take into consideration the presence of uncertainties in the model to achieve robust performance and constraint satisfaction. The controller should provide good performance despite the need for robustness. Min-max MPC is one of the earliest robust approaches (Campo and Morari, 1987). It optimizes the closed-loop behavior for the worst-case cost of the objective and obtains a sequence of control inputs that satisfy the constraints for all realizations of the uncertainty. Open-loop min-max MPC has been shown to be quite conservative because it does not account for the presence of feedback in the predictions.

Tube-based NMPC uses two controllers, a nominal controller and an ancillary controller (Mayne et al., 2011; Rawlings and Mayne, 2009). The nominal controller solves the optimization problem using the nominal model of the system with tightened constraints and the ancillary controller tracks the trajectory of the nominal controller. A key advantage of the approach is that the problem size remains close to the nominal NMPC problem. A major drawback of the approach is its conservativeness because of the tightened constraints for large disturbances.

The multi-stage NMPC approach offers a promising alternative to the approaches discussed above (Lucia et al., 2013, 2014). It formulates the decision problem on a scenario tree. It is assumed that the uncertainty takes one of the possible discrete realizations at every step in the prediction horizon and accounts for the fact that at the next time-step, measurement information is available and the future control inputs can act as recourse to counteract the effects of the uncertainty that has realized up to this point in time. This makes the approach less conservative than open-loop min-max NMPC. The drawback of the approach however is the exponential growth in problem size with respect to the number of uncertainties.

In this paper, we combine the strengths of multi-stage NMPC and tube-based NMPC and formulate a novel tube-based multi-stage NMPC that is real-time implementable, robust to a wide range of uncertainties and non-conservative. The idea behind the approach is to counteract the large uncertainties using multi-stage NMPC while counteracting small-magnitude disturbances using the ideas of tube-based NMPC. This results in a practically implementable scheme that is less conservative with the following advantages when compared to multi-stage NMPC and tube-based NMPC independently:

(1) The growth in problem complexity is limited because the small uncertainties are not considered in the scenario tree.
(2) The recourse that is modeled in the prediction for the realizations of the large uncertainties in the scenario tree reduces the conservatism compared to tube-based NMPC.

Usually, not all states of the plant are measured but have to be estimated from the measured outputs using a state estimator. An output feedback MPC methodology therefore must take into account the estimation errors at every time-step in addition to the model uncertainties to guarantee robust constraint satisfaction. Lee and Ricker (1994) proposed an Extended Kalman Filter (EKF) based approach that uses the measurements that are available to predict the future states. Findeisen et al. (2003) used high gain observers and proved stability of the output feedback scheme but the resulting controller can lead to a poor performance in the presence of measurement errors. Copp and Hespanha (2014) proposed an output feedback

* The research leading to these results has received funding from the European Commission under grant agreement number 291458 (MOBOCON)

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MPC scheme that combines Moving Horizon Estimation and MPC into a single min-max optimization problem. We have proposed multi-stage output feedback NMPC approaches using the EKF (Subramanian et al., 2014) and the Unscented Kalman Filter (UKF) (Subramanian et al., 2015). The schemes were shown to be robust to the estimation errors as well as to the model uncertainties. The problem size however increases exponentially with respect to the number of measurements.

In this work, we show that the proposed tube-based multi-stage NMPC scheme can be extended to an output feedback strategy for small estimation errors. In addition, the output feedback formulation that we propose is independent of the estimation method employed and keeps the complexity of the problem equal to that of a full-state information problem making the approach practical. The advantages of the proposed approach is demonstrated for an industrial fed-batch polymerization reactor.

2. INVESTIGATED SYSTEM

A nonlinear system is assumed to be given by

$$x_{k+1} = f(x_k, u_k, d_k) + w_k, \tag{1a}$$

$$y_k = h(x_k, u_k) + r_k, \tag{1b}$$

where $x_k \in \mathbb{R}^{n_x}$ denotes the state vector, $u_k \in \mathbb{R}^{n_u}$ represents the input vector, $d_k \in \mathbb{D} \subseteq \mathbb{R}^{n_d}$ denotes the parametric uncertainty and $w_k \subseteq \mathbb{W} \in \mathbb{R}^{n_x}$ denotes additive disturbances. The parametric uncertainties d_k and the additive disturbances w_k are assumed to be bounded by the sets $\mathbb{D}$ and $\mathbb{W}$. For simplicity, we assume that the parametric uncertainties are of large magnitude and the additive disturbances are of small magnitude throughout this paper. $f : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \times \mathbb{R}^{n_d} \to \mathbb{R}^{n_x}$ is the model of the system dynamics. In the equation (1b), y_k represents the output vector with dimension n_y, $h : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \to \mathbb{R}^{n_y}$ is the measurement model and $r_k \in \mathbb{R}^{n_y}$ represents the measurement noise. An estimation scheme must be employed to estimate the states to initialize the controller. Because of the presence of parametric uncertainties, additive disturbances and measurement noise, estimation errors $e_k := x_k - \hat{x}_k$ are unavoidable in reality. Here $\hat{x}_k$ denotes the state estimates. Hence the estimation errors must be accounted for in addition to model uncertainties in the robust output feedback NMPC framework. We discuss the proposed tube-based multi-stage NMPC for the case of full-state information problem and later extend it to the case of output feedback problem that accounts for the presence of estimation errors.

3. TUBE-BASED MULTI-STAGE NMPC SCHEME WITH FULL-STATE INFORMATION

To handle plant-model mismatch, we propose two controllers: a primary controller and an ancillary controller.

3.1 Primary controller

The primary controller is designed such that it is robust to the large uncertainties $d_k \in \mathbb{D}$. The system considered by the primary controller can then be given by

$$z_{k+1} = f(z_k, v_k, d_k), \tag{2}$$

where $z_k \in \mathbb{R}^{n_x}$ denotes the state and $v_k \in \mathbb{R}^{n_u}$ denotes the input of the model (2). The primary controller does not consider the small disturbances $w_k \in \mathbb{W}$. Hence the predictions using (2) will

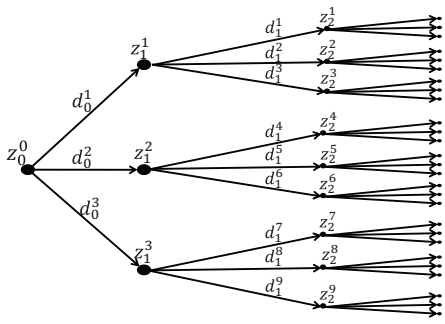

Fig. 1. Scenario tree representation of the uncertainty evolution for multi-stage NMPC.

in general be different from the plant (1). The robustness against the large uncertainties are obtained using multi-stage NMPC as a primary controller. In multi-stage NMPC, the future evolution of the plant for different realizations of the uncertainty is modeled by a scenario tree as shown in Fig. 1. Nodes in Fig. 1 denote the states of the system at discrete time instants. The root node represents the initial condition of the tree. From the root node, the system can reach different states at the next time-step depending on the realization of the uncertainty. The branches of the tree denote the different possible realizations of the uncertainty in the future. The tree grows until the end of the prediction horizon N_p. A path from the root node to a leaf node of the tree is called a scenario. For a nonlinear system if an uncertainty is given as a continuous interval, theoretically we need infinitely many branches at every stage to guarantee satisfaction of constraints. Often, the satisfaction of the constraints can be achieved by considering only the extreme realizations of the uncertainty as branches in practice (Lucia et al., 2014). But even for a small number of branches, the problem size grows exponentially with respect to the number of prediction steps. The increase in the growth of the tree can be limited by branching the tree until a certain stage called robust horizon N_r beyond which the uncertainty can be assumed to be constant. The general formulation of the optimization problem that is solved at each sampling time can be written as:

$$\min_{z_{k+1}^j, v_k^j, \forall (j,k) \in I} \sum_{i=1}^{N} \omega_i \sum_{k=0}^{N_p-1} \mathscr{L}_i(z_{k+1}^j, v_k^j) \tag{3a}$$

subject to:

$$z_{k+1}^j = f(z_k^{p(j)}, v_k^j, d_k^{r(j)}), \qquad \forall (j, k+1) \in I, \tag{3b}$$

$$z_{k+1}^j \in \mathbb{Z},\ v_k^j \in \mathbb{V}, \qquad \forall (j,k) \in I, \tag{3c}$$

$$v_k^j = v_k^l \text{ if } z_k^{p(j)} = z_k^{p(l)}, \qquad \forall (j,k), (l,k) \in I, \tag{3d}$$

where N denotes the number of scenarios. The set of indices (j,k) in the scenario tree is defined as I. z_k^j and v_k^j denote the predicted states and inputs of the primary controller for all $(j,k) \in I$. Each state vector z_{k+1}^j at stage $k+1$ and position j in the scenario tree depends on the parent state $z_k^{p(j)}$ at stage k, the control inputs v_k^j and the corresponding realization r of the uncertainty $d_k^{r(j)}$. Equation (3b) denotes the model of the plant and (3c) represents the state and input constraints of the primary controller. The tightened state and input constraints of the primary controller must satisfy the property $\mathbb{Z} \subseteq (\mathbb{X} \ominus \mathbb{W})$ and $\mathbb{V} \subseteq \mathbb{U}$, where $\mathbb{X}$ and $\mathbb{U}$ are the true state and input constraints. Here $\ominus$ denotes the Pontryagin difference.

Since the measurement information is available at every time-step, the future control inputs can be adapted depending on

the realization of the uncertainty. However, the control inputs for the branches with the same parent node need to be the same because the realization of the uncertainty at the future is not known. This is enforced by the so-called non-anticipativity constraints as shown in (3d). Each scenario is assigned a certain weight ω_i that can be updated online to improve the performance of the controller. If the weights are not known, they can be assumed to be uniform. The uncertain parameters are assumed to take discrete values $d_k \in \mathbb{D}$. $\mathscr{L}_i(z^j_{k+1}, v^j_k)$ represents the stage cost of the ith scenario and can be chosen according the control objective. The scenario tree of the primary controller is initialized using the initial conditions at the first time-step and using one of the predicted states of the primary controller $z_k \in \{z^{j*}_1\}, \forall (j,1) \in I$ in the following time steps.

3.2 Ancillary controller

The ancillary controller is used to counteract the effects of all the uncertainties that are not part of the primary controller. The ancillary controller keeps the trajectory of the system close to one of the predicted trajectories of the primary controller. This is achieved by tracking all the trajectories of the primary controller for all $d_k \in \mathbb{D}$. The general formulation of the optimization problem to be solved at each sampling instance together with the primary controller can be written as:

$$\min_{x^j_{k+1}, u^j_k, \forall (j,k) \in I} \sum_{i=1}^{N} \omega_i \sum_{k=0}^{N_p-1} \mathscr{J}_i(x^j_{k+1} - z^{j*}_{k+1}, u^j_k - v^{j*}_k) \tag{4a}$$

subject to:

$$x^j_{k+1} = f(x^{p(j)}_k, u^j_k, d^{r(j)}_k), \qquad \forall (j,k+1) \in I, \tag{4b}$$

$$x^j_{k+1} \in \mathbb{X} \ominus \mathbb{W},\ u^j_k \in \mathbb{U}, \qquad \forall (j,k) \in I, \tag{4c}$$

$$u^j_k = u^l_k \text{ if } x^{p(j)}_k = x^{p(l)}_k, \qquad \forall (j,k),(l,k) \in I, \tag{4d}$$

where z^{j*}_k and v^{j*}_k are the predicted optimal state and input trajectories of the primary controller for all $(j,k) \in I$. The prediction horizon, the robust horizon and the tree structure remain the same as that of the primary controller with a different initialization. The current state of the system x_k forms the root node of the scenario tree of the ancillary controller. The objective function is chosen such that the distance between the predicted state and input trajectories of the primary controller and the true system are minimized. A possible choice for the stage cost is $\mathscr{J}_i = (x^j_{k+1} - z^{j*}_{k+1})^T Q (x^j_{k+1} - z^{j*}_{k+1}) + (u^j_k - v^{j*}_k) R (u^j_k - v^{j*}_k)$, where $Q \in \mathbb{R}^{n_x \times n_x}$ and $R \in \mathbb{R}^{n_u \times n_u}$ are the weighting matrices whose values can be tuned. There is no back-off necessary for the input constraints of the ancillary controller (see (4c)). However the constraints of the states are tightened to account for the presence of additive disturbances $w_k \in \mathbb{W}$. This is necessary to achieve robust constraint satisfaction because the presence of additive disturbances is not explicitly considered in the predictions. Note that the state constraints do not apply to the initial state of the system which is always initialized using the feedback x_k from the plant. The non-anticipativity constraints are also enforced using (4d) for the ancillary controller. The control input u^{1*}_0 is applied to the plant.

A schematic representation of the proposed approach compared to a standard tube-based NMPC is shown in Figure 2. In contrast to a single nominal path predicted in tube-based NMPC, we predict different scenarios for different realizations of the large uncertainties. The predicted trajectories are tracked using the ancillary controller.

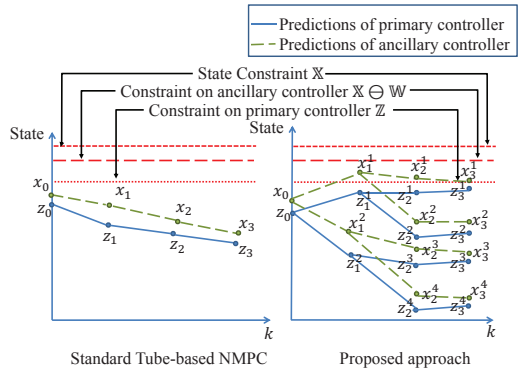

Fig. 2. Schematic representation of the proposed approach in comparison with a standard tube-based NMPC.

4. OUTPUT FEEDBACK CONTROLLER

An estimation scheme must be employed to estimate the states of the system if the full-state information is not available. To keep the scheme general, we do not assume a specific type of estimation method employed but only that an estimation scheme is available and that the estimation error is bounded. The estimation error bounds are denoted by $\mathbb{E}_k, \forall k$. The bounds on the estimation error can be obtained by assuming a confidence interval on the covariance matrices of the estimation schemes or can be rigorously obtained using set-based estimation schemes. An output feedback NMPC scheme was proposed using tube-based NMPC framework in Rawlings and Mayne (2009) for systems with additive uncertainties assuming invariant estimation error bounds. In this paper, we propose an output feedback scheme using the proposed new tube-based multi-stage NMPC framework. Key differences in the proposed formulation compared to Rawlings and Mayne (2009) include the consideration of parametric uncertainties and a new formulation of an output feedback framework for time-varying estimation error bounds.

If the current state x_k of the system is not known, the estimate $\hat{x}_k$ can be used to initialize the ancillary controller. If the ancillary controller is initialized using $\hat{x}_k$, the effect of the estimation error must be considered in addition to the plant-model mismatch for the method to be robust. The predicted states x^j_{k+1} at the next time-step $k+1$ are

$$x^j_{k+1} = f(\hat{x}_k, u_k, d^{r(j)}_k), \tag{5}$$

where $d^{r(j)}_k \in \mathbb{D}$ denotes one of the realizations of the uncertainty considered in the tree. The state of the plant evolves from the not precisely known state x_k for the applied input u_k and the realization of the parametric uncertainty d_k and the additive disturbance w_k as in (1a). The difference between the predicted state x^j_{k+1} that corresponds to the actual realization of the large uncertainty $d^{r(j)}_k = d_k$ and the true state x_{k+1} of the plant is realized as follows:

$$\gamma_{k+1} = (f(x_k, u_k, d_k) - f(\hat{x}_k, u_k, d_k)) + w_k. \tag{6}$$

The prediction error γ_{k+1} depends on the realization of the estimation error $e_k \in \mathbb{E}_k$ and the additive disturbance $w_k \in \mathbb{W}$. We know that the estimator satisfies the relation $x_k \in \hat{x}_k \oplus \mathbb{E}_k$, where $\oplus$ denotes the Minkowski-sum. The bounds on γ_{k+1} can then be given by $\Gamma_{k+1} = \{\gamma_{k+1} : \gamma_{k+1} = (f(x_k, u_k, d_k) - f(\hat{x}_k, u_k, d_k)) + w_k, \forall x_k \in \hat{x}_k \oplus \mathbb{E}_k, \forall w_k \in \mathbb{W}\}$. From (1a) and (6), the state x_{k+1} at the next time-step can be given by

$$x_{k+1} = f(\hat{x}_k, u_k, d_k) + \gamma_{k+1}, \tag{7}$$

for a realization of $\gamma_{k+1} \in \Gamma_{k+1}$. Since the estimation error satisfies the relations $e_{k+1} = x_{k+1} - \hat{x}_{k+1}$ and $e_{k+1} \in \mathbb{E}_{k+1}$, the following relation holds for the estimate $\hat{x}_{k+1}$ at the next time-step:

$$\hat{x}_{k+1} = f(\hat{x}_k, u_k, d_k) + \gamma_{k+1} - e_{k+1}, \tag{8}$$

for realizations of $\gamma_{k+1} \in \Gamma_{k+1}$ and of $e_{k+1} \in \mathbb{E}_{k+1}$. If the estimation error is sufficiently small for all k, the influence of the estimation error on the predictions of the ancillary controller can be considered along with the set of small disturbances $w_k \in \mathbb{W}$ and the constraints of the primary controller can be tightened for the enlarged additive bound $-\mathbb{E}_{k+1} \oplus \Gamma_{k+1}, \forall k$. Hence the optimization problem for the primary controller remains the same as that of the full-state information problem (3) with more stringent state and input constraints ($\mathbb{Z}$ and $\mathbb{V}$). $\mathbb{Z}$ must satisfy the property $\mathbb{Z} \subseteq \mathbb{X} \ominus (-\mathbb{E}_{k+1} \oplus \Gamma_{k+1}), \forall k$ to account for the estimation errors in addition to the additive disturbances. The constraints of the ancillary controller must be tightened to accommodate the presence of estimation errors $e_k \in \mathbb{E}_k$ and the error propagation $\gamma_k \in \Gamma_k$.

The optimization problem of the ancillary controller for the output feedback problem is given by

$$\min_{x^j_{k+1}, u^j_k, \forall (j,k) \in I} \sum_{i=1}^{N} \omega_i \sum_{k=0}^{N_p-1} \mathscr{J}_i(\hat{x}^j_{k+1} - z^{j*}_{k+1}, u^j_k - v^{j*}_k) \tag{9a}$$

subject to:

$$\hat{x}^j_{k+1} = f(\hat{x}^{p(j)}_k, u^j_k, d^{r(j)}_k), \qquad \forall (j, k+1) \in I, \tag{9b}$$

$$\hat{x}^j_{k+1} \in \mathbb{X} \ominus (-\mathbb{E}_{k+1} \oplus \Gamma_{k+1}), u^j_k \in \mathbb{U}, \qquad \forall (j,k) \in I, \tag{9c}$$

$$u^j_k = u^l_k \text{ if } \hat{x}^{p(j)}_k = \hat{x}^{p(l)}_k, \qquad \forall (j,k), (l,k) \in I, \tag{9d}$$

where (9b) denotes the prediction of the estimates for the large uncertainties. The state constraints (9c) are tightened to accommodate the presence of estimation errors.

The time varying uncertainties can be simplified to an invariant bound if the estimation error converges to a constant error bound $\mathbb{E}_\infty$ or by assuming outer bounds for all k for both $\mathbb{E}_k$ and Γ_k. This results in some conservatism but simplifies the implementation.

5. ALGORITHM AND IMPLEMENTATION DETAILS

The algorithm that implements the proposed approach is given in Algorithm 1. In Algorithm 1, only the primary controller is run at the first time-step. This is because the initial conditions for both the primary and the ancillary controllers are the same (i.e. $\hat{x}_0$). The optimal solution for the ancillary controller therefore is $u^{j*}_k = v^{j*}_k, \forall (j,k) \in I$.

5.1 Cost functions and initialization

The cost function of the primary controller can be an economic or a tracking objective. The cost function of the ancillary controller is chosen to track of the predictions of the primary controller. The primary controller is always initialized using one of the states that is predicted by it in the previous time-step (except at $k = 0$ where it is initialized using the estimate). The set of all states predicted one-step ahead by solving (3) is given by $\{z^{j*}_1\}, \forall (j,1) \in I$. The nearest point to the estimate $\hat{x}_k$ in the set $\{z^{j*}_1\}, \forall (j,1) \in I$ is then used to initialize the primary controller. The ancillary controller is always initialized using the state estimate $\hat{x}_k$.

Algorithm 1 Proposed Tube-based multi-stage NMPC

Require:	$\mathbb{Z}$, $\mathbb{V}$, $\hat{x}_k$, $\mathbb{E}_k$, Γ_k.
If $(k = 0)$**:**	Initialize primary controller with $\hat{x}_0$. Solve (3), apply v^{1*}_0 to the plant and store $\{z^{j*}_1\}, \forall (j,1) \in I$.
If $(k > 0)$**:**	
Step 1	Estimate the states $\hat{x}_k$.
Step 2	Find the nearest point to $\hat{x}_k$ in $\{z^{j*}_1\}, \forall (j,1) \in I$ solved at previous time-step $(k-1)$ and store it as z_0.
Step 3	Initialize the primary controller with z_0 and solve (3).
Step 4	Store $\{z^{j*}_k\}$ and $\{v^{j*}_k\}, \forall (j,k) \in I$.
Step 5	Initialize the ancillary controller with $\hat{x}_k$, pass the stored values $\{z^{j*}_k\}$ and $\{v^{j*}_k\}, \forall (j,k) \in I$ and solve (9).
Step 6	Apply u^{1*}_0 as input to the plant and go to step 1 at the next sampling time.

5.2 Constraint tightening

As noted in (3), the constraints of the primary controller must be tighter than the true constraints of the system. An easy way to tighten the constraints using simulation studies was proposed in Mayne et al. (2011). It is achieved with the help of the parameters α and β such that $\mathbb{Z} = \alpha\mathbb{X}$ and $\mathbb{V} = \beta\mathbb{U}$ with $\alpha \leq 1$ and $\beta \leq 1$. If the constraints are violated in the simulations, α and β must be reduced. Similarly if the states/estimates of the plant are far away from the constraints but the primary controller predictions are at the tightened constraints, α and β can be made larger.

6. CASE STUDY

An industrial polymerization process proposed in Lucia et al. (2014) is used as the case study. In Lucia et al. (2014), the authors studied the effect of plant-model mismatch using the multi-stage NMPC approach assuming full-state information. In this paper, we test the proposed approach using this case study that includes additive disturbances and estimation errors in addition to parametric uncertainties. There are three mass balances and five energy balances that models the process. The reaction is highly exothermic and the jacket and the heat exchanger are used to regulate the temperature inside the reactor. Full details of the model along with the parameter values can be found in Lucia et al. (2014). There are two important state constraints. The constraints on the reactor temperature T_R helps in achieving the desired quality of the end product and the safety constraint on T_ad assures safe operation of the plant. There are three manipulated variables: Monomer feed $\dot{m}_\mathrm{F}$, jacket inlet temperature T^IN_M and the inlet temperature of the heat exchanger $T^\mathrm{IN}_\mathrm{AWT}$. The bounds on the states and the inputs are given in Table 1. The tightened bounds for the primary and the ancillary controllers of the proposed scheme are also given.

The goal is to maximize the production of polymer m_P. The economic objective function of the primary controller is defined as follows:

$$J_\mathrm{eco}(x^j_{k+1}, u^j_k) = \sum_{i=1}^{N} \omega_i \sum_{k=0}^{N_p-1} -m^j_{\mathrm{P},k+1} + r_1(\Delta\dot{m}^j_{\mathrm{F},k})^2 + r_2(\Delta T^{\mathrm{IN},j}_{\mathrm{M},k})^2 + r_3(\Delta T^{\mathrm{IN},j}_{\mathrm{AWT},k})^2 \ ,$$

Table 1. Important state and input constraints

State/ Input	Min.	Max.	Primary controller bounds	Ancillary controller bounds	Unit
T_R	88.0	92.0	[88.7, 91.3]	[88.35, 91.65]	°C
T_{ad}	0	109	[0, 106]	[0, 107.5]	°C
$\dot{m}_F$	0	30000	[0, 29800]	[0, 30000]	$\frac{kg}{h}$
T_M^{IN}	60	100	[61, 99]	[60, 100]	°C
T_{AWT}^{IN}	60	100	[61, 99]	[60, 100]	°C

where r_1, r_2 and r_3 are tuning parameters that penalize the change in the control moves. For this case study, the values of the tuning parameters are chosen as $r_1 = 0.125$, $r_2 = 4$ and $r_3 = 0.25$. The ancillary controller is designed to track the critical states of the system m_P and T_R and the inputs. The Q matrix of (9a) is chosen as $\mathrm{diag}(0,0,0.01,100,0,0,0,0)$ and the R matrix is chosen as an identity matrix $\mathtt{I}^{3\times 3}$. The tightened constraints of the primary and the ancillary controllers were obtained by conducting simulation studies and are given in Table 1. The reaction rate k_0 is uncertain by $\pm 30\%$ and the enthalpy of the reaction ΔH_R is uncertain by $\pm 10\%$ relative to the nominal value. The nominal value of k_0 is 7 and the bounds are given by $k_0 \in [4.9, 9.1]$. The nominal value of ΔH_R is 950 kJ kg^{-1} and the bounds are given by $\Delta H_R \in [855, 1045]$ kJ kg^{-1}. Only two states, T_R and T_M are measured. The measurement noise was assumed to be zero-mean Gaussian with standard deviation of 0.3°C. The bounds on the additive disturbances for the mass balances at every discrete time instants is given by ± 2 kg/h and for the energy balances is given by ± 0.1°C.

We consider the extreme values and the nominal value of the parametric uncertainty in the scenario tree and assume the weights of the branches to be the same. A prediction horizon $N_p = 20$ and a robust horizon $N_r = 1$ are used. This results in 9 scenarios for both the primary and ancillary controllers. The sampling time of the controller is 50 s. The state constraints are formulated as soft constraints for the standard certainty-equivalence multi-stage MPC to avoid infeasible optimization problems. The proposed tube-based multi-stage NMPC is implemented with hard constraints. We use orthogonal collocation on finite elements for the discretization of the nonlinear dynamics and CasADi (Andersson et al., 2012) for the automatic generation of first and second order exact derivatives of the resulting nonlinear programming problem which are solved using IPOPT (Wächter and Biegler, 2006). An Extended Kalman Filter is used to estimate the states. The uncertain parameters are also estimated along with the states. The initial value of the parameters for the EKF algorithm are chosen as their nominal values. The batch is considered finished if the mass of the polymer reaches the desired quantity.

7. RESULTS

We compare the proposed new output feedback scheme with three alternatives: standard multi-stage NMPC assuming certainty equivalence with given constraints (CE-MS), standard multi-stage NMPC assuming certainty equivalence with tightened constraints (CE-MSTC) and the tube-based NMPC. Standard multi-stage NMPC assuming certainty equivalence ignores estimation errors and additive disturbances. The tube-based NMPC (Mayne et al., 2011) was proposed for additive disturbances and for tracking objectives. Since the application considered here is a batch process with parametric uncertainties and the objective is economic, the formulation proposed

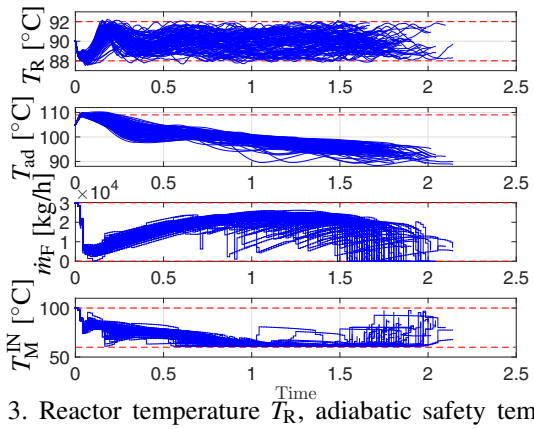

Fig. 3. Reactor temperature T_R, adiabatic safety temperature T_{ad}, Monomer Feed $\dot{m}_F$, jacket inlet temperature T_M^{IN} trajectories of the polymerization reactor obtained for 100 batch runs controlled using CE-MS NMPC.

in Mayne et al. (2011) cannot be directly applied. To facilitate a comparison, we have replaced the robust multi-stage NMPC controller with the nominal NMPC controller in the proposed scheme and refer to it as tube-based NMPC. The bounds of the reactor temperature of the nominal controller of the tube-based NMPC were chosen as $[89.9, 90.1]$ and of the adiabatic safety temperature were chosen as $[0, 104.5]$. To quantify the differences, 100 simulation runs were performed. The initial conditions for the estimates and the additive disturbances are randomly chosen and the uncertain parameter is also randomly chosen but assumed to be constant for a given simulation run.

In Figure 3, the results of 100 simulations are provided for CE-MS NMPC. For many cases both the constraints on the reactor temperature T_R and the adiabatic safety temperature T_{ad} are violated. The control inputs are oscillatory because of the presence of unanticipated errors in the form of estimation errors and additive disturbances. The results of 100 simulation runs using the tube-based NMPC is given in Figure 4. The tube-based NMPC scheme does not result in constraint satisfaction despite stringent tightening of the constraints. The constraints of the reactor temperature T_R in some cases are violated by as much as 5°C. This makes the tube-based NMPC impractical to apply for this case study. Figure 5 shows the results of the proposed output feedback scheme for 100 simulation runs. The constraints are satisfied for all cases by imposing suitable back-off from the constraints to account for estimation errors and additive disturbances in addition to parametric uncertainties. This is because in the proposed scheme, the effects of the most important uncertainties are predicted explicitly using the primary controller and the small disturbances are rejected using the ancillary controller.

The critical state constraints are satisfied for CE-MSTC NMPC because of the tightened constraints. However, the inputs are oscillatory similar to Fig. 3 (the figure of state and input trajectories of CE-MSTC NMPC is not shown here because of lack of space). The oscillatory inputs result in an increased batch time. Since the controller always tries to maximize the product in CE-MS NMPC with tightened constraints, the control inputs are chosen such that the state trajectories are always at the constraints. The unaccounted disturbances, despite being small, lead to violation of soft constraints and result in a suboptimal input sequence. The proposed new scheme performs better than the certainty equivalence multi-stage NMPC with tightened constraints because the ancillary controller does not maximize

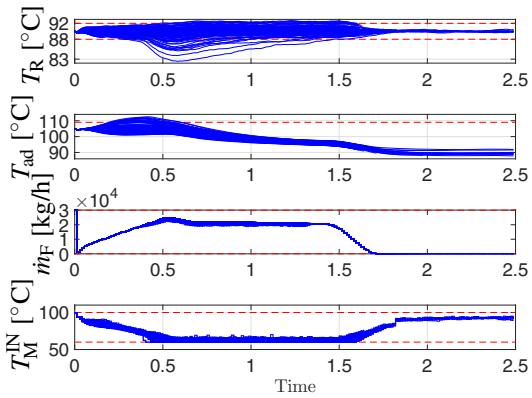

Fig. 4. Reactor temperature T_R, adiabatic safety temperature T_{ad}, Monomer Feed $\dot{m}_F$, jacket inlet temperature T_M^{IN} trajectories of the polymerization reactor obtained for 100 batch runs controlled by the tube-based NMPC scheme.

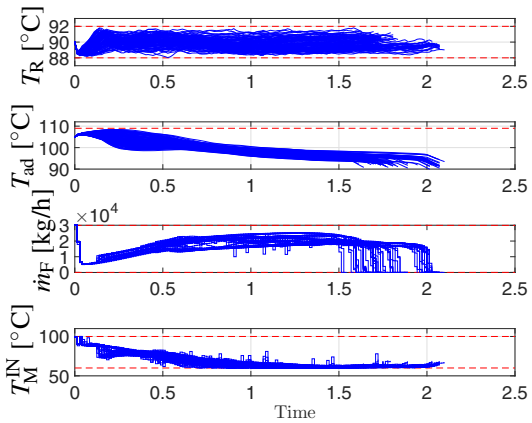

Fig. 5. Reactor temperature T_R, adiabatic safety temperature T_{ad}, Monomer Feed $\dot{m}_F$, jacket inlet temperature T_M^{IN} trajectories of the polymerization reactor obtained for 100 batch runs controlled by the proposed scheme.

Table 2. Performance comparison between multi-stage NMPC that assumes certainty equivalence (CE-MS), CE-MS NMPC with tightened constraints (CE-MSTC), the tube-based MPC (Tube) and the proposed new robust output feedback NMPC scheme (proposed)

NMPC Controller	CE-MS	CE-MSTC	Tube	Proposed
Batch time [h]	1.76	1.91	1.73 [1]	1.81
Max. violation of T_R	0.3°C	0	5.55°C	0
Max. violation of T_{ad}	1°C	0	3°C	0
Avg. comp. time per iter. [s]	0.48	0.48	0.07	0.98

the product yield but aims to track the trajectories of the primary controller. The primary controller maximizes the yield but is not influenced directly by unaccounted disturbances. This results in a robust strategy that is neither conservative nor computationally expensive and thus suitable for industrial applications.

Table 2 presents a summary of the results. It can be seen that the maximum violation of the reactor temperature T_R is 0.3°C for CE-MS NMPC and 5.55°C for the tube-based NMPC. The maximum violation of the adiabatic safety constraint is 1°C for CE-MS NMPC and 3°C for the tube-based NMPC. The proposed scheme satisfied the constraints at all times. Compared to the advantage of no constraint violations, a slightly higher average batch and computation times are reasonable. CE-MS NMPC with tightened constraints also satisfies the original constraints, but the batch time is approx. 5% larger than the proposed new approach on the average. The longest batch is approx. 10% larger than the proposed new approach.

[1] Not all batches were finished with the required amount of polymer.

8. CONCLUSION

We proposed a new robust NMPC scheme by combining two-well known approaches to robust NMPC and showed that they complement each other in handling uncertainties of different magnitudes. The resulting scheme has a better trade-off between problem complexity and performance when compared to the two schemes independently. We showed that robust constraint satisfaction can be achieved for different kinds of uncertainties that are present in a real nonlinear system.

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