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Integration of model predictive control and backstepping approach and its stability analysis

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Abstract: Backstepping controller (BS) and model predictive controller (MPC) have been widely used for many applications by virtue of their own merits. BS works even with non-minimum phase and finite-time escape and MPC can handle state and input constraints explicitly. Nevertheless, BS requires repeated differentiations of the virtual control, whereas high computational loads of MPC are obstacles to practical implementation. This study proposes a control strategy that combines BS and MPC for nonlinear systems in strict-feedback form. It is proven that the controller renders the closed-loop system asymptotically stable. The proposed MPC-BS requires less computational load than that of MPC, since it only optimizes the virtual input of the first step and computes the input by backstepping approach. The explosion of terms caused by the consecutive differentiation in BS approach is also addressed.

Keywords: Model predictive control, Backstepping control, Stability analysis, Nonlinear control systems

1. INTRODUCTION

Since backstepping control (BS) was developed in 1990s, there have been numerous applications of this nonlinear controller. Ezal et al. (2000) suggests locally optimal and robust backstepping design which starts with a linear H_∞-optimality. Tanner and Kyriakopoulos (2003) applies BS to a unicycle driven by a new discontinuous kinematic controller. Design of backstepping controller has fast calculation speed and flexibility. Zhao and Kanellakopoulos (1998) show that the virtual control of the first step and the Lyapunov functions can be chosen differently, affecting the performance of controller. In addition, the value of gain and the form of Lyapunov function can be flexible. However, BS often suffers from the explosion of terms because of the continuous differentiation of the virtual control (Yang et al. (2007)). Users should define a function for the virtual control design in the first step.

Model predictive control (MPC) is an open-loop optimal control with feedback update implemented in a receding horizon fashion. It can consider both state and input constraints explicitly in an online optimization. However, high computational load is still a challenge for the nonlinear MPC (NMPC) of systems with fast dynamics. There have been many efforts to reduce the computational load of NMPC, and among them are suboptimal NMPC and fast NMPC. Suboptimal NMPC computes approximate solution of the dynamic optimization problem (Stewart et al. (2010), Zeilinger et al. (2011)). Fast NMPC improves the optimization algorithm for NMPC-inspired optimization problems (Lopez-Negrete et al. (2013), Jschke et al. (2014)). Coordinate transformation is another approach to reducing the computational load of NMPC. When the system is input-output feedback linearizable, the controller designed by feedback linearization yields a linear system with $\tilde{u}$, part of the real input, and $\tilde{u}$ is determined by linear MPC (Simon et al. (2013)). The main drawback of the coordinate transformation approach is that the simple state and input constraints are converted to nonlinear constraints.

Both BS and MPC can synergistically complement each other through proper integration of the two methods. The computational advantage of BS and the explicit consideration of constraints of MPC can address the issues of its counterpart: the excessive computational load of MPC and the explosion of terms of BS. Several studies (Gouta et al. (2015), Ouali et al. (2012)) compares the performances of BS and MPC. However, there is no previous study that tried to combine BS and MPC to the best knowledge of the authors. In this study, integrated backstepping and model predictive controller is first proposed and its asymptotic stability is proven. The virtual input of first step in backstepping approach is designed using MPC with piecewise constant control, allowing for addressing the explosion of terms in BS approach and alleviating the computational load compared with MPC-only approach.

The paper is organized as follows: Section 2 presents a class of nonlinear systems considered in this paper and a preliminary theorem about the stability of the closed loop system when the controller is applied in a sample-and-hold fashion. The design of the proposed controller and its stability analysis are shown in Section 3. The results of applying the proposed controller to an illustrative example

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and the comparison between BS and MPC are discussed in Section 4.

2. PRELIMINARIES

Consider the nonlinear continuous-time system in a strict-feedback form in (1) and (2), which is obtained by coordinate transformation.

$$\dot{x} = f_0(x) + g_0(x)z_1 \tag{1}$$

$$\begin{aligned} &\dot{z}_i = f_i(x, z_1, ..., z_i) + g_i(x, z_1, ..., z_i)z_{i+1} \\ &\text{ for } 1 \leq i \leq \text{N-1 and N} \geq 2 \\ &\dot{z}_N = f_N(x, z_1, ..., z_N) + g_N(x, z_1, ..., z_N)u \end{aligned} \tag{2}$$

where $x \in \mathbf{R}^n$, $z_i \in \mathbf{R}$, $u \in \mathbf{R}$

The overall system is denoted by

$$\dot{\mathbf{X}} = F(\mathbf{X}(t), u(t)), \;\; \mathbf{X} = [x, z_1, ..., z_N]^T \tag{3}$$

Without loss of generality, the steady-state is assumed to be $(\mathbf{X}_s, u_s) = (\mathbf{0}, 0)$

2.1 Backstepping control

Assumption 1. The functions f_i, g_i for all $i = 1, ..., N$ are smooth and f_0 to f_N vanish at the origin, and $g_i(x, z_1, ..., z_N) \neq 0$ over the domain of interest.

Under the strict-feedback structure and Assumption 1, the backstepping approach is possible for designing the controller. The procedure of designing the controller is well described in Khalil (2015).

2.2 Sample-and-hold MPC

We uses the MPC with piecewise constant input for designing the virtual input of the first step in backstepping controller. This sample-and-hold MPC is described using the same notation in (3).

Assumption 2. There exists a locally Lipschitz feedback controller $z_1 = h(x)$ with $h(0) = 0$ such that the origin of the closed loop system 1 is locally exponentially stable and globally asymptotically stable.

If Assumption 2 holds and let $R > 0$, there exist a radially unbounded Lyapunov function $V : \mathbf{R}^n \rightarrow \mathbf{R}_{\geq 0}$ and constants $c_i > 0$ ($i = 1, 2, 3,$ and 4) satisfying (4) for all $x \in \Omega_\rho$ where $\rho := \max\{V(x) : x \in B_R\}$:

$$\begin{aligned} c_1 \|x\|^2 \leq V(x) \leq c_2 \|x\|^2 \\ \|\nabla V(x)\| (f_0(x) + g_0(x)h(x)) \leq -c_4 \|x\| \\ \|\nabla V(x)\| \leq c_4 \|x\| \end{aligned} \tag{4}$$

Here, B_R denotes a closed ball with radius R and center at the origin. $\|\cdot\|$ denotes the Euclidean norm.

Lemma 1. (Ellis et al. , 2014) If Assumption 2 holds, there exist $\Delta^* > 0$ and M, $\sigma > 0$ such that for every partition $\{t_k\}_{k=0}^{\infty}$ of $\mathbf{R}_{\geq 0}$ with $sup_{k \geq 0}(t_{k+1} - t_k) \leq \Delta^*$, the closed loop system of (1) with the input trajectory $u(t) = h(x(t_k))$ for $t \in [t_k, t_{k+1}), k \in \mathbf{Z}_{\geq 0}$ and for $x_0 \in B_R$ satisfies the estimate $\|x(t)\| \leq M \exp(-\sigma t) \|x_0\|$ for all $t \geq 0$.

Since $h : \mathbf{R}^n \rightarrow \mathbf{R}$ is a locally Lipschitz mapping with $h(0, 0) = 0$ and f_0 and g_0 are smooth with $f_0(0) = 0$, there exist constants L, M> 0 such that

$$\begin{aligned} \|g_0(x)h(x) - g_0(x)h(z)\| \leq L \|x - z\|, \\ \|f_0(x) + g_0(x)h(z)\| \leq M \|x\| + M \|z\| \end{aligned} \tag{5}$$

By letting $\Delta^* > 0$ sufficiently small such that the following inequality holds

$$c_4 L \frac{2M\Delta^* exp(M\Delta^*)}{1 - 2M\Delta^* exp(M\Delta^*)} < c_3, \tag{6}$$

the inequality of (7) holds for $i \in \mathbf{Z}_{\geq 0}$

$$\nabla V(x(t)) f(x(t), h(x(t_k))) \leq -\frac{q}{2} \|x(t)\|^2 \tag{7}$$

, where $q := c_3 - c_4 L \frac{2M\Delta^* exp(M\Delta^*)}{1 - 2M\Delta^* exp(M\Delta^*)} > 0$. $\mathbf{R}_{\geq 0}$ and $\mathbf{Z}_{\geq 0}$ are the nonnegative real number and integer, respectively. $\{t_k\}_{k=0}^{\infty}$ denotes a partitioning of $\mathbf{R}_{\geq 0}$, where t_k is a strictly increasing sequence with $t_0 = 0$ and $\lim_{k \rightarrow \infty} t_k = \infty$.

3. INTEGRATED MPC AND BACKSTEPPING APPROACH

This section presents the integrated design of BS and MPC for the system (3). First, the virtual input of Step 1 in BS controller is designed by MPC.

Step 1 : Finite-horizon optimal control problem (FHOCP)

$$\begin{aligned} &\min_{\bar{z_1} \in S(T_s)} \int_{t_k}^{t_k + T_p} (\|\bar{x}(s)\|_Q + \|\bar{z}_1(s)\|_R) ds \\ &\text{subject to } \dot{\bar{x}}(t) = f_0(\bar{x}(t)) + g_0(\bar{x}(t))\bar{z}_1(t) \\ &\qquad \bar{x}(t_k) = x(t_k) \\ &\qquad \frac{\partial V_0}{\partial x}(f_0(x(t_k)) + g_0(x(t_k))\bar{z}_1(t_k)) \\ &\qquad \leq \frac{\partial V_0}{\partial x}(f_0(x(t_k)) + g_0(x(t_k))h(t_k)) \end{aligned} \tag{8}$$

, where $\bar{x}$ is the predicted state, $S(T_s)$ is the set of piecewise constant functions with the period T_s, and T_p is the prediction horizon. The optimal desired z_1 obtained by FHOCP is denoted as $z^*_{1,des}(t)$ and for $t \in [t_k, t_{k+1})$, $z_{1,des}(t) = z^*_1(t_k)$. Sampling time ($T_s$) is chosen such that $T_s \leq \Delta^*$ with Δ^* satisfying (6).

Given $V_0 = \frac{p_0}{2}x^2$, $z^*_{1,des}(t_k)$ yields (9) by Lemma 1:

$$\dot{V_0} = p_0 x(f_0 + g_0 z_{1,des}(t)) \leq -\frac{q}{2} \|x(t)\|^2 \tag{9}$$

Since there exists error between $z_{1,des}$ and real z_1,

$$\begin{aligned} \dot{V_0} &= p_0 x(f_0(x) + g_0(x)(e_1 + z_{1,des})) \\ &\leq p_0 x(f_0(x) + g_0(x)h(t_k)) + p_0 x g_0(x) e_1 \\ &\leq -\frac{q}{2} \|x(t)\|^2 + p_0 x g_0(x) e_1 \end{aligned} \tag{10}$$

, where $e_1 = z_1 - z_{1,des}$.

Step 2:
Letting $V_1 = \frac{p_0}{2}x^2 + \frac{p_1}{2}e_1^2$, its derivative is given by:

$$\dot{V_1} \leq -\frac{q}{2} \|x(t)\|^2 + p_0 x g_0(x) e_1 + p_1 e_1 (\dot{e_1}) \tag{11}$$

Since $z_{1,des}$ is constant for $t \in [t_k, t_{k+1})$, $\dot{e_1}$ is simply equal to $\dot{z_1}$. This leads to

$$\begin{aligned}\dot{V_1} &\le -\frac{q}{2}\|x(t)\|^2 + p_0 x g_0(x) e_1 + p_1 e_1(\dot{z_1}) \\ &= -\frac{q}{2}\|x(t)\|^2 + e_1(p_0 x g_0(x) + p_1(f_1(x,z_1) + g_1(x,z_1)z_2))\end{aligned} \tag{12}$$

By letting $z_{2,des} = \frac{1}{g_1(x,z_1)}(-f_1(x,z_1) - \frac{p_0}{p_1} x g_0(x) - \frac{a_1}{p_1}e_1)$ and $e_2 = z_2 - z_{2,des}$,

$$\begin{aligned}\dot{V_1} &\le -\frac{q}{2}\|x(t)\|^2 + p_0 x g_0(x_1) e_1 + p_1 e_1(\dot{z_1}) \\ &= -\frac{q}{2}\|x(t)\|^2 - a_1 e_1^2 + e_1 p_1 g_1(x,z_1) e_2\end{aligned} \tag{13}$$

Step 3:
Given $V_3 = \frac{p_0}{2}x^2 + \frac{p_1}{2}e_1^2 + \frac{p_2}{2}e_2^2$,

$$\begin{aligned}\dot{V_3} &\le -\frac{q}{2}\|x(t)\|^2 - a_1 e_1^2 + p_1 e_1 g_1(x,z_1) e_2 + p_2 e_2 \dot{e_2} \\ &= -\frac{q}{2}\|x(t)\|^2 - a_1 e_1^2 + e_2(p_1 e_1 g_1(x,z_1) + p_2(\dot{z}_2 - \dot{z}_{2,des})) \\ &= -\frac{q}{2}\|x(t)\|^2 - a_1 e_1^2 + e_2(p_1 e_1 g_1(x,z_1) \\ &\quad + p_2(f_2(x,z_1,z_2) + g_2(x,z_1,z_2)z_3 - \dot{z}_{2,des}))\end{aligned} \tag{14}$$

By letting $z_{3,des} = \frac{1}{g_2(x,z_1,z_2)}(-f_2(x,z_1,z_2) + \dot{z}_{2,des} - \frac{p_1}{p_2}e_1 g_1(x,z_1) - \frac{a_2}{p_2}e_2)$ and $e_3 = z_3 - z_{3,des}$,

$$\dot{V_3} \le -\frac{q}{2}\|x(t)\|^2 - a_1 e_1^2 - a_2 e_2^2 + p_2 e_2 g_2(x_1,z_1,z_2) e_3 \tag{15}$$

Step N:
Finally, taking $V_N = \frac{p_0}{2}x^2 + \frac{p_1}{2}e_1^2 + \dots + \frac{p_N}{2}e_N^2$ as a Lyapunov function candidate for the overall system,

$$\begin{aligned}\dot{V_N} &\le -\frac{q}{2}\|x(t)\|^2 - a_1 e_1^2 + \dots - a_{N-1}e_{N-1}^2 \\ &\quad + p_{N-1}e_{N-1}g_{N-1}(x_1,z_1,\dots,z_{N-1})e_N + p_N e_N \dot{e_N} \\ &= -\frac{q}{2}\|x(t)\|^2 - a_1 e_1^2 + \dots - a_{N-1}e_{N-1}^2 \\ &\quad + p_{N-1}e_{N-1}g_{N-1}(x_1,z_1,\dots,z_{N-1})e_N \\ &\quad + p_N e_N(f_N(x,z_1,\dots,z_N) + g_N(x,z_1,\dots,z_N)u - \dot{z}_{N,des})\end{aligned} \tag{16}$$

By letting

$$\begin{aligned}u =& \frac{1}{g_N(x,z_1,\dots,z_N)}(-f_N(x,z_1,\dots,z_N) + \dot{z}_{N,des} \\ &- \frac{p_{N-1}}{p_N}e_{N-1}g_{N-1}(x,z_1,\dots,z_{N-1}) - \frac{a_N}{p_N}e_N)\end{aligned} \tag{17}$$

and $e_N = z_N - z_{N,des}$,

$$\dot{V_N} \le -\frac{q}{2}\|x(t)\|^2 - a_1 e_1^2 + \dots - a_i e_N^2 \tag{18}$$

Since we uses MPC with piecewise constant functions for designing the $z_{1,des}$ and backstepping approach, $V_N(\chi(t))$ with $\chi = [x, e_1, \dots, e_n]^T$ decreases for $t \in [t_k, t_{k+1})$ for all $k \in \mathbf{Z}_{\ge 0}$. However, at t_k for all $k \in \mathbf{Z}_{\ge 0}$, $V_N(\chi(t_k^+)) > V_N(\chi(t_k^-))$ can occur depending on the $z_{1,des}(t_k^+)$ determined by FHOCP at t_k, which requires Assumption 3:

Assumption 3. There exists $0 < \Delta_2^* \le T_s$ such that FHOCP has a feasible solution under the following additional constraints

$$c_{N,2}\left\|\chi(t_k^+)\right\|^2 \le c_{N,1}\left\|\chi(t_k^-)\right\|^2 + c_{N,4}\left\|\chi(t_k^-)\right\|^2 (T_s - \Delta_2^*) \tag{19}$$

Theorem 1. If Assumption 1 holds, there exists $\delta^* > 0$ such that $c_4 L \frac{2M\Delta^* exp(M\Delta^*)}{1 - 2M\Delta^* exp(M\Delta^*)} < c_3$. In addition, if Assumptions 2 and 3 hold, the controller designed by integrated MPC and backstepping approach (FHOCP and (17)) with the additional constraint of (20) for FHOCP renders the closed-loop system of (3) asymptotically stable.

The additional constraint for FHOCP with a constant τ such that $0 \le \tau \le \Delta_2^*$ is given by

$$c_{N,2}\left\|\chi(t_k^+)\right\|^2 \le c_{N,1}\left\|\chi(t_k^-)\right\|^2 + c_{N,4}\left\|\chi(t_k^-)\right\|^2 (T_s - \tau) \tag{20}$$

Proof:

By Lemma 1, during each $[t_k, t_{k+1})$ for all $k \in \mathbf{Z}_{\ge 0}$, (18) is satisfied. Thus, we only need to show that for all $k \in \mathbf{Z}_{\ge 0}$, $V_N(\chi(t_{k+1}^-)) \le V_N(\chi(t_k^-))$, with $\chi = [x, e_1, \dots, e_n]^T$.

There exist some positive constants $c_{N,1}, c_{N,2}$ and $c_{N,4}$ such that

$$c_{N,1}\|\chi\|^2 \le V_N(\chi) \le c_{N,2}\|\chi\|^2 \tag{21}$$

$$\frac{\partial V_N}{\partial t} + \frac{\partial V_N}{\partial x}F(t,\chi) \le -c_{N,4}\|\chi\|^2 \tag{22}$$

For all $0 \le \omega \le T_s$, we can obtain

$$\begin{aligned}V_N(\chi(t_{k+1}^-)) &\le V_N(\chi(t_{k+1}^- - \omega)) \\ &\le V_N(\chi(t_k^+)) - c_{N,4}\left\|\chi(t_{k+1}^- - \omega)\right\|^2 (T_s - \omega)\end{aligned} \tag{23}$$

By virtue of

$$V_N(\chi(t_k^+)) \le V_N(\chi(t_k^-)) + c_{N,2}\left\|\chi(t_k^+)\right\|^2 - c_{N,1}\left\|\chi(t_k^-)\right\|^2,$$

we obtain

$$\begin{aligned}&V_N(\chi(t_{k+1}^-)) \\ &\le V_N(\chi(t_k^-)) + c_{N,2}\left\|\chi(t_k^+)\right\|^2 - c_{N,1}\left\|\chi(t_k^-)\right\|^2 \\ &- c_{N,4}\left\|\chi(t_{k+1}^- - \omega)\right\|^2 (T_s - \omega)\end{aligned} \tag{24}$$

If $\left\|\chi(t_{k+1}^- - \Delta_2^*)\right\| \ge \left\|\chi(t_k^-)\right\|$, under the additional constraint and by (24),

$$\begin{aligned}&V_N(\chi(t_{k+1}^-)) \le V_N(\chi(t_{k+1}^- - \tau)) \le V_N(\chi(t_{k+1}^- - \Delta_2^*)) \\ &\le V_N(\chi(t_k^-)) + c_{N,2}\left\|\chi(t_k^+)\right\|^2 - c_{N,1}\left\|\chi(t_k^-)\right\|^2 \\ &- c_{N,4}\left\|\chi(t_{k+1}^- - \Delta_2^*)\right\|^2 (T_s - \Delta_2^*) \\ &\le V_N(\chi(t_k^-)) + c_{N,2}\left\|\chi(t_k^+)\right\|^2 - c_{N,1}\left\|\chi(t_k^-)\right\|^2 \\ &- c_{N,4}\left\|\chi(t_k^-)\right\|^2 (T_s - \Delta_2^*) \\ &\le V_N(\chi(t_k^-))\end{aligned}$$

(25)

If $\left\|\chi(t_{k+1}^{-}-\Delta_2^*)\right\| < \left\|\chi(t_k^{-})\right\|$,

$$\begin{aligned} V_N(\chi(t_{k+1}^{-})) \leq V_N(\chi(t_{k+1}^{-}-\tau)) \leq & V_N(\chi(t_{k+1}^{-}-\Delta_2^*)) \\ & \leq V_N(\chi(t_k^{-})) \end{aligned} \tag{26}$$

□

4. ILLUSTRATIVE EXAMPLE

In this section, we apply the proposed controller to the example of (27) (Khalil , 2015), and compare the results with those of the conventional BS controller and MPC.

$$\begin{aligned} \dot{x} &= x^2 - x^3 + z_1 \\ \dot{z_1} &= z_2 \\ \dot{z_2} &= u \end{aligned} \tag{27}$$

The simulations are conducted with the initial condition of $\chi(0) = (0.3, 0.1, 0.2)$.

4.1 Backstepping controller

From Khalil (2015),

$$\begin{aligned} z_{1,des,BS} &= -x_1^2 - x_1 \\ z_{2,des,BS} &= -x_1 + \dot{z}_{1,des,BS} - a_1 e_1 \\ &= -x_1 - (1+2x_1)(-x_1 - x_1^3 + e_1) - e_1 \\ u_{BS} &= -e_1 + \dot{z}_{2,des} - a_2 e_2 \end{aligned} \tag{28}$$

, with

$$\begin{aligned} \frac{\partial z_{2,des,BS}}{\partial x} &= 2x_1 - 2e_1 + (2x_1+1)(3x_1^2+1) + 2x_1^3 - 1 \\ \frac{\partial z_{2,des,BS}}{\partial e_1} &= -a_1 - 2x_1 - 1 \end{aligned} \tag{29}$$

We set all the constants $(a_1, ..., a_N, p_0, ..., p_N)$as one in both BS control design and the proposed control design.

4.2 The proposed controller

We have the $z_{1,des,MPC}(t) = z_{1,des,MPC}(t_k)$ for $t \in [t_k, t_{k+1})$ at each sampling time t_k by solving the following FHOCP:

$$\begin{aligned} &\min_{\bar{z}_1 \in S(T_s)} \int_{t_k}^{t_k+T_p} (\|\bar{x}(s)\|_Q + \|\bar{z}_1(s)\|_R) ds \\ &\text{subject to } \dot{\bar{x}}(t) = \bar{x}^2 - \bar{x}^3 + \bar{z}_1 \\ &\qquad \bar{x}(t_k) = x(t_k) \\ &\qquad x(t_k)(x(t_k)^2 - x(t_k)^3 + \bar{z}_1(t_k)) \\ &\qquad \leq -x(t_k)^2 - x(t_k)^4 \\ &\qquad c_{N,2}\left\|\chi(t_k^{+})\right\|^2 \\ &\qquad \leq c_{N,1}\left\|\chi(t_k^{-})\right\|^2 + c_{N,4}\left\|\chi(t_k^{-})\right\|^2 (T_s/2) \end{aligned} \tag{30}$$

, where $Q = 1$, $R = 2$, and $T_p = 0.5$

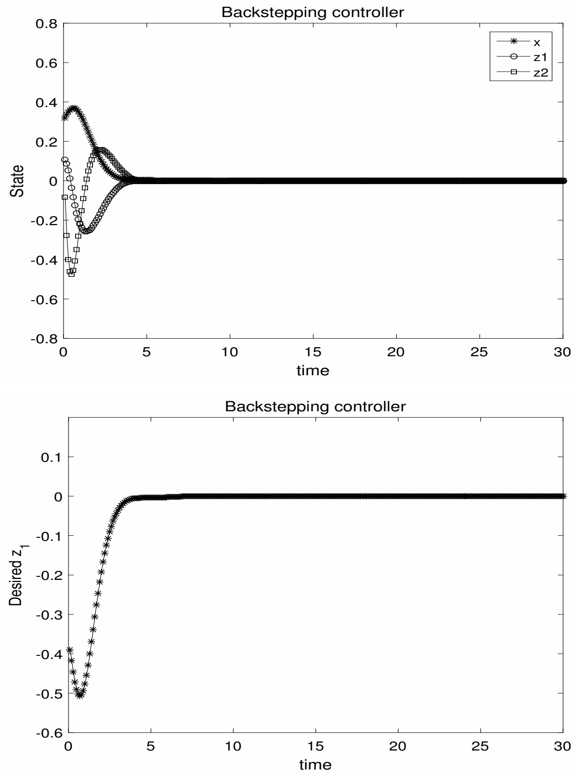

Fig. 1. The trajectories of state and desired z_1 under BS

and

$$\begin{aligned} z_{2,des,MPC} &= -x - a_1 e_1 \\ u_{MPC} &= -e_1 + \dot{z}_{2,des} - a_2 e_2 \end{aligned} \tag{31}$$

with

$$\begin{aligned} \frac{\partial z_{2,des,MPC}}{\partial x} &= -1 \\ \frac{\partial z_{2,des,MPC}}{\partial e_1} &= -a_1 \end{aligned} \tag{32}$$

For $x \in [-0.5, 0.5]$ with $h(x) = -x_1^2 - x_1$, T_s can be set as 0.1 and then $q = 0.2849$, since $L = 1, M = 1.75, c_3 = 1$, and $c_4 = 1$.

4.3 MPC

$$\begin{aligned} &\min_{\bar{u} \in S(T_s)} \int_{t_k}^{t_k+T_p} (\|\bar{\chi}(s)\|_Q + \|\bar{u}(s)\|_R) ds \\ &\text{subject to } \dot{\bar{\chi}} = \mathbf{F}(\bar{\chi}, \bar{u}) \\ &\qquad \bar{\chi}(t_k) = \chi(t_k) \\ &\qquad \frac{\partial V_N}{\partial \chi}\mathbf{F}(\chi, \bar{u}) \leq \frac{\partial V_N}{\partial \chi}\mathbf{F}(\chi, \bar{u}_{BS}) \end{aligned} \tag{33}$$

We set $Q = [1 \quad 0; 0 \quad 2]$ and $R = 1$ to make the objective function similar to that of the proposed controller.

4.4 Results and discussion

To solve the optimization problems, IPOPT (Wchter and Biegler , 2006) was used and computation was performed on an Intel i5-4670 3.40 GHz processor. To compare the settling time, the first time to maintain $\|\chi(t))\|$ less than

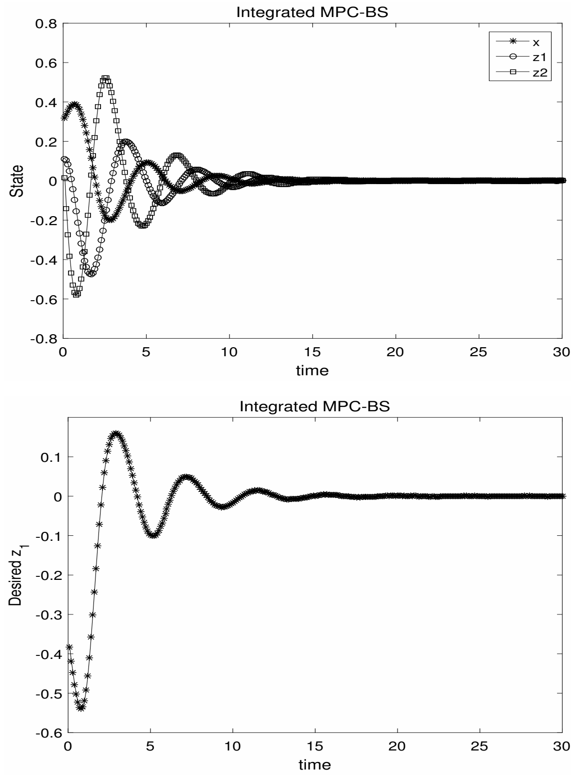

Fig. 2. The trajectories of state and desired z_1 under MPC-BS

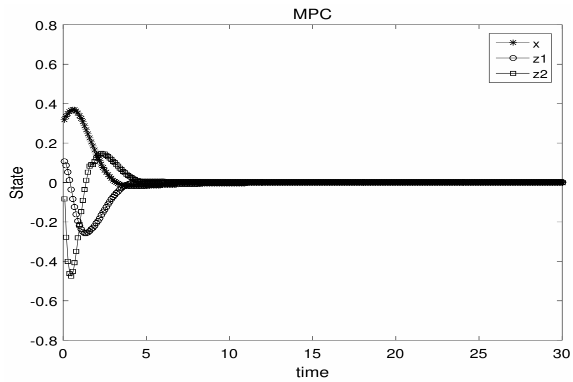

Fig. 3. The trajectories of state under MPC

$\|\chi(0)\| \times 0.03$ and the first time to maintain $\|\chi(t)\|_\infty$ less than $\|\chi(0)\|_\infty \times 0.03$ are shown in Table 1. The mean of the maximum elapsed time and average elapsed time were obtained by running the simulation for 10 times.

Table 1. The simultaion results of MPC-BS, MPC, BS

	MPC-BS	MPC	BS
The first time of maintaining $\|\chi\| < \|\chi(0)\| \times 0.03$	15.7	7	4.2
The first time of maintaining $\|\chi\|_\infty < \|\chi(0)\|_\infty \times 0.03$	15.7	7.3	4.2
$\sum_{k=1}^{T_F/T_s}(\|u(t_k)\| \times T_s)$	3.42	1.45	1.18
Mean of Maximum Elasped time [s]	0.165	0.245	-
Mean of Average Elasped time [s]	0.089	0.147	-

Table 1 shows the simulation results of MPC-BS, MPC, and BS. In terms of the first time of maintaining $\|\chi\| < \|\chi(0)\| \times 0.03$, the value increased in the order of BS, MPC, and MPC-BS. The ratio between the three values is 3.74:1.67:1, respectively. Since $z_{1,des,MPC}(t) = z_{1,des,MPC}(t_k)$ for $t \in [t_k, t_{k+1})$, $\dot{z}_{1,des,MPC} = 0$ in the proposed controller. This causes fluctuation as shown in Figure 2 and leads to a longer convergence time than the conventional methods. The same tendency is observed for the first time of maintaining $\|\chi\|_\infty < \|\chi(0)\|_\infty \times 0.03$, and the ratio of MPC-BS, MPC, and BS is 3.74:1.74:1, respectively. The larger $\sum_{k=1}^{T_F/T_s}(\|u(t_k)\| \times T_s)$ value of MPC-BS came from the longer settling time. In the case of the elasped time, MPC-BS showed the smaller elapsed time than that of MPC. The mean of the maximum elapsed time and average elapsed time are 0.67 and 0.6 smaller than those of MPC, respectively. This is because the dimension of the optimization problem in MPC-BS is smaller than that of MPC. From (29) and (32), MPC-BS had much simpler form of input than that of BS, addressing the explosion of terms.

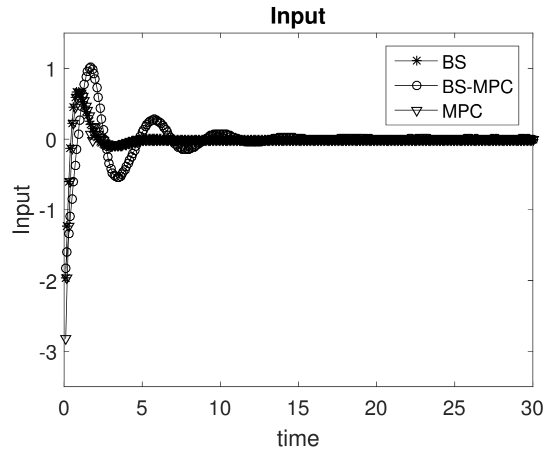

Fig. 4. The trajectories of input under BS, BS-MPC, and MPC

Since the performance of MPC-BS is not better than those of both MPC and BS, it is not useful for this simple stabilization problem. However, in the case of control problems which need the trajectory determined by optimization and the constraints for states of (1), the proposed method can be used instead of MPC or BS. Nonlinear systems in strict-feedback form of (1) and (2) usually have z_1 as the output (y). In this case, (1) is called internal dynamics and the proposed method determines the optimal trajectory z_1 under the user-defined objective function of x and z_1 using MPC and an observer of x. Then, the input (u) is designed by backstepping approach.

5. FUTURE WORK

This is the first work that integrates MPC and BS and proves its stability. In the future, we plan to check the applicability of the proposed controller on practical problems. Since the general output of the nonlinear systems in the strict-feedback form is z_1, the observer will be designed and its stability analysis will be further investigated.

6. CONCLUSION

This work integrates MPC optimizing with piecewise constant functions and BS. The asymptotic stability of the proposed controller is proven. Compared with backstepping controller, the slower stabilization speed caused by removing the time derivative term of the virtual control of the first step was observed, when MPC-BS controller was used. As a result, performance was not better than MPC and BS in two respects: the settling time and total amount of the input. However, the elapsed time of MPC-BS is less than that of MPC because of its reduced size of optimization problem. Furthermore, the explosion of complexity caused by the repeated differentiations of the virtual control is addressed. Given the problems of determining optimal trajectories and tracking them simultaneously, the proposed method is applicable, whereas the BS approach-only is not appropriate.

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