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Quasi-Infinite Adaptive Horizon Nonlinear Model Predictive Control

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Abstract: In this work we present a new method for calculating terminal conditions for nonlinear model predictive control (NMPC) that is non-conservative and scalable via the quasi-infinite horizon methodology. Then, we introduce adaptive-horizon NMPC, a new method for updating prediction horizon lengths online via nonlinear programming sensitivity calculations. Finally, we show how these methods work together to provide an adaptive horizon NMPC implementation for a quad-tank simulation example.

Keywords: predictive control, process control, robust stability, nonlinear control, nonlinear programming, model-based control, multivariable feedback control

1. INTRODUCTION

Model predictive control (MPC) has seen a great deal of success in the chemical industry, as it can naturally handle multiple-input-multiple-output systems with operating constraints. A survey of industrial applications of MPC is given in Qin and Bagdwell (2003), and a thorough theoretical treatment of MPC is given in Rawlings and Mayne (2009). Nonlinear model predictive control (NMPC) has the added advantage of being able to capture nonlinear effects and thus provides higher accuracy across a wide range of states (Grüne and Pannek, 2011). Fast NMPC implementations for large systems are enabled by noting that an exact solution of the associated nonlinear programming (NLP) problem is not necessary (Pannocchia et al., 2011; Zavala and Biegler, 2009).

Terminal conditions are an important aspect of ensuring the stability of NMPC. However, calculating appropriate terminal constraints and costs for the nonlinear case is not straightforward. In Chen and Allgöwer (1998), a quasi-infinite horizon approach is proposed in which the terminal cost is computed based on a controller for the linearized system, and the terminal region represents a region of attraction for the linear controller applied to the nonlinear system. This method was applied to an experimental quad-tank system in Raff et al. (2006) and further extended in Rajhans et al. (2016). Furthermore, this method was extended to a discrete time analysis in Rajhans et al. (2017), which eliminates the need for a small discretization step upon implementation. The main drawback of these methods is in the necessity of either finding a Lipschitz constant for the nonlinear part of the system or solving a series of nonconvex optimization problems to global optimality, either of which makes application to a large system very cumbersome. We instead propose a method of bounding the nonlinear effects of the system that is more practical, and which leads to a method of calculating terminal conditions that is scalable.

We then consider another major issue in NMPC design, which is the selection of horizon length. In particular, we note a significant trade-off in this choice. The longer the horizon length, the larger the computational burden of the NLP that is solved online. The shorter the horizon, the smaller the region of the state space from which the terminal region is N-reachable. Moreover, we recognize that this trade-off can vary with the state of the system. Thus, it is desirable to have a method for updating horizon lengths online. One method for updating horizon lengths is known as variable horizon MPC (Scokaert and Mayne, 1998). Here, the horizon length is treated as a decision variable in the optimization problem. However, in the nonlinear case, this leads to solving a mixed-integer nonlinear program (MINLP) online, which is currently impractical for large systems with significant nonlinearities. Another idea is that of adaptive horizon NMPC. Here, the prediction horizon is updated online based on current state estimates. We propose a method that utilizes sensitivity updates from sIPOPT (Pirnay et al., 2012) in order to choose a sufficient horizon length in real time.

In this work we combine the technologies of quasi-infinite horizon NMPC and adaptive horizon NMPC in order to provide a flexible NMPC formulation that is asymptotically stable under reasonable assumptions. Finally, we show our methods applied to a quad tank example from Raff et al. (2006).

2. NOTATION AND DEFINITIONS

We consider the system:

$$x_{k+1} = f(x_k, u_k) \tag{1}$$

where $x_k \in \mathcal{X}$ is a vector of states that fully define the system at time k, and $u_k \in \mathbb{U}$ is the vector of control actions implemented at time k. We use $|\cdot|$ as the 2-norm, $\mathbb{R}$ as the set of real numbers, $\mathbb{Z}$ as the set of integers, and the subscript $+$ to indicate their nonnegative counterparts. We make the following basic assumptions and definitions.

Assumption 1. (A)The set $\mathcal{X} \subseteq \mathbb{R}^{n_x}$ is positive invariant for $f(\cdot,\cdot)$. That is, $f(x,u) \in \mathcal{X}$ holds for all $x \in \mathcal{X},\ u \in \mathbb{U}$. (B) The set $\mathcal{X} \subset \mathbb{R}^{n_x}$ is closed and bounded (C) The setpoint $(x_s, u_s) = (0,0)$ satisfies $0 = f(0,0)$. (D) The set $\mathbb{U}$ is closed and bounded, and contains zero in its interior.

Definition 2. (Comparison Functions) A function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ is of class $\mathcal{K}$ if it is continuous, strictly increasing, and $\alpha(0) = 0$. A function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ is of class $\mathcal{K}_\infty$ if it is a $\mathcal{K}$ function and $\lim_{s\to\infty} \alpha(s) = \infty$.

Definition 3. (Stable Equilibrium Point) The point $x = 0$ is called a stable equilibrium point of (1) if, for all $k_0 \in \mathbb{Z}_+$ and $\epsilon_1 > 0$, there exists $\epsilon_2 > 0$ such that $|x_{k_0}| < \epsilon_2 \Rightarrow |x_k| < \epsilon_1$ for all $k \geq k_0$.

Definition 4. (Asymptotic Stability) The system (1) is asymptotically stable on $\mathcal{X}$ if $\lim_{k\to\infty} x_k = 0$ for all $x_0 \in \mathcal{X}$ and $x = 0$ is a stable equilibrium point.

Definition 5. (Control Lyapunov function) A function $V : \mathcal{X} \to \mathbb{R}_+$ that satisfies the following:

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|) \tag{2a}$$
$$V(f(x, u_c(x))) - V(x) \leq -\alpha_3(|x|), \tag{2b}$$

for some $\mathcal{K}_\infty$ functions $\alpha_1, \alpha_2, \alpha_3$ and some control law $u_c : \mathcal{X} \to \mathbb{U}$, is said to be a control Lyapunov function for (1).

Theorem 6. If system (1) admits a control Lyapunov function for some control law u_c, then u_c is asymptotically stabilizing on $\mathcal{X}$.

See Appendix B of Rawlings and Mayne (2009) for proof of the preceding.

3. NONLINEAR MODEL PREDICTIVE CONTROL

First we consider the traditional terminal cost / terminal region NMPC formulation:

$$V_N(x) = \min_{z_i, v_i} \quad \sum_{i\in\mathcal{N}} L(z_i, v_i) + \psi(z_N) \tag{3a}$$
$$\text{s.t.} \;\; z_{i+1} = f(z_i, v_i) \;\; \forall\, i = 0 \dots N-1 \tag{3b}$$
$$z_0 = x_k \tag{3c}$$
$$v_i \in \mathbb{U} \;\; \forall\, i = 0 \dots N-1 \tag{3d}$$
$$z_N \in \mathcal{X}_f \tag{3e}$$

where $z \in \mathbb{R}^{n_x}$ and $v \in \mathbb{R}^{n_u}$ are the predicted states and controls, respectively. The mapping $L : \mathcal{X} \times \mathbb{U} \to \mathbb{R}_+$ is the tracking stage cost penalizing deviations from the setpoint, and $\psi : \mathcal{X}_f \to \mathbb{R}_+$ is the terminal cost. At each time k, the NLP is solved for x_k, and the first control is implemented to the system, that is $u_k := v_0|k$. The following assumption imposes a basic requirement on the nature of the tracking stage cost and other basic assumptions for tracking NMPC formulations.

Assumption 7. (A) There exist $\alpha_U, \alpha_L, \alpha_{U,\phi}, \alpha_{L,\phi} \in \mathcal{K}_\infty$ such that $\alpha_U(|x|) \geq L(x,u) \geq \alpha_L(|x|)\ \forall\ x \in \mathcal{X},\ u \in \mathbb{U}$ and $\alpha_{U,\phi}(|x|) \geq \phi(x) \geq \alpha_{L,\phi}(|x|)\ \forall\ x \in \mathcal{X}_f$. (B) A solution to (3) exists for all $x_k \in \mathcal{X}$. (C) The functions $L(\cdot,\cdot)$, $f(\cdot,\cdot,\cdot)$, and $\psi(\cdot)$ are twice continuously differentiable. (D) There exists $\alpha_\psi \in \mathcal{K}_\infty$ and a control law $u_f(x)$ such that $\psi(f(x, u_f(x))) - \psi(x) \leq -\alpha_\psi(|x|)\ \ \forall\ x \in \mathcal{X}_f$.

Definition 8. Weak controllability (Diehl et al., 2011) is satisfied for a given NMPC formulation if there exists a control trajectory $v_i, i = 0 \dots N-1$ satisfying

$$\sum_{i=0}^{N-1} |v_i| \leq \alpha_{wc}(|x|) \tag{4}$$

for some $\alpha_{wc} \in \mathcal{K}_\infty$.

The upper bound $\alpha_U(|x|) \geq L(x,u)$ holds if weak controllability holds, since $|v_i| \leq \alpha(|x|)$ holds $\forall\ i$. The tracking stage cost usually has the form $L(z,v) = z^T Q z + v^T R v$, where Q, R are positive semidefinite matrices but other norms can also be used to satisfy Assumption 7A. The following result is standard.

Theorem 9. Under Assumptions 1 and 7, $V_N(x)$ satisfies the conditions of a control Lyapunov function (2b), and thus the system (1) under control by NMPC (3) is asymptotically stable for all $x_0 \in \mathcal{X}$.

Assuming a good initialization for the NLP (3), the proof of Theorem 9 is analogous to that of linear MPC (Pannocchia et al., 2011).

4. QUASI-INFINITE HORIZON NMPC

Establishing Assumption 7 (D) is a key difficulty in ensuring the stability of (3). This assumption is satisfied if there exists a stabilizing controller in the terminal region with $\psi(x)$ as a control Lyapunov function. Chen and Allgöwer (1998) propose an infinite-horizon LQR applied to the linearized system as the stabilizing controller in the terminal region. Finding the size of the terminal region is then a question of finding the largest region around the setpoint in which the LQR is stabilizing for the nonlinear system. This is done in previous works by finding a Lipschitz constant for the nonlinear system and analyzing the descent of the Lyapunov function, or by solving a sequence of global optimization problems. The terminal cost is then the cost function of the LQR, $\psi(x) = x^T P x$. The main issue with this method is that finding the terminal region via a Lipschitz constant bound or by solving a sequence of global optimization problems can be cumbersome when applied to a large system. In the next section we propose a more practical method of finding the terminal region size via a bound on the nonlinear effects of the system that more easily applies to large systems. We also do the analysis in discrete time, as in Rajhans et al. (2017).

4.1 Deriving a Terminal Cost and Region

Consider (1) broken down into linear and nonlinear parts with the terminal control law $u_f(x) = -Kx$ applied, so that

$$x_{k+1} = f(x_k, -Kx_k) = A_K x_k + \phi(x_k, -Kx_k) \tag{5}$$

where $A_K = A - BK$, the pair (A, B) is assumed to be stabilizable, and $\phi : \mathcal{X} \times \mathbb{U} \to \mathcal{X}$ is the nonlinear part of the system dynamics. For the terminal control law u_f

we choose infinite horizon LQR applied to the linearized system, so that

$$\psi(x) = x^T P x = \min \sum_{i=0}^{\infty} L(z_i, v_i) \tag{6a}$$

$$s.t.\ z_{i+1} = Az_i + Bv_i \quad \forall\, i = 0 \dots \infty, \quad z_0 = x \tag{6b}$$

In order to show a stability region of the linear controller for the nonlinear system, it is necessary to show a bound on the nonlinear system effects. To that end, we show the existence of a bound of the following form.

Theorem 10. There exists $M, q \in \mathbb{R}_+$ such that

$$|\bar{\phi}(x)| \leq M|x|^q \quad \forall\, x \in \mathcal{X} \tag{7}$$

where $\bar{\phi}(x) = \phi(x, -Kx)$.

Proof. Define $\bar{f}(x) := f(x, -Kx)$. Then the nonlinear part of the system is $\bar{\phi}(x) := \bar{f}(x) - A_K x$. By Taylor's Theorem we have

$$\bar{\phi}_j(x) = \bar{\phi}_j(0) + \nabla\bar{\phi}_j(0)^T x$$
$$+\frac{1}{2}\int_0^1 x^T \nabla^2 \bar{\phi}_j(x\tau) x \, d\tau \quad \forall\, i = 1 \dots n \tag{8}$$

where j is indexed over each state. Note that, at $x = 0$, $\bar{\phi}_j(x) = 0$ and $\nabla\bar{\phi}_j(x)^T = \nabla\bar{f}_j(x)^T - \nabla(A_{K,j}, x_j) = A_{K,j} - A_{K,j} = 0$. Then

$$\bar{\phi}_j(x) = \frac{1}{2}\int_0^1 x^T \nabla^2 \bar{\phi}_j(x\tau) x \, d\tau \tag{9}$$

Given Assumptions 1B and 7C, we can define

$$\lambda_m := \max_{x \in \mathcal{X}, j \in 1 \dots n} |\ \nabla^2 \bar{\phi}_j(x)| \tag{10}$$

and from (8):

$$|\bar{\phi}(x)| \leq \sqrt{n}\frac{\lambda_m}{2}|x|^2 \tag{11}$$

Thus (7) is satisfied with $M = \sqrt{n}\frac{\lambda_m}{2}$ and $q = 2$. □

Note that, in general, (5) represents an implicit discretization of a set of differential and algebraic equations (DAEs) and ϕ cannot be obtained explicitly. Furthermore, actually quantifying (7) by finding a bound on the Hessians of (5) may be tedious. Instead, we find M and q in (7) explicitly via simulations from a sampling of initial conditions in the state space, as shown in Section 7. Here the key advantages of this method are apparent, in that we only need to solve a series of one step simulations using the linear control, and do not need to iterate on regions in which a Lipschitz constant is valid. To find the terminal region, we consider the LQR controller (6) applied to the fully nonlinear system (1). From the optimality conditions for (6) the infinite horizon cost matrix $P \in \mathbb{R}^{n\times n}$ satisfies the discrete-time Riccati equation

$$A^T PA - P - (A^T PB)(B^T PB + R)^{-1}(B^T PA) + Q = 0 \tag{12}$$

This also gives the gain matrix $K = (R+B^T PB)^{-1}B^T PA$ such that $u_f(x) = -Kx$. Defining $W = Q + K^T RK$, we satisfy the Lyapunov equation:

$$A_K^T P A_K - P + W = 0 \tag{13}$$

Since $P = \sum_{k=0}^{\infty}(A_K^T)^k W (A_K)^k$ solves this equation, we can write

$$\|P\| \leq \sum_{k=0}^{\infty} \|(A_K^T)^k W (A_K)^k\| \leq \frac{\lambda_W^{max}}{1-\hat{\sigma}^2} \tag{14}$$

where λ_W^{max} and λ_W^{min} are maximum and minimum eigenvalues of W, respectively, and we assume the maximum singular value of A_K, $\hat{\sigma} \in [0, 1)$. Similarly, we have:

$$\|A_K^T P\| \leq \|A_K\| \sum_{k=0}^{\infty} \|(A_K^T)^k W (A_K)^k\| \leq \frac{\hat{\sigma}\lambda_W^{max}}{1-\hat{\sigma}^2} \tag{15}$$

To show the descent of the Lyapunov function under evolution of (5) in the terminal region we have:

$$\psi(x_{k+1}) - \psi(x_k) \tag{16a}$$
$$= x_{k+1}^T P x_{k+1} - x_k^T P x_k \tag{16b}$$
$$= (A_K x_k + \bar{\phi}(x_k))^T P (A_K x_k + \bar{\phi}(x_k)) - x_k^T P x_k \tag{16c}$$
$$= x_k^T (A_K^T P A_K - P) x_k + 2x_k^T A_K^T P \bar{\phi}(x_k) + \bar{\phi}(x_k)^T P \bar{\phi}(x_k) \tag{16d}$$
$$= -x_k^T W x_k + 2x_k^T A_K^T P \bar{\phi}(x_k) + \bar{\phi}(x_k)^T P \bar{\phi}(x_k) \tag{16e}$$
$$\leq -\lambda_W^{min}|x_k|^2 + 2\hat{\sigma}\frac{\lambda_W^{max}}{1-\hat{\sigma}^2}M|x_k|^{q+1} + \frac{\lambda_W^{max}}{1-\hat{\sigma}^2}M^2|x_k|^{2q} \tag{16f}$$
$$\leq -\epsilon_\psi |x_k|^2 \quad \forall\, x_k \in \mathcal{X}_f \tag{16g}$$

which gives the stability condition

$$-\lambda_W^{min} + 2\hat{\sigma}\Lambda_P M|x_k|^{q-1} + \Lambda_P M^2|x_k|^{2(q-1)} \leq -\epsilon_\psi < 0 \tag{17}$$

where $\Lambda_P = \frac{\lambda_W^{max}}{1-\hat{\sigma}^2}$ and $\epsilon_\psi > 0$ is an arbitrarily small constant. Then by the quadratic formula:

$$|x_k| \leq c_f$$
$$:= \left(\frac{-\hat{\sigma}\Lambda_P M + \sqrt{(\hat{\sigma}\Lambda_P M)^2 + (\lambda_W^{min} - \epsilon_\psi)\Lambda_P M^2}}{\Lambda_P M^2}\right)^{\frac{1}{q-1}} \tag{18}$$

which will be used to define $\mathcal{X}_f = \{x \mid |x| \leq c_f\}$. Note that it must also be verified that control constraints are satisfied in the terminal region, that is $-Kx \in \mathbb{U} \quad \forall\, x \in \mathcal{X}_f$. If this does not hold then $\mathcal{X}_f$ must be decreased in size until control constraints are satisfied.

5. QUASI-INFINITE ADAPTIVE HORIZON NMPC (QIAH-NMPC)

We now consider the problem of choosing a horizon length N. We propose an algorithm for finding a *sufficient* horizon length using sensitivity updates from sIPOPT (Pirnay et al., 2012), shown in Figure 1. The first step is to determine N_{min}, the minimum horizon length that will be discussed in the next section, N_s, a safety factor chosen through simulation, and N_{max}, a sufficiently long horizon length that guarantees feasibility of (3) and serves as an initialization for N. Then, at each time point k, solve the NMPC problem (3) which we call $P(x_k)$. Next, solve $P_s(x_{k+1})$, the sensitivity prediction using the successor state x_{k+1} as the initial condition. If $P_s(x_{k+1})$ gives a feasible solution, that is, the terminal region is reached in N time steps, then determine S_T, the time step at which the state reaches the terminal region. Then, set $N_{k+1} = S_T + N_s$ and $k = k + 1$, and proceed to the next NMPC problem. If $P_s(x_{k+1})$ does not give a feasible solution, then set $N_{k+1} = N_{max}$ and $k = k+1$, and proceed to the next NMPC problem. In this fashion, the horizon length is chosen based on a sensitivity prediction plus a safety factor, and N_{max} is chosen as a default in case this calculation fails.

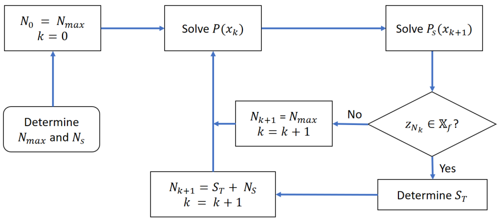

Fig. 1. Algorithm to Determine N_k

6. PROPERTIES OF QIAH-NMPC

First we establish relationships between the linear and nonlinear systems. Consider the cost of the LQR control evaluated for (3):

$$V_N^K(x) = \sum_{i=0}^{N-1} L(z_i, -Kz_i) + \psi(z_N) \tag{19}$$

where $z_{i+1} = Az_i + Bv_i + \phi(z_i, v_i) \;\; \forall\, i = 0 \dots N-1$.

Lemma 11. There exists $\alpha_{NL} \in \mathcal{K}_\infty$ such that $|V_N^K(x) - \psi(x)| \le \alpha_{NL}(|x|) \;\; \forall x \in \mathcal{X}_f, \; N \in \mathcal{N}$

Lemma 11 holds since the nonlinear system dynamics are uniformly continuous. Now consider the following parameterized problem:

$$V_N^p(x, p) = \min \sum_{i=0}^{N-1} L(z_i, v_i) + \psi(x_N) + \rho\, p\, \epsilon_i \tag{20a}$$

$$s.t.\; z_{i+1} = Az_i + Bv_i + \gamma_i \;\; \forall\, i = 0 \dots N-1 \tag{20b}$$

$$|\gamma_i - \phi(z_i, v_i)| \le \epsilon_i, \; \epsilon_i \ge 0 \;\; \forall\, i = 0 \dots N-1 \tag{20c}$$

$$|\gamma_i| \le p \;\; \forall\, i = 0 \dots N-1 \tag{20d}$$

$$z_0 = x \tag{20e}$$

with $\rho > 0$. We now use (20) to relate (3) and (6).

Lemma 12. There exists $\alpha_V \in \mathcal{K}_\infty$ such that $|\psi(x) - V_N(x)| \le \alpha_V(|x|) \;\; \forall\, x \in \mathcal{X}_f, \; N \in \mathcal{N}$

Proof. Note that $V_N^p(x, 0) = \psi(x)$ from (6). Also, from weak controllability there exists some $p = \alpha_p(|x|)$ such that $V_N^p(x, \alpha_p(|x|)) = V_N(x)$ from (3), when ρ is chosen sufficiently large. Thus, (20) is parameterized in the evolution of the nonlinearities of the system. Furthermore, (20) satisfies the Mangasarian-Fromovitz Constraint Qualification (MFCQ) since any control v_i is feasible. MFCQ ensures that $V_N^p(x, p)$ is uniformly continuous in p (and x), and therefore Lemma 12 holds.

See Yang et al. (2015) for more information on constraint qualifications and continuity of parameterized nonlinear programming problems in the context of NMPC.

6.1 Asymptotic Stability of QIAH-NMPC

We now consider the stability of (1) under control according to (3) with terminal conditions described in Section 4, and a horizon length that is updated adaptively and assumed to be feasible. For now, we assume no plant model mismatch, i.e., $w_k = 0 \;\; \forall\, k \in \mathbb{I}_+$. Define the bounded set of acceptable horizon lengths $\mathcal{N} = \{N | N_s \le N \le N_{max}, \; N \in \mathbb{I}_+\}$, and the subset of horizon lengths that define feasible problems (3) at time k that we denote as $\mathcal{N}_k \subset \mathcal{N}$. Furthermore, define some process (e.g. Figure 1) that determines horizon lengths $H : \mathbb{R}^n \times \mathcal{N} \times \mathbb{R}^n \to \mathcal{N}$ so that $N_{k+1} = H(x_k, N_k, x_{k+1}) \in \mathcal{N}_{k+1}$.

Assumption 13. If problem (3) at time k with x_k and N_k is feasible, then so is problem (3) solved at time $k+1$ with x_{k+1} and $N_{k+1} = H(x_k, N_k, x_{k+1})$. That is, $H(x_k, N_k, x_{k+1}) \in \mathcal{N}_{k+1} \;\; \forall x_k, x_{k+1} \in \mathcal{X}, \; N_k \in \mathcal{N}_k$.

Assumption 14. There exists a value of the parameter N_s such that the solution of (3) with horizon $N_k \ge N_s$ satisfies

$$\alpha_L(|x_k|) - \alpha_{NL}(|z_{N_k|k}|) \ge \alpha_3(|x_k|) \tag{21a}$$

$$\alpha_L(|x_k|) - \alpha_V(|z_{N_{k+1}+1|k}|) \ge \alpha_3(|x_k|) \tag{21b}$$

for some $\alpha_3 \in \mathcal{K}_\infty$, where $N_{k+1} = H(x_k, N_k, x_{k+1})$, α_L satisfies Assumption 7(A), and α_{NL}, α_V satisfy Lemmas 11 and 12, respectively.

Essentially, this condition means that costs due to nonlinear effects in the terminal region must be small compared to the stage cost of the initial condition, and therefore (3) is a good approximation of the infinite horizon problem. Note that, in the case of a lengthening horizon, the usual assumption that $V_f(x) \ge V_\infty(x) \;\; \forall\; x \in \mathcal{X}_f$ would also suffice. However, we instead employ Assumption 14 so that the horizon may be lengthened or shortened freely.

Assumption 14 may need to be checked through simulation and enforced by selection of N_s. We recognize that a value of N_s that rigorously guarantees Assumption 14 may be difficult or impossible to find. However, in the case of our examples, it is straightforward to find a value that leads to satisfactory simulation results. We now show that AH-NMPC is asymptotically stable.

Theorem 15. Under Assumptions 1, 7, 13, and 14 there exist $\alpha_1, \alpha_1, \alpha_3 \in \mathcal{K}_\infty$ with $w_k = 0 \;\; \forall\, k \in \mathbb{I}_+$ such that:

$$\alpha_1(|x_k|) \le V_{N_k}(x_k) \le \alpha_2(|x_k|) \tag{22a}$$

$$V_{H(x_k, N_k, x_{k+1})}(x_{k+1}) - V_{N_k}(x_k) \le -\alpha_3(|x_k|) \tag{22b}$$

$$\forall\, x_k \in \mathcal{X}, \; N_k \in \mathcal{N}_k$$

Proof. The inequalities (22a) are satisfied by the form of the objective function and weak controllability. The descent inequality (22b) is not as simple in the case of a variable horizon. We consider this in two separate cases.

Increasing or constant horizon, $\mathbf{N_{k+1} \ge N_k}$

In the case of an increasing horizon we define the initialization for (3) solved at time $k+1$ as the following:

$$\hat{v}_{i|k+1} = \begin{cases} v_{i+1|k} \;\forall\; i = 0 \dots N_k - 2 \\ -Kz_i \;\forall\; i = N_k - 1 \dots N_{k+1} - 1 \end{cases} \tag{23}$$

$$\hat{z}_{0|k+1} = z_{1|k} \tag{24a}$$

$$\hat{z}_{i+1|k+1} = f(\hat{z}_{i|k}, \hat{v}_{i|k}) \;\forall\; i = 0 \dots N_{k+1} \tag{24b}$$

with the value function $\hat{V}_{N_{k+1}}(x_{k+1})$. Then the descent inequality of the Lyapunov function is given as follows:

$$V_{N_{k+1}}(x_{k+1}) - V_{N_k}(x_k) \le \hat{V}_{N_{k+1}}(x_{k+1}) - V_{N_k}(x_k) \tag{25a}$$

$$\begin{aligned} &= \sum_{i=0}^{N_{k+1}-1} L(\hat{z}_{i|k+1}, \hat{v}_{i|k+1}) + \psi(\hat{z}_{N_{k+1}|k+1}) \\ &- \sum_{i=0}^{N_k-1} L(z_{i|k}, v_{i|k}) - \psi(z_{N_k|k}) \\ &= -L(x_k, u_k) \\ &+ \sum_{i=0}^{N_k-2} \big(L(\hat{z}_{i|k+1}, \hat{v}_{i|k+1}) - L(z_{i+1|k}, v_{i+1|k}) \big) \end{aligned} \tag{25b}$$

$$+ \sum_{i=N_k-1}^{N_{k+1}-1} L(\hat{z}_{i|k+1}, \hat{v}_{i|k+1}) + \psi(\hat{z}_{N_{k+1}|k+1}) - \psi(z_{N_k|k}) \tag{25c}$$

$$= -L(x_k, u_k) + \sum_{i=N_k-1}^{N_{k+1}-1} L(\hat{z}_{i|k+1}, \hat{v}_{i|k+1})$$

$$+ \psi(\hat{z}_{N_{k+1}|k+1}) - \psi(z_{N_k|k}) \tag{25d}$$

$$= -L(x_k, u_k) + V^K_{N_{k+1}-N_k+1}(z_{N_k|k}) - \psi(z_{N_k|k}) \tag{25e}$$

$$\leq -\alpha_L(|x_k|) + \alpha_{NL}(|z_{N_k|k}|) \leq -\alpha_3(|x_k|) \tag{25f}$$

where (25f) follows from Lemma 11 and (21a).

Decreasing horizon, $\mathbf{N_{k+1} < N_k}$

In the case of a decreasing horizon we define the initialization for (3) solved at time $k+1$ as the following:

$$\hat{v}_{i|k+1} = v_{i+1|k} \; \forall \; i = 0 \ldots N_{k+1} - 1, \tag{26}$$

again with the state initialization given by (24) and the value function denoted as $\hat{V}_{N_{k+1}}(x_{k+1})$. Then the descent inequality of the Lyapunov function is given as follows:

$$V_{N_{k+1}}(x_{k+1}) - V_{N_k}(x_k) \leq \hat{V}_{N_{k+1}}(x_{k+1}) - V_{N_k}(x_k) \tag{27a}$$

$$= \sum_{i=0}^{N_{k+1}-1} L(\hat{z}_{i|k+1}, \hat{v}_{i|k+1}) + \psi(\hat{z}_{N_{k+1}|k+1})$$

$$- \sum_{i=0}^{N_k-1} L(z_{i|k}, v_{i|k}) - \psi(z_{N_k|k}) \tag{27b}$$

$$= -L(x_k, u_k) + \sum_{i=0}^{N_{k+1}-1} \left(L(\hat{z}_{i|k+1}, \hat{v}_{i|k+1}) - L(z_{i+1|k}, v_{i+1|k})\right)$$

$$+ \psi(\hat{z}_{N_{k+1}|k+1}) - \sum_{i=N_{k+1}+1}^{N_k-1} L(z_{i|k}, v_{i|k}) - \psi(z_{N_k|k}) \tag{27c}$$

$$= -L(x_k, u_k) + \psi(\hat{z}_{N_{k+1}|k+1})$$

$$- \sum_{i=N_{k+1}+1}^{N_k-1} L(z_{i|k}, v_{i|k}) - \psi(z_{N_k|k}) \tag{27d}$$

$$= -L(x_k, u_k) + \psi(z_{N_{k+1}+1|k})$$

$$- V_{N_k - N_{k+1}-1}(z_{N_{k+1}+1|k}) \tag{27e}$$

$$\leq -\alpha_L(|x_k|) + \alpha_V(|z_{N_{k+1}+1|k}|) \leq -\alpha_3(|x_k|) \tag{27f}$$

where we use $\hat{z}_{N_{k+1}|k+1} = z_{N_{k+1}+1|k}$ in (27e), and (27f) follows from Lemma 12 and (21b). Thus $V_N(x)$ satisfies (22), and QIAH-NMPC is asymptotically stable.

7. QUAD TANK EXAMPLE

We consider the experimental quad tank system from Raff et al. (2006) described by the following equations, ignoring state constraints for this work:

$$\dot{x}_1 = -\frac{a_1}{A_1}\sqrt{2gx_1} + \frac{a_3}{A_1}\sqrt{2gx_3} + \frac{\gamma_1}{A_1}u_1 \tag{28a}$$

$$\dot{x}_2 = -\frac{a_2}{A_2}\sqrt{2gx_2} + \frac{a_4}{A_2}\sqrt{2gx_4} + \frac{\gamma_2}{A_2}u_2 \tag{28b}$$

$$\dot{x}_3 = -\frac{a_3}{A_3}\sqrt{2gx_3} + \frac{(1-\gamma_2)}{A_3}u_2 \tag{28c}$$

$$\dot{x}_4 = -\frac{a_4}{A_4}\sqrt{2gx_4} + \frac{(1-\gamma_1)}{A_4}u_1 \tag{28d}$$

$$-43.4 \leq u_1 \leq 16.6 \tag{28e}$$

Table 1. Example parameters and results

	h	Q	R	M	q	c_f
Quad tank	10	$1.5I_4$	I_2	0.005	2.1	28.3

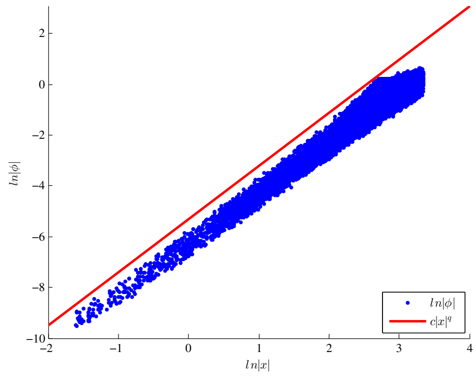

Fig. 2. Quad Tank, nonlinearity bound

$$-35.4 \leq u_2 \leq 24.6 \tag{28f}$$

$$x \geq 0 \tag{28g}$$

$$x_{ss} = [14, 14, 14.2, 21.3]^T \tag{28h}$$

The valve parameters are held constant at $\gamma_1 = \gamma_2 = 0.4$.

7.1 Terminal Region Calculations

The LQR parameters are shown in Table 1. We then use these parameters to define the LQR, which is then used to simulate the system and find the upper bound for ϕ. This is done by simulating one step forward from many initial conditions and subtracting the linear part of the system. The results for 10,000 such simulations are shown in Figure 2, and the bound parameters are shown in Table 1. In this case the terminal region is given by $|z_N| \leq c_f = 28.1$, and we confirm that the control constraints are satisfied for $u = -Kx$ in this region. This region has a volume $\frac{1}{2}\pi^2 c_f^4 = 3.15 \times 10^6$, which is significantly larger than the region of volume 3×10^4 given in Raff et al. (2006) using the method of Chen and Allgöwer (1998). We attribute the improvement in our method to a more accurate approximation of nonlinearity due to (7), as well as not having to iterate on regions in which a given Lipschitz constant is valid.

7.2 Simulation Results

We have implemented both standard and adaptive horizon NMPC for (28) using IPOPT on an Intel i7-4770 3.4 GHz CPU. For standard NMPC, we set $N = 25$, and for AH-NMPC we set $N_{max} = 25, N_s = 5$. Also, sIPOPT is used with the initial condition as the sensitivity parameter p for updated NMPC calculations. However, because our computed terminal region is so large, we may artificially reduce it to $|z_N| \leq 1$ in order to more adequately test the adaptive horizon algorithm. Also, in order to simulate a disturbance for which the sensitivity prediction is infeasible, we set the states to large predefined values at $k = 0, 50, 100$. The norm of the state trajectories over time is shown in Figure 3. The tracking behaviors of standard NMPC and AH-NMPC are nearly identical. The difference between the two methods is in the horizon lengths shown

Table 2. Predefined state values

k	x_1	x_2	x_3	x_4
0	40	40	0	0
50	40	0	40	0
100	40	0	0	40

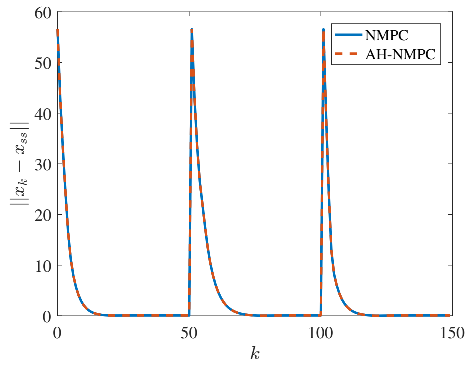

Fig. 3. Quad tank, norm of states

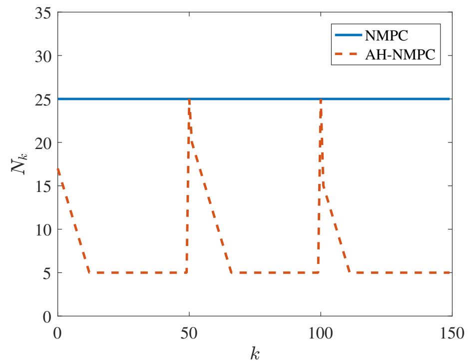

Fig. 4. Quad tank, Horizon Lengths

in Figure 4 and corresponding solve times. The standard NMPC case has a constant horizon length and a higher average solve time. For the adaptive horizon case the horizon length tends to decrease as the system approaches the steady state, leading to faster average solution times. In the case of constant horizon NMPC, the average solve time is $0.0115s$, while the average solve time in the case of AH-NMPC is $0.0062s$, which shows a decrease of 46%. When the sensitivity prediction is infeasible ($z_N \notin \mathcal{X}_f$) due to a large disturbance, the algorithm detects this and defaults to $N_k = N_{max}$, which guarantees feasibility of (3) for all $x_k \in \mathcal{X}$. Thus, the horizon can be updated adaptively under normal conditions, allowing for faster average solve times, but still retains robustness in the case of large disturbances.

8. CONCLUSIONS AND FUTURE WORK

This study presents a new method to calculate terminal costs and regions for NMPC via the quasi-infinite horizon framework that is both non-conservative and scalable. We have also developed a method for updating prediction horizon lengths adaptively online via a framework that retains stability properties of the NMPC under assumptions of the quality of linear control in the terminal region. Together, these methods bridge a large gap in the practicality of applying NMPC to real systems. Finally, we have shown success on a computational example. The next steps of this project will be to extend this methodology to provide a rigorous robust stability property via reformulation of the NLP, and to apply the quasi-infinite and adaptive horizon methods to a large scale distillation system.

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