Preprints, 10th IFAC International Symposium on
Advanced Control of Chemical Processes
Shenyang, Liaoning, China, July 25-27, 2018

Design of Robust Input-Constrained Feedback Controllers for Nonlinear Systems

Diego A. Muñoz * **Wolfgang Marquardt** **

* *Optimización Matemática de Procesos ÓPTIMO, Centro de Ciencia Básica, Universidad Pontificia Bolivariana, Circular 1, 70-01, Medellin, Colombia, (e-mail: diego.munoz@upb.edu.co).*
** *Aachener Verfahrenstechnik, Process Systems Engineering, RWTH Aachen University, Templergraben 56, D-52056 Aachen, Germany, (e-mail: Wolfgang.Marquardt@avt.rwth-aachen.de)*

Abstract: This work contributes to the optimal design of closed-loop nonlinear systems with input saturation in the presence of unknown uncertainty. Stability conditions based on contractive constraints were developed for a general class of nonlinear systems under some Lipschitz assumptions. Closed-loop robust stability and robustly optimal performance can be guaranteed in the presence of input bounds, if the solution of the design problem, formulated as a nonlinear semi-infinite program (SIP) with differential equation constraints, can be guaranteed to be feasible. In this work, the SIP is solved by means of a local reduction approach, which requires a local representation of the so-called lower level problems associated with the SIP. The suggested design method is illustrated by means of chemical reactor control problem.

Keywords: input saturation; MIMO; robustness; bounded disturbances; uncertainty; transient stability; semi-infinite optimization.

1. INTRODUCTION

In practice, most control problems are subject to input, state or output constraints for a variety of reasons. Safe operation of some technical system, for example, often requires to confine states or outputs, i.e., temperatures, pressures, velocities, or voltages within given bounds to make sure that the system is not suffering form damage. Furthermore, any physical input to a system including force, torque, current, mass or energy flow, has a limited range of operation imposed by the actuator devices. In some cases, inputs have to be restricted to guarantee appropriate of operation. As a consequence, controllability of the system is reduced. This restriction can be partially overcome, if actuators were over-designed, for example by choosing a more powerful device, such that input bounds are never reached during normal operation. However, such an ad hoc strategy does not guarantee that the inputs may saturate causing the over-design approach to fail (Bak, 2000).

The design of input-constrained nonlinear control system has been addressed by various authors. Chen and Chang (1985) were the first to develop an algorithm for designing globally stable closed-loop SISO systems with controller saturation for open-loop stable nonlinear plants. Several design criteria were established for the elimination of attracting equilibrium points considering P, PI and PID controllers. This technique is based on the projection of the open-loop trajectory onto the x_1-u plane. An extension of this approach to MIMO systems seems to be impossible, however. For input-output linearizing (IOL) SISO systems, a general result has been proposed by Kapoor and Daoutidis (2000) under the assumption that the origin of the so-called "zero dynamics" is locally exponentially stable. Based on a linear transformation of variables , Kapoor and Daoutidis (2000) defined an appropriate region in the state space, where a control law is locally stabilizing in the absence of constraints, but has the ability to guarantee closed-loop stability even when the controller saturates. Instead of using input-output linearization to design IOL controllers, a specific plant comprising a chain of integrators has been exploited to construct a *Control Lyapunov Function* (CLF) (Artstein, 1983), which allows to derive several stabilizing control laws that can handle input bounds, such as Sontag's universal formula (Lin and Sontag, 1991). Inspired by Lin and Sontag (1991), Jankovic and Kolmanovsky (2000) applied the so-called *domination redesign* to guarantee a stronger robustness property when uncertainties affect directly the inputs, while El-Farra and Christofides (2003) derived explicit analytical formulas to consider additive uncertainty.

In contrast to the design methodologies discussed so far, model-predictive control (MPC) takes a completely different approach to address the constraint control problem (Mayne et al., 2000). Rather than designing a controller which can properly deal with constraints, a control law is implicitly defined by an optimization problem. The capability of MPC comes from model-based prediction of the system behavior as well as incorporating constraints on future input, output or state variables which are embedded in a quadratic (QP) or nonlinear program (NLP) to be solved in real time. MPC is a mature technology that has been successfully applied in practice (Piechottka and Hagenmeyer, 2014), mainly relying on linear models. However, the computational requirements still represents a severe limitation of current MPC theory and technology (Xi et al., 2013).

In this work we aim at the design of a robust controller with guaranteed closed-loop stability, which does not require real-time optimization, but which can successfully deal with unknown uncertainty in the presence of input constraints. This article is structured as follows. In Section 2, the problem formulation is stated. In Section 3, we propose a result inspired by stability analysis in MPC to design constrained control laws which do not require real-time optimization. Subsequently, in Section 4 the developed stability conditions are embedded in an optimization-based design problem where unknown disturbances are considered. The feasibility of the suggested design method and the performance of the resulting controller are illustrated in Section 5. Conclusions are given finally in Section 6.

2. PROBLEM FORMULATION

We consider a class of nonlinear systems represented by the state-space description

$$\begin{aligned}\dot{x}(t) &= \tilde{f}\big(x(t), u^{\mathrm{sat}}(t)\big) + \tilde{g}\big(x(t), u^{\mathrm{sat}}(t), d(\alpha, t)\big),\\ x(t_0) &= x_0,\end{aligned} \qquad (1)$$

where $x(t) \in \mathbb{R}^{n_x}$ denotes the vector of state variables with corresponding initial conditions x_0, and

$$u^{\mathrm{sat}}(t) = \left[u_1^{\mathrm{sat}}(t), \ldots, u_{n_u}^{\mathrm{sat}}(t)\right]^T$$

is the vector of constrained manipulated inputs defined by

$$u_i^{\mathrm{sat}}(t) = \begin{cases} u_i^+ & \text{if } \; u_i(t) \geq u_i^+ \\ u_i(t) & \text{if } \; u_i^- < u_i(t) < u_i^+ \\ u_i^- & \text{if } \; u_i(t) \leq u_i^- \end{cases}, \quad i = 1, \ldots, n_u. \qquad (2)$$

$d(\alpha, t) \in \mathbb{R}^{n_d}$ represents unmeasurable disturbances parameterized by a set of uncertain parameters $\alpha \in \mathcal{A} \subset \mathbb{R}^{n_\alpha}$ and time $t \in \mathbb{R}$. Note that the uncertain parameters α are introduced to represent a family of bounded disturbances which are interpreted as the most plausible disturbance scenarios against which robustness is required. The vector fields $\tilde{f}(\cdot)$ and $\tilde{g}(\cdot)$ map from some open subsets $\Omega_f \subset \mathbb{R}^{n_x} \times \mathbb{R}^{n_u}$ and $\Omega_g \subset \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \times \mathbb{R}^{n_d}$ into $\mathbb{R}^{n_x}$, and are assumed to be continuous in Ω_f, Ω_g, respectively. The following assumption is formulated for the uncertain parameters α.

Assumption 1. The set of uncertain parameters α is

$$\mathcal{A} = \{\alpha \in \mathbb{R}^{n_\alpha} |\ 0 \leq \beta(\alpha)\}\,, \qquad (3)$$

with differentiable function $\beta(\cdot)$ mapping from $\mathbb{R}^{n_\alpha}$ into $\mathbb{R}$.

For box-type uncertainty region used in this work, $\alpha_i \in [-\Delta\alpha_i, \Delta\alpha_i]$, $i = 1, \ldots, n_\alpha$, , the smooth approximation

$$\beta(\alpha) := n_\alpha - \sum_{i=1}^{n_\alpha} \left(\frac{\alpha_i}{\Delta\alpha_i}\right)^{2j}, \quad j \in \mathbb{N}, \qquad (4)$$

is used which is obviously an instance of the function $\beta(\cdot)$ of set (3).

Because $\tilde{g}(\cdot)$ strongly affects the stability analysis, it is important to distinguish between cases where the disturbances vanish when $t \to \infty$ or where they persist $\forall t > t_0$. Thus, an additional assumption is introduced for the vector field $\tilde{g}(\cdot)$:

Assumption 2. The vector field $\tilde{g}\big(x(t), u^{\mathrm{sat}}(t), d(\alpha, t)\big)$: $\mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \times \mathbb{R}^{n_d} \to \mathbb{R}^{n_x}$ satisfies the inequality

$$\begin{aligned}\|\tilde{g}\big(x(t), u^{\mathrm{sat}}(t), d(\alpha, t)\big)\| \leq \gamma_1(&\|x(t)\| + \|u^{\mathrm{sat}}(t)\| \\ &+\|d(\alpha, t)\|) \quad \forall t \geq t_0,\, \forall \alpha \in \mathcal{A},\end{aligned}$$

where γ_1 is a nonnegative constant. Additionally, the disturbances $d(\alpha, t)$ satisfy the bounding condition

$$\begin{aligned} d_k(\alpha, t)\, :\, \|d_k(\alpha, t)\| \leq \gamma_{2\,k}, \quad &\text{for } \gamma_{2\,k} \in [0, \infty), \\ &t \in [t_k, t_{k+1}],\, \forall k \in \mathbb{Z}_+,\end{aligned}$$

where $d_k(\alpha, t) := d(\alpha, t)$ for $[t_k, t_{k+1}]$, $\forall k \in \mathbb{Z}_+$. Furthermore,

(1) for any $\epsilon > 0$, $\exists \bar{k}(\epsilon) \in \mathbb{Z}_+$ such that $\gamma_{2\,k} \leq \epsilon$, $\forall k \geq \bar{k}(\epsilon)$, and $\bar{k}(\epsilon) \to \infty$ if $\epsilon \to 0$ in case of decaying disturbances, or
(2) there exists a nonnegative constant γ_2 such that $\|d(\alpha, t)\| \leq \gamma_2$, $\forall t > t_0$, in case of persistent time-varying disturbances.

The vector field $\tilde{g}(x(t), u^{\mathrm{sat}}(t), 0) = 0$, for $d(\alpha, t) = 0$, i. e., the nominal system is described by

$$\dot{x}^*(t) = \tilde{f}\big(x^*(t), u^{\mathrm{sat}}(t)\big), \quad x^*(t_0) = x_0^*. \qquad (5)$$

Without loss of generality, the origin is assumed to be the desired operating point in case of decaying disturbances. Note that in case of persisting time-varying disturbances, the origin may not be an equilibrium point of the perturbed system (1).

Our aim is to design a constrained state-feedback control law

$$u^{\mathrm{sat}}(t) = k(x(t), p) := [k_1(x(t), p),\, \ldots,\, k_{n_u}(x(t), p)]^T, \qquad (6)$$

with

$$k_i(x(\cdot), p) : \mathbb{R}^{n_x} \times \mathbb{R}^{n_p} \to [u_i^-, u_i^+],\ i = 1, \ldots, n_u, \qquad (7)$$

such that, for the perturbed nonlinear system (1), $k(x(\cdot), p)$ guarantees closed-loop stability even in the presence of uncertainty according to Eq. (3). Control parameters p are considered as time-invariant parameters to be tuned. In order to achieve this aim, an optimization-based design problem is formulated for which a novel stability condition must be considered to guarantee closed-loop stability in the presence of uncertainty.

3. ROBUST STABILITY CONDITION

Before to formulate the optimization-based design problem, in this section we propose a novel stability result for the closed-loop system Eqs. (1), (2) which is further specified by Assumption 2. This result is based on the concept of contractive constraints, which has been used to guarantee exponential stability of MPC (de Oliveira Kothare and Morari, 2000). These constraints require that the states at the end of the prediction horizon are contracted in norm compared to the beginning of the horizon. Based on this idea, we will introduce a stability condition which guarantees that a neighborhood of the desired operating point is not left despite unknown disturbances and in the presence of input saturation.

In order to illustrate this idea, Figure 1 sketches the transient response of the perturbed system (solid line) for two unknown disturbances parameterized by uncertain parameters, $d(\alpha_1, t)$ (left) and $d(\alpha_2, t)$ (right). Dashed lines represent the transient response of the nominal system, i. e., for $d(\alpha, t) = 0$, $\forall t > t_0$, taking the states reached by the perturbed system at t_k as initial condition. This sketch gives rise to the following observation: *If the trajectory of the nominal system, Eq.* (5)*, starting at t_k with initial conditions provided by the states of the perturbed system* (1) *reached at t_k, goes away from the desired operating point, we cannot expect that the perturbed system approaches the operating point asymptotically.*

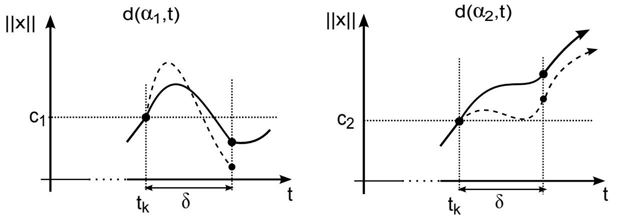

Fig. 1. State transients in the interval $[t_k, t_k + \delta]$, $\delta > 0$, for two unknown disturbances parameterized by uncertain parameters, $d(\alpha_1, t)$ (left) and $d(\alpha_2, t)$ (right). The dashed lines represent the state transient of the nominal system, Eq. (5), starting at the states reached by the perturbed system at t_k. The solid lines represent the trajectories of the perturbed system, Eq. (1), with the same initial conditions.

The rigorous formulation of this idea is presented in the following theorem.

Theorem 3. Suppose that the nonlinear system admits the form of Eq. (1). Let the vector field $\tilde{f}\big(x(\cdot), u^{\text{sat}}(\cdot)\big) : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \to \mathbb{R}^{n_x}$ and the bounded state feedback control law

$$u^{\text{sat}}(t) = k(x(t), p) := [k_1(x(t), p), \ldots, k_{n_u}(x(t), p)]^T, \quad (8)$$

with

$$k_i(x(\cdot), p) : \mathbb{R}^{n_x} \times \mathbb{R}^{n_p} \to [u_i^-, u_i^+], \quad i = 1, \ldots, n_u,$$

be Lipschitz continuous in $x(\cdot)$. p is the vector of time-invariant control parameters. Let the vector field $\tilde{g}(\cdot)$ satisfy Assumption 2, and let $\delta > 0$ be some constant for which the independent variable t is discretized by the sequence $t_{k+1} = t_k + \delta$. Let $x^*(t)$ denote the state transients of the nominal system, Eq. (5), with initial conditions provided by the perturbed system at time t_k, i. e., $x^*(t_k) = x(t_k)$. If, for each interval $[t_k, t_{k+1}]$, there exists a contractive constant $\kappa(\delta) \in [0, 1)$, such that

$$\begin{aligned} \|x^*(t_{k+1})\| \le \kappa(\delta)\|x(t_k)\|, \quad t_{k+1} = t_k + \delta, \\ \forall \alpha \in \mathcal{A}, \forall t_k, \, k \in \mathcal{K} = \{0, 1, 2, \ldots\}, \end{aligned} \quad (9)$$

then the following statements hold:

(1) For decaying disturbances satisfying the asymptotic property described in (1) of Assumption 2, the origin is an asymptotically stable equilibrium point of the perturbed system (1) with bounded state-feedback control, Eq. (8).
(2) For persistent time-varying disturbances $\|d(\alpha, t)\| \le \gamma_2$, $\forall t > t_0$, the perturbed system (1) with bounded state-feedback control, Eq. (8), is bounded around the origin.

Proof: in (Muñoz, 2015).

The constant $\delta > 0$, required to define the sequence $t_{k+1} = t_k + \delta$, should be computed such that $\tilde{\kappa} = \kappa(\delta) + \Delta_1 < 1$, where $\kappa(\delta) \in [0, 1)$, according to the proof of Theorem 3 (Muñoz, 2015). To this end, the procedure to compute δ is summarized in Table 1.

Table 1. Procedure to compute the constant δ required in Theorem 3.

Compute constants:
γ_1, $\gamma_{2\,k}$ satisfying Assumption 2,
Solve $\tilde{\kappa} = \kappa(\delta) + \gamma_1(1 + L_1)\frac{e^{(L+\gamma_1(1+L_1))\delta} - 1}{L + \gamma_1(1 + L_1)} e^{L\delta} < 1$,
for the chosen contractive constant $\kappa(\delta) \in [0, 1)$.

4. OPTIMIZATION-BASED DESIGN OF STABLE FEEDBACK CONTROLLERS

Let us consider the nonlinear systems described in Eq. (1) for which Assumption 2 holds. The constrained feedback control law is given by Eq. (8). In order to design a stable feedback controller for this input-constrained system in the presence of unknown disturbances, the closed-loop trajectories must satisfy the contractive constraints (9) according to Theorem 3 for each interval $[t_k, t_{k+1}]$, $k \in \mathcal{K} = \{0, 1, 2, \ldots\}$.

To design controllers that satisfy the conditions stated in Theorem 3, the following constrained semi-infinite optimization problem (SIP) is formulated:

$$\min_{p} \quad J(x^*(\tau), p) \quad (10a)$$

$$\begin{aligned} \text{s.t.} \quad 0 \le \psi_{\text{cc},k}\big(x(t_k), x^*(t_{k+1})\big) \\ := \kappa(\delta)\big\|x(t_k)\big\| - \big\|x^*(t_{k+1})\big\|, \quad (10b) \\ [t_k, t_{k+1}], \, k \in \mathcal{K}, \end{aligned}$$

$$x(t_k) = \varphi(t_k; x_0, t_0, p, d(\alpha, \cdot)), \quad (10c)$$

$$x^*(t_{k+1}) = \varphi^*(t_{k+1}; x(t_k), t_k, p, 0), \quad (10d)$$

$$\forall \alpha \in \mathcal{A} = \{\alpha \in \mathbb{R}^{n_\theta} \,|\, 0 \le \beta(\alpha)\}, \forall t > t_0, \quad (10e)$$

Eq. (10a) is the merit function, which is not subject to uncertainty, and which can be formulated by an objective function of Mayer-type, i. e.,

$$J(x^*(\tau), p) := \phi(x^*(\tau), p) = \int_{t_0}^{\tau} L(x^*(t), p) dt,$$

where $\phi(\cdot)$ and $L(\cdot)$ are smooth functions mapping from $\mathbb{R}^{n_x} \times \mathbb{R}^{n_p}$ to $\mathbb{R}$. The contractive constraints, Eq. (10b), are referred to as $\psi_{\text{cc},k}(\cdot)$. Eq. (10c) represents the trajectories of the constrained closed-loop nonlinear system, defined by Eqs. (1) and (8). Eq. (10d) represents the nominal system at $t = t_{k+1}$ with $x(t_k)$ as initial conditions. Since the closed-loop trajectories are not available in an analytic form but have to be evaluated by numerical integration, we denote the closed-loop trajectories by the flow

$$x(t) = \varphi(t; x_0, t_0, p, d(\alpha, \cdot)), \quad (11)$$

for given initial conditions $x_0 = x(t_0)$ and control parameters p.

As a consequence of Theorem 3 and the formulation of optimization problem (10), the following result can be stated:

Proposition 4. Suppose that Assumptions 1 and 2 are satisfied for systems represented by Eq. (1) and constrained feedback control law Eq. (8) with given functions

$k_i(x(\cdot), p)$ with controller parameters p. Let $\bar{p}$ be a robustly feasible point of SIP (10), i. e.,

$$\bar{p} \in \mathcal{F}_{CC} := \Big\{ p \in \mathbb{R}^{n_p} \,|\, 0 \le \psi_{\mathrm{cc,k}}\big(x(t_k), x^*(t_{k+1})\big), \\ [t_k, t_{k+1}],\, \forall k \in \mathcal{K},\, \forall \alpha \in \mathcal{A},\, \forall t \ge t_0 \Big\}. \quad (12)$$

Then, closed-loop stability of system Eq. (1) under constrained state-feedback Eq. (8) in a neighborhood of the equilibrium point is guaranteed.

Note that $\bar{p}$ is not necessarily a (globally) optimal solution of SIP (10), but rather just a feasible point. Closed-loop performance, however, is improved the closer the feasible point $\bar{p}$ is to the globally optimal solution p^*. The proof of Proposition 4 is obviously trivial. The remaining non-trivial problem is rather finding an algorithm, which is constructive and provides such a feasible solution. We want to point out that the solution algorithm applied to problem (10) has to provide a feasible but not a globally optimal solution. The determination of the solution of the robust feasibility problem (12), however, also requires the application of global optimization techniques. In order to solve problem (10), finitely many degrees of freedom p are optimized on a feasible set described by infinitely many constraints (10b)-(10e).

The fundamental idea to solve numerically such SIP (Hettich and Kortanek, 1993; Stein, 2012) is to reduce the infinitely-constrained to a finitely-constrained problem, such that standard nonlinear programming can be applied. Solution strategies include discretization and local reduction approaches (Reemtsen and Görner, 1998). For the first category, a finite approximation of the original infinite problem is available when the uncertain set is replaced by its discretization or by a sequence of successively refined grids (Reemtsen and Görner, 1998), while for local reduction methods the infinitely many constraints are reduced considering a local description for the feasible set (12) (Hettich and Kortanek, 1993). In the sequel, we pragmatically focus our attention on the local reduction approach (Stein, 2012), which only provides a locally optimal and feasible solution. This choice of algorithm is justified, because there are no mature algorithms for nonlinear SIP, where solutions come with the guarantees requested in Proposition 4.

Using the mathematical developments introduced by Muñoz and Marquardt (2013), SIP (10) is locally equivalent to the reduced problem

$$\begin{aligned} &\min_{p} \quad J(x^*(\tau), p) \\ &\text{s.t.} \\ &0 \le \psi_{\mathrm{cc}}\Big(x^{(i,k)}(p), x^{*(i,k)}(p)\Big), \\ &x^{(i,k)}(p) = \varphi\Big(t_k; x_0, t_0, p, d\big(\pi^{\alpha\,(i,k)}(p), \cdot\big)\Big), \\ &x^{*(i,k)}(p) = \varphi^*(\pi^{t\,(i,k)}(p); x^{(i,k)}(p), t_k, p, 0), \\ &k \in \mathcal{K}_0, \quad i \in \mathcal{I}_k, \end{aligned} \quad (13)$$

where the locally defined C^1-functions

$$\begin{aligned} \Pi^{(i,k)}(p) : \Omega_\Pi \subset \mathbb{R}^{n_p} &\to \mathbb{R}^{n_\alpha+1} \\ p &\mapsto \begin{bmatrix} \pi^{\alpha\,(i,k)}(p) \\ \pi^{t\,(i,k)}(p) \end{bmatrix}, \\ \Lambda^{(i,k)}(p) : \Omega_\Pi \subset \mathbb{R}^{n_p} &\to \mathbb{R} \end{aligned} \quad (14)$$

with $\pi^{\alpha\,(i,k)}(\bar{p}) = \bar{\alpha}^{(i,k)}$, $\pi^{t\,(i,k)}(\bar{p}) = \bar{t}^{(i,k)}$, $\Lambda^{(i,k)}(\bar{p}) = \bar{\lambda}^{(i,k)}$, and Ω_Π a neighborhood of $\bar{p}$, such that $\Pi^{(i,k)}(\bar{p}) = \left[\bar{\alpha}^{(i,k)}, \bar{t}^{(i,k)}\right]^T$ is the locally unique minimizer, with multipliers $\Lambda^{(i,k)}(\bar{p}) = \bar{\lambda}^{(i,k)}$.

The remaining problem is providing suitable functions $\Pi^{(i,k)}(\cdot)$ and sets $\mathcal{K}_0$, $\mathcal{I}_k$ to formulate the reduced problem (13). To this end, we derive this information using the concept of *manifolds of critical points* (Mönnigmann and Marquardt, 2002; Gerhard, 2010; Muñoz et al., 2012). The solution strategy described by Muñoz and Marquardt (2013) does not have to be modified. Since the numerical integration can only be carried out for a finite span of time, the detection is limited to a finite time horizon, which must be chosen to establish a compromise between computational cost and the risk of missing a constraint violations.

5. ILLUSTRATIVE CASE STUDY

Let us consider a CSTR where two exothermic consecutive reactions A $\rightarrow$ B and B $\rightarrow$ C take place, with B being the desired product. The CSTR model consists of nonlinear state equations which represent the material balances of species A and B and the energy balances for the reactor and cooling jacket assuming perfect level control (Panjapornpon et al., 2006). The model equations are

$$\begin{aligned} \dot{c}_{\mathrm{A}}(t) &= \frac{q(t)}{V}\big(c_{\mathrm{Aq}} - c_{\mathrm{A}}(t)\big) - r_1(t), \\ \dot{c}_{\mathrm{B}}(t) &= -\frac{q(t)}{V} c_{\mathrm{B}}(t) + r_1(t) - r_2(t), \\ \dot{T}(t) &= \frac{q(t)}{V}\big(T_q - T(t)\big) - \frac{\Delta H_1}{\rho C_p} r_1(t) - \frac{\Delta H_2}{\rho C_p} r_2(t) \\ &\quad + \frac{UA}{V \rho C_p}\big(T_j(t) - T(t)\big), \\ \dot{T}_{\mathrm{j}}(t) &= \frac{q_{\mathrm{j}}(t)}{V_{\mathrm{j}}}\big(T_{q_{\mathrm{j}}} - T_{\mathrm{j}}(t)\big) - \frac{UA}{V_{\mathrm{j}} \rho_{\mathrm{j}} C_{p\mathrm{j}}}\big(T_{\mathrm{j}}(t) - T(t)\big), \end{aligned} \quad (15)$$

with reaction rates $r_1(t)$ and $r_2(t)$ defined by

$$\begin{aligned} r_1(t) &= k_{10} \exp\left(-\frac{E_1}{RT(t)}\right) (c_{\mathrm{A}}(t))^2, \\ r_2(t) &= k_{20} \exp\left(-\frac{E_2}{RT(t)}\right) c_{\mathrm{B}}(t). \end{aligned}$$

$c_{\mathrm{A}}(\cdot)$ $\left[\frac{\mathrm{mol}}{\mathrm{l}}\right]$ and $c_{\mathrm{B}}(\cdot)$ $\left[\frac{\mathrm{mol}}{\mathrm{l}}\right]$ denote the concentrations of reactant A and product B, $T(\cdot)$ [K] and $T_{\mathrm{j}}(\cdot)$ [K] the temperatures in the reactor and the cooling jacket, respectively. $q(\cdot)$ $\left[\frac{\mathrm{l}}{\mathrm{h}}\right]$ and $q_{\mathrm{j}}(\cdot)$ $\left[\frac{\mathrm{l}}{\mathrm{h}}\right]$ are the feed rates of the reactant and coolant, respectively. All the parameter values of the model are taken from Panjapornpon et al. (2006).

The feed concentration c_{Aq} and feed temperature T_q are realized by decaying disturbances modeled by

$$T_q(t) = \begin{cases} T_q^{(0)}, & t \le t_0, \\ T_q^{(0)} + dT_q e^{-\beta t} \sin(\omega_{T_q}(t - t_0)), & t > t_0, \end{cases} \quad (16a)$$

$$c_{\mathrm{Aq}}(t) = \begin{cases} c_{\mathrm{Aq}}^{(0)}, & t \le t_0, \\ c_{\mathrm{Aq}}^{(0)} + dc_{\mathrm{Aq}} e^{-\beta t} \sin(\omega_{c_{\mathrm{Aq}}}(t - t_0)), & t > t_0. \end{cases}, \quad (16b)$$

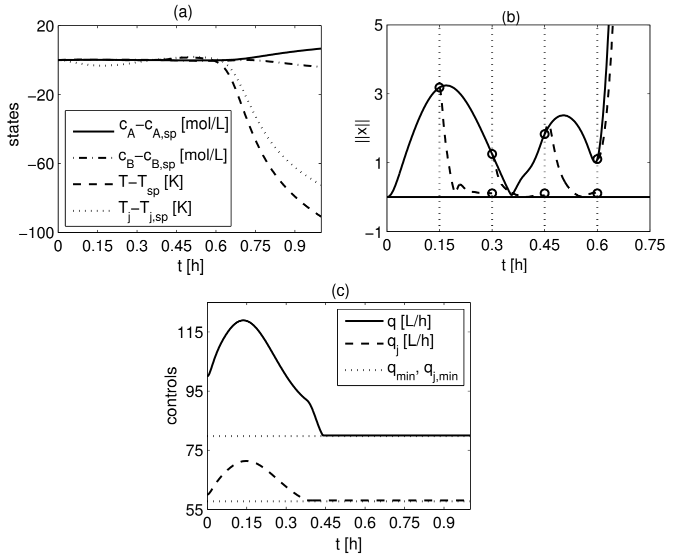

Fig. 2. Transient behavior of the CSTR model, Eqs. (15), in the presence of vanishing disturbances defined by Eqs. (16) but with $dT_q = 10.85\,\mathrm{h}$. (a) Closed-loop trajectories for deviations of the states $c_{\mathrm{A}}(\cdot)$, $c_{\mathrm{B}}(\cdot)$, $T(\cdot)$ and $T_{\mathrm{j}}(\cdot)$ from the desired unstable set-point. (b) Norm of the nominal system evolution (dashed) with initial conditions provided by the perturbed system at t_k represented by ($\circ$). Norm of the perturbed system evolution (solid). (c) Input trajectories for the feed rate $q(\cdot)$ of reactant A (solid) and of the coolant rate $q_{\mathrm{j}}(\cdot)$ (dashed).

with, $\beta = 1.0$, $\omega_{T_q} = 9.58\,\frac{\mathrm{rad}}{\mathrm{h}}$, $\omega_{c_{Aq}} = 3.8$. A bounded multi-variable PI controller is considered with input bounds $80 \leq q(t) \leq 120$, $58 \leq q_{\mathrm{j}}(t) \leq 100$. Here, the vector $p \in \mathbb{R}^{10}$ corresponds to the control parameters. Using the desired steady state, a change of variables is applied to transfer the operating point to the origin.

This example was introduced by Gerhard (2010) to show how input saturation can be avoided by a suitable controller design. Before applying our methodology, we illustrate the closed-loop behavior with control parameters taken from Gerhard (2010). Figure 2 shows the effect of a small change in dT_q from 10.8 K to 10.85 K. The closed-loop system is unstable (Figure 2(a)). As expected, the nominal system, with initial conditions reached by the perturbed system at some t_k, violates the contractive constraint, Eq. (9), at $t_k = 0.6\,\mathrm{h}$, cf. Figure 2(b). Hence, we cannot expect that the perturbed system converges asymptotically to the desired set-point, since the nominal system diverges from the operating point. Figure 2(c) shows the inputs which saturate at their lower bounds at some point in time, making it impossible to drive the system to the set-point.

This example vividly demonstrates that input saturation is vital for closed-loop stability. In general, it is not sufficient to assure stability of the constrained system just by guaranteeing stability neglecting possible input saturation. In order to properly solve this problem, we apply the methodology presented in this work, which relies on the formulation and the solution of the constrained SIP (10).

Table 2. Parameters and constants.

parameter	value	parameter	value
γ_1	2.06	$\gamma_{2\,k}\|_{k=1}$	0.035
L_1	0.015	L	0.01
Δ_1	0.370	Δ_2	0.015
$\tilde{\kappa}$	0.99	δ	0.15

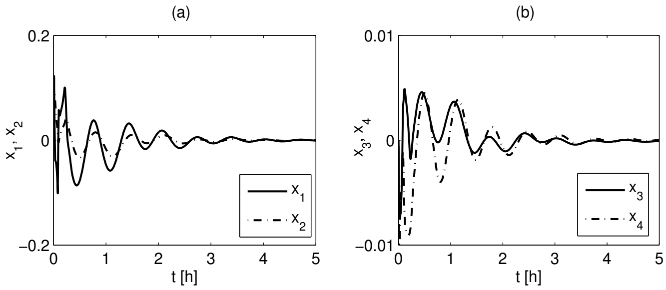

Fig. 3. Time evolution of states $x(\cdot)$ at the critical point summarized in Table 3.

Table 3. Solution of the control design problem (10) for the CSTR, Eqs. (15).

p	value	unit	p	value	unit
$\alpha_1^{(1,4)}$	0.12	$\frac{\mathrm{mol}}{\mathrm{l}}$	$\alpha_2^{(1,4)}$	10.8	K
K_{11}	−315.10	$\frac{\mathrm{m}^6}{\mathrm{kmol\,h}}$	K_{12}	−90.25	$\frac{\mathrm{m}^6}{\mathrm{kmol\,h}}$
K_{13}	−18.33	$\frac{\mathrm{m}^3}{\mathrm{K\,h}}$	K_{14}	−8.81	$\frac{\mathrm{m}^3}{\mathrm{K\,h}}$
K_{21}	−112.98	$\frac{\mathrm{m}^6}{\mathrm{kmol\,h}}$	K_{22}	−57.01	$\frac{\mathrm{m}^6}{\mathrm{kmol\,h}}$
K_{23}	−9.59	$\frac{\mathrm{m}^3}{\mathrm{K\,h}}$	K_{24}	−4.19	$\frac{\mathrm{m}^3}{\mathrm{K\,h}}$
K_{I12}	20.93	$\frac{\mathrm{m}^6}{\mathrm{kmol\,h}^2}$	K_{I23}	−1.13	$\frac{\mathrm{m}^3}{\mathrm{K\,h}^2}$

Using the procedure described in Table 1, γ_1 and $\gamma_{2\,k}$ are estimated and summarized in Table 2. The decaying disturbances and the perturbed term $\tilde{g}(\cdot)$ satisfy Assumption 2. Hence, the stability condition (9) can be formulated.

The goal here is to find control parameters p for which the cost function

$$J(x^*(\tau), p) = \int_0^{\tau} (x_2^*(t))^2 dt \tag{17}$$

is minimized and, at the same time, the stability condition (9) is satisfied, in the presence of uncertainties. At the optimal solution, there is one active normal vector constraint corresponding to the critical manifold of contractive constraints for $k = 4$, i. e., for the interval $[t_4, t_5] = [0.6, 0.75]$, $\mathcal{I}_4 = \{1\}$. Figures 3 and 4 show the time response of the states $x(\cdot)$ and the bounded manipulated variables $u^{\mathrm{sat}}(\cdot)$, respectively, corresponding to the nearest critical point summarized in Table 3. Both manipulated variables rest at bounds for some short period of time without affecting closed-loop stability. The values of the parameters p and the uncertain parameters α at the optimal point are summarized in Table 3.

6. CONCLUSIONS

The results presented in this work contributes to the synthesis of closed-loop nonlinear systems with optimal performance guaranteeing closed-loop robust stability in the presence of input bounds and unknown (parameterized) disturbances. However, we want to point out that Assumption 2 formulated for system (1), (8) could be difficult to satisfy, especially because there is not yet a

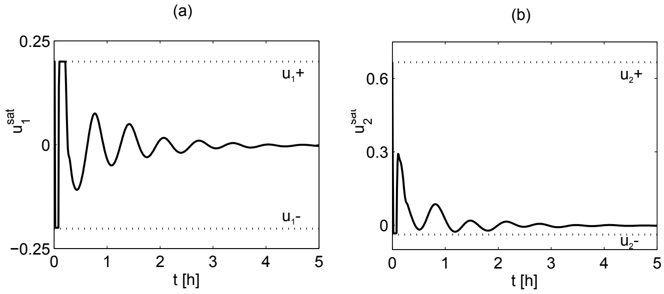

Fig. 4. Time evolution of the bounded multi-variable PI controller with parameters p summarized in Table 3. Both manipulated variables saturate for a period of time without affecting the closed-loop stability. Dotted lines correspond to the scaled upper and lower bounds for manipulated variables $u^{\mathrm{sat}}(\cdot)$.

systematic procedure to compute the Lipschitz constants. The control design problem results in a SIP, which is reformulated using the local reduction framework (Muñoz and Marquardt, 2013). In theory, the suggested design methodology guarantees robust stability and robust optimal performance for the system class considered. In practice, however, the SIP problem proposed for robust control system design is difficult to solve to global optimality due to its non-convexity and semi-infinite nature. The current implementation of our algorithm cannot guarantee that all global minimizers can be detected and that the solution of the SIP (10) computed by the suggested local reduction approach is always robustly feasible. Therefore, a different and more advanced solution strategy has to be developed which allows to fully leverage the potential of the novel design method. The development of fully satisfactory methods presents an enormous challenge due to the nature of the problem and requested guarantee of a robustly feasible, but not necessarily globally optimal solutions of SIP with embedded nonlinear differential equations and inequality constraints. Appropriate solution methods are not yet available. Significant progress has been made on both, bi-level methods for SIP (e.g. Stein (2012)) as well as optimal control methods (Esposito and Floudas (2000); Singer and Barton (2006)) solved to global optimality. The development of globally optimal solution methods for SIP with embedded differential equations is still in its infancy.

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