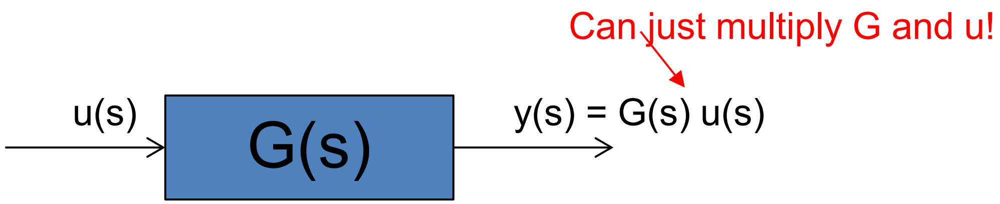
Transfer function



G(s) = transfer function of linear dynamic system
u and y: deviation variables
s: Laplace variable (replaces t as independt variable).
Note (may be confusing): s has units s^{-1} = $second^{-1}$

Some typical transfer functions:

1. First-order with delay process, $G(s)=k\, e^{-\theta s}/(\tau s+ 1)$
   – Example: Heated tank with delay in heater, y=T, u=Q,
     $V\, dT/dt = q_{in}(T_{in}-T(t)) + Q(t-\theta)/(\rho c_P)$ Get $\tau = V/q_{in}$ and $k=1/(\rho c_P V)$
2. Integrating process, $G(s)=k'/s$
   – Example: y = level (V) and u=qin, $dV/dt = q_{in} - q_{out}$ Get k'=1
3. PID-controller , $C(s) = K_c(1 + 1/(\tau_I s) + \tau_D\, s)$ («ideal PID»)

$$= K_c \frac{\tau_I \tau_D\, s^2 + \tau_I\, s + 1}{\tau_I\, s}$$

First-order system

$y(s) = G(s)\, u(s)$

Two standard forms of first-order system:

1. $G(s) = b/(s-a)$ where a = pole («state space» form)

   Follow from general case with A=a, B=b, C=1, D=0.

   Note:

   k'=b=initial slope

   Time response to step M in u(t):

   $$y(t) = \frac{bM}{-a}\left(1 - e^{at}\right)$$

   - Stable for a<0 since exp(at) =0 as t->∞
   - Unstable for a>0 since exp(at) -> ∞ as t->∞

2. $G(s) = k/(\tau s+1)$ (time constant form for stable system)

   $\tau = -1/a$

   $k=g(0) = -b/a$ = steady-state gain

   Time response to step M in u(t): $y(t) = kM\,(1 - \exp(-t/\tau))$

General procedure, matrix state-space form

1. General* Nonlinear dynamic model:

   $dx/dt = f(x,u,d)$, $y = g(x,u,d)$ (x is vector of states, y is vector of "outputs")

2. Steady state model. $dx^*/dt=0 \rightarrow f(x^*,u^*,d^*)=0$
   - Find steady state *. Typically, use to find missing data
3. Introduce deviation variables and linearize
   - $dx/dt = \Delta f = A\,\Delta x(t) + B\,\Delta u(t) + B_d\,\Delta d(t)$. $A = \left(\frac{\partial f}{\partial x}\right)_*$ etc.
   - $\Delta y(t) = \Delta g = C\,\Delta x(t) + D\,\Delta u(t) + D_d\,\Delta d(t)$
4. Laplace** of both sides of linear model* (t -> s)
   - $sx(s) = A\,x(s) + B\,u(s) + B_d\,d(s)$
   - $y(s) = C\,x(s) + D\,u(s) + D_d\,d(s)$
5. Algebra (eliminate x(s), see next page)
   - $y(s) = G(s)\,u(s) + G_d\,d(s)$
   - Transfer matrix, $G(s) = C\,(sI-A)^{-1}B + D$,
   - $G_d(s) = C\,(sI-A)^{-1}B_d + D_d$
6. Block diagram
7. Controller design

*State-space form (differential equations) is not completely general:

1) Cannot handle time delay.

2) Let $g(s)= n(s)/d(s)$. Must assume: order $d(s) \geq$ order $n(s)$ (so cannot handle ideal PID)

**We will only use Laplace for linear systems!

General* Transfer Matrix

General system with n differential equations in n state variables $x(t)$ (where x, u, y are vectors and A, B, C, D are matrices):

$$\begin{aligned}\frac{dx(t)}{dt} &= A\,x(t) + B\,u(t)\\ y(t) &= C\,x(t) + D\,u(t)\end{aligned}$$

Laplace transform with zero intitial condition, $x(0) = 0, u(0) = 0$ (deviation variables):

$$\begin{aligned}sI\ x(s) &= A\,x(s) + B\,u(s)\\ (sI - A)\,x(s) &= B\,u(s)\\ x(s) &= (sI - A)^{-1}\ B\,u(s)\end{aligned}$$

Get $y(s) = G(s)u(s)$ where transfer matrix is:

$$G(s) = C\ (sI - A)^{-1}\ B + D$$

Here

$$(sI - A)^{-1} = \frac{adj(sI - A)}{det(sI - A)}$$

where $det(SI - A) =$

$$d(s) = a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0$$

is a n'th order polynomial in n,

The n roots (generally complex) of the polynomial d(s),

d(s) = det(sI-A)=0

are the same as the eigenvalues of the state matrix A, and are known as the «poles» of the system.

A system with n states (so A is a nxn matrix) has n roots = eigenvalues = poles

*Warning: Not completely general. Does not include time delay, which cannot be written as a polynomial in s, and must assume order d(s) ≥ order n(s)..

Plan for next two weeks

1. **First-order systems (SiS5)**

3. Second-order systems

– Can have oscillations (complex poles)

4. **Closed-loop transfer function (with control)**

5. Poles and zeros

– Including inverse response (RHP-zeros)

6. **Slow oscillations for PI-control of integrating process**

7. Approximating transfer functions

– Time delay

– Half rule

8. Derivation of SIMC PID rules (SiS6)

Start here lecture 2 in week 5 (19/9-24)

Initial and final values for step response

- Transfer function g(s)
  - y(s) = g(s) u(s)
- Deviation variables for y(t) and u(t)
- Consider response y(t) to **step** of magnitude M in input.
  - u(t) = 0 for t>0, u(t) = M for t≥0 ⇒ u(s)=M/s
- From g(s) we get directly final and intital part of time response:

$$\text{Steady-state gain: } \frac{y(\infty)}{M} = g(0)$$

$$\text{Initial gain: } \frac{y(0^+)}{M} = g(\infty)$$

$$\text{Initial slope: } \frac{y'(0^+)}{M} = \lim_{s\to\infty} sg(s)$$

Proof: Note that $y(s) = g(s)\frac{M}{s}$

Final value theorem: $\lim_{t\to\infty} y(t) = \lim_{s\to 0} sy(s) = \lim_{s\to 0} sg(s)\frac{M}{s} = g(0)M$

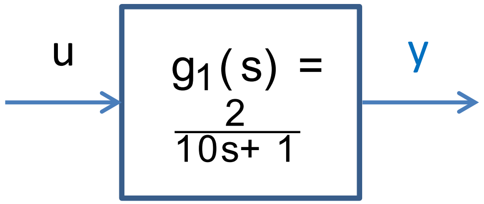
Initial value theorem: $\lim_{t\to 0} y(t) = \lim_{s\to\infty} sy(s) = g(\infty)M$

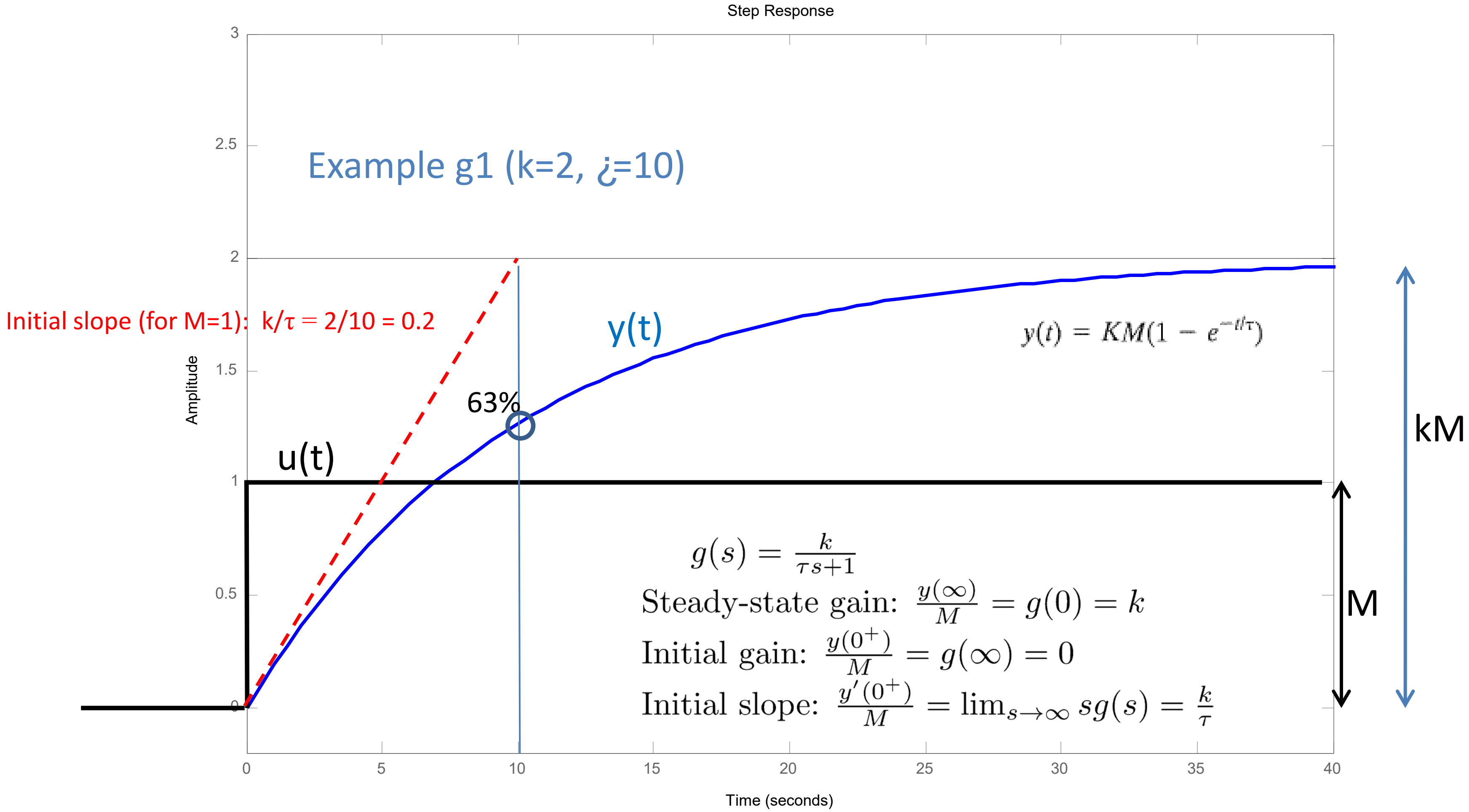
Initial value theorem: $\lim_{t\to 0} y'(t) = \lim_{s\to\infty} s(sy(s)) = \lim_{s\to\infty} sg(s)M$

Initial value theorem: $\lim_{t\to 0} y^{(n)}(t) = \lim_{s\to\infty} s^n(sy(s)) = \lim_{s\to\infty} s^n g(s)M$

Example: First-order system

Time response: $y(t) = kM\,(1 - \exp(-t/\tau))$

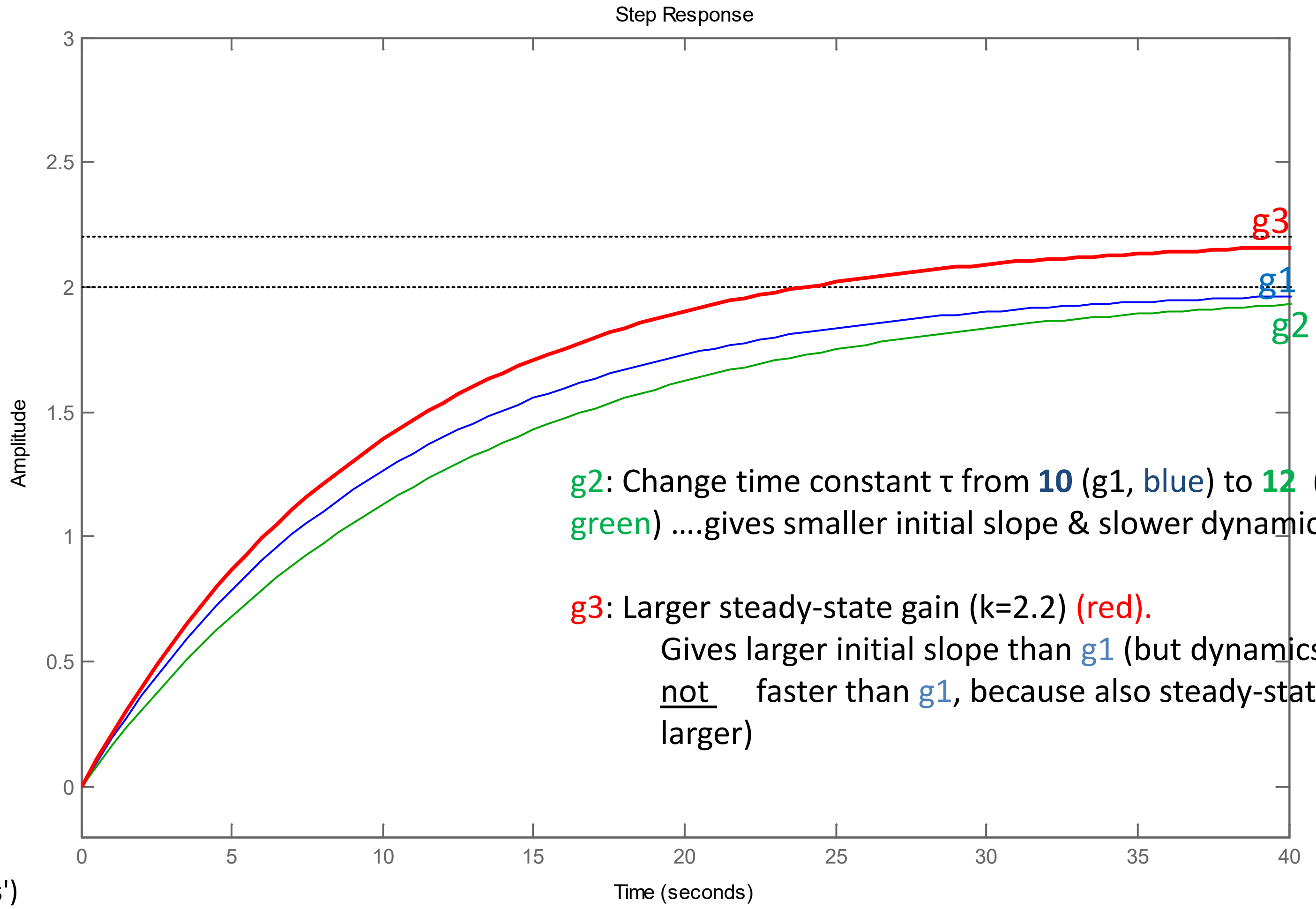





```
s=tf('s')
g1 = 2/(10*s+1)
step(g1,50)
axis([0 40 -0.2 3])
```

First-order system responses

Time response: $y(t) = kM\,(1 - \exp(-t/\tau))$



g2: Change time constant τ from **10** (g1, blue) to **12** (g2, green) ….gives smaller initial slope & slower dynamics

g3: Larger steady-state gain (k=2.2) (red).

Gives larger initial slope than g1 (but dynamics are <u>not</u> faster than g1, because also steady-state is larger)

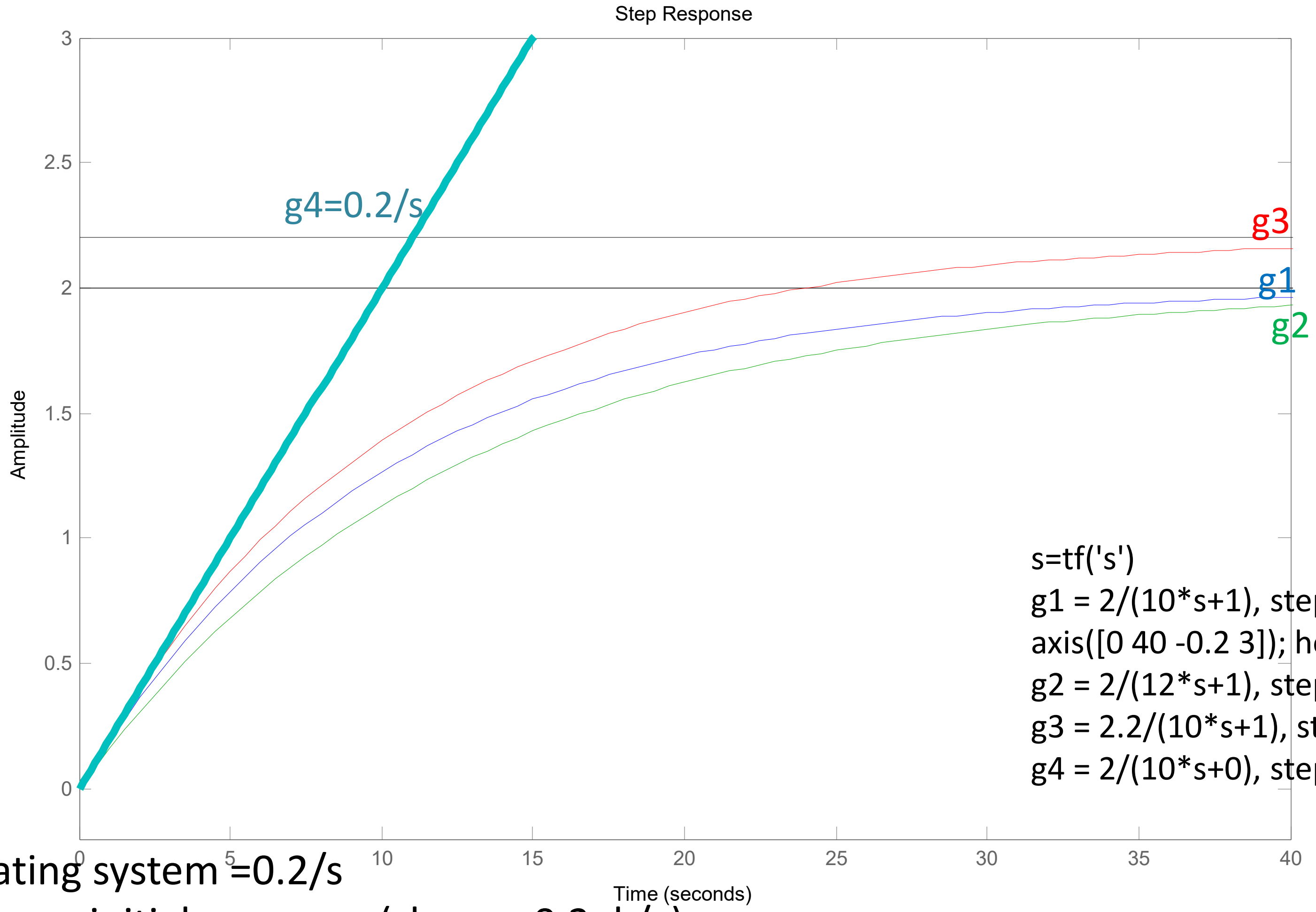
```
s=tf('s')
g1 = 2/(10*s+1), step(g1,50)
axis([0 40 -0.2 3]); hold on,
g2 = 2/(12*s+1), step(g2,50)
g3 = 2.2/(10*s+1), step(g3,50)
```

Integrating system, g(s)=k’/s.

Special case of first-order system with ¿=∞and k=∞ but slope k’=k/¿ is finite

Large ¿: g(s)=k/(¿ s+1) ≈ k/(¿ s) = k’/s

Step response (u=M): y(t)/M = k’t (ramp)



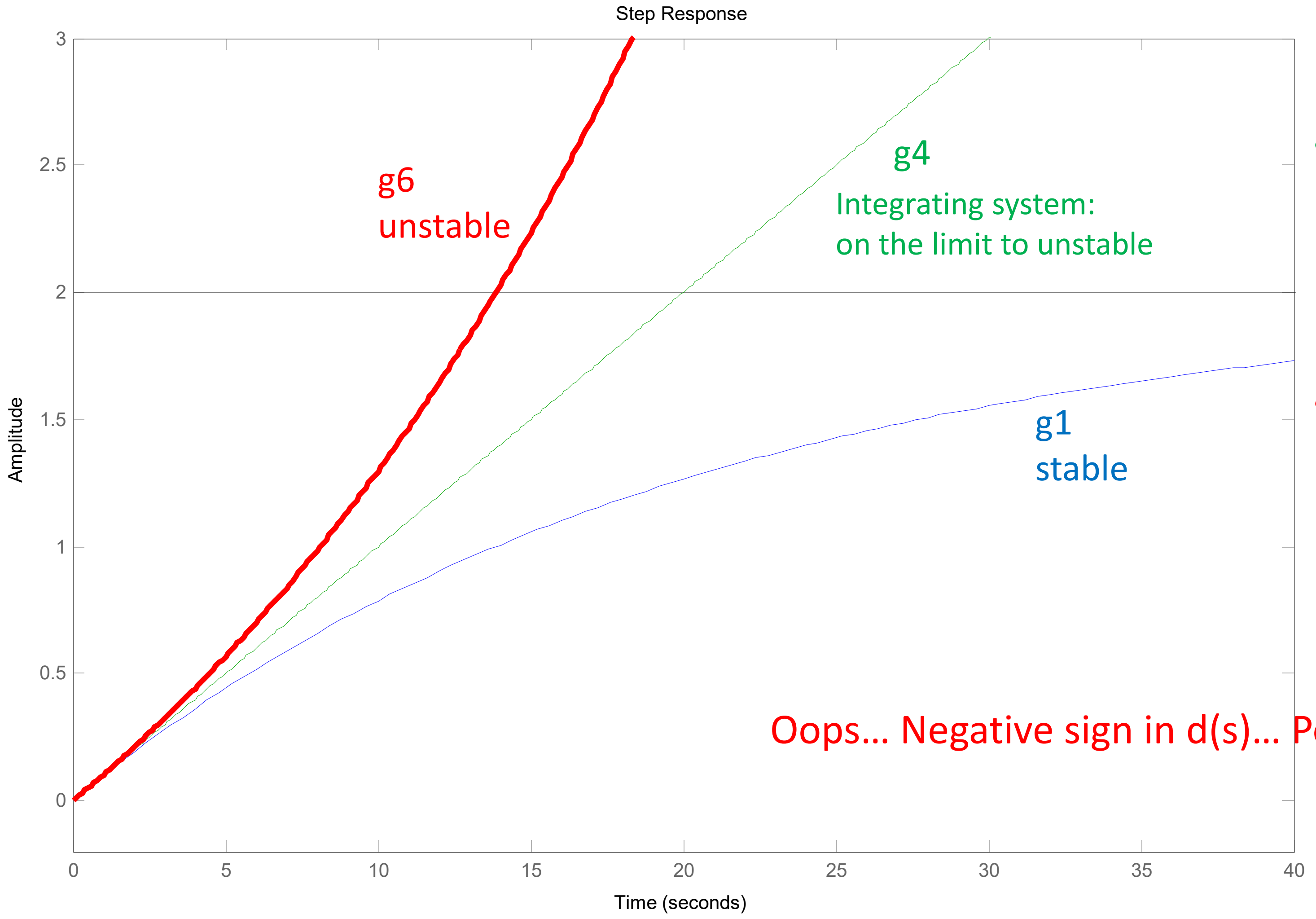

```
s=tf('s')
g1 = 2/(10*s+1), step(g1,50)
axis([0 40 -0.2 3]); hold on,
g2 = 2/(12*s+1), step(g2,50)
g3 = 2.2/(10*s+1), step(g3,50)
g4 = 2/(10*s+0), step(g4,50)
```

Example:

g4: Integrating system =0.2/s

g1 & g4: Same initial response (slope = 0.2=k/¿)

g6: Unstable system (e.g., exothermic reactor):
Note: Sign change in denominator d(s)



$$g1 = \frac{2}{10\,s + 1}$$

$$g4 = \frac{2}{10\,s}$$

$$g6 = \frac{2}{10\,s - 1}$$

Oops... Negative sign in d(s)... Pole p=0.1 Unstable!

2nd order system.
Special case: Two first-order in series

$$G(s)=\frac{K}{(\tau_1 s+1)(\tau_2 s+1)}$$

Example: Temperature in two tanks in series, $\tau_1=V_1/q$, $\tau_2 = V_2/q$

(1) $y(s) = G(s)\, u(s)$ with $u(s)=M/s$ (step).

Partial fraction expansion of (1)

(2) $y(s) = kM\left(\frac{C_0}{s}+\frac{C_1}{s+1/\tau_1}+\frac{C_2}{s+1/\tau_2}\right)$ where $C_0=1$, $C_1=-\tau_1/(\tau_1-\tau_2)$, $C_2=\tau_2/(\tau_1-\tau_2)$

(Find C_0, C_1 and C_2 from (1)=(2)).

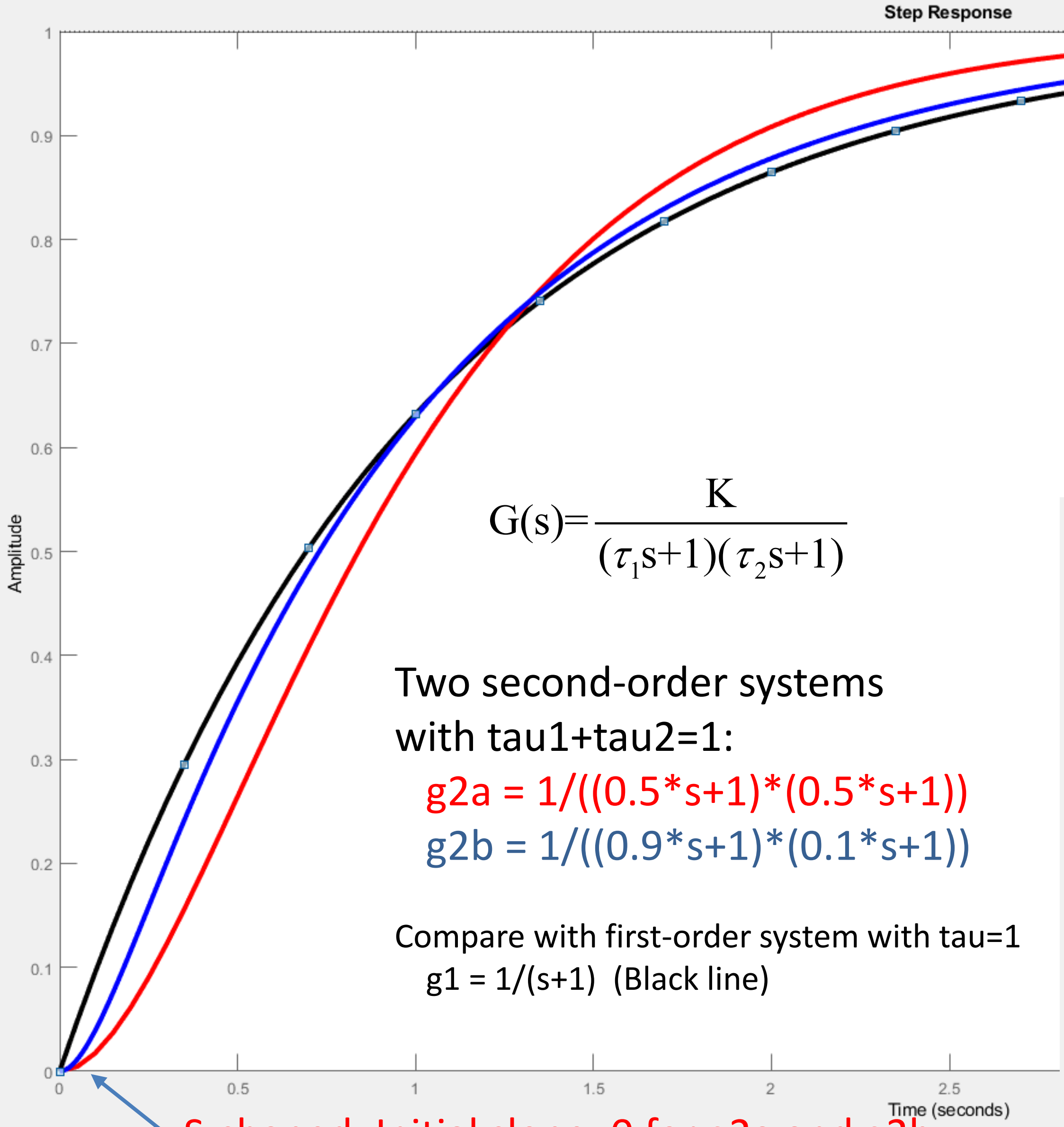
Inverse Laplace of (2)

$y(t) = kM\,(C_0+C_1 e^{-t/\tau 1}+C_2\, e^{-t/\tau 2})$

Conclusion: Step response (M = change in input):

$$y(t) = KM\left(1-\frac{\tau_1 e^{-t/\tau_1}-\tau_2 e^{-t/\tau_2}}{\tau_1-\tau_2}\right) \qquad (5\text{-}47)$$

Step response for two first-order in series: S-shaped response



$$G(s)=\frac{K}{(\tau_1 s+1)(\tau_2 s+1)}$$

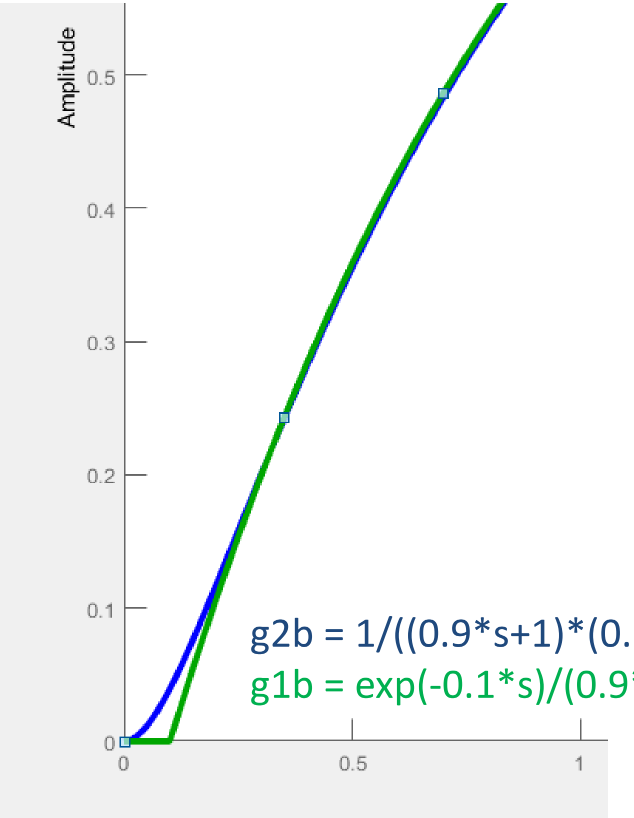
Two second-order systems
with tau1+tau2=1:

g2a = 1/((0.5*s+1)*(0.5*s+1))
g2b = 1/((0.9*s+1)*(0.1*s+1))

Compare with first-order system with tau=1
g1 = 1/(s+1) (Black line)

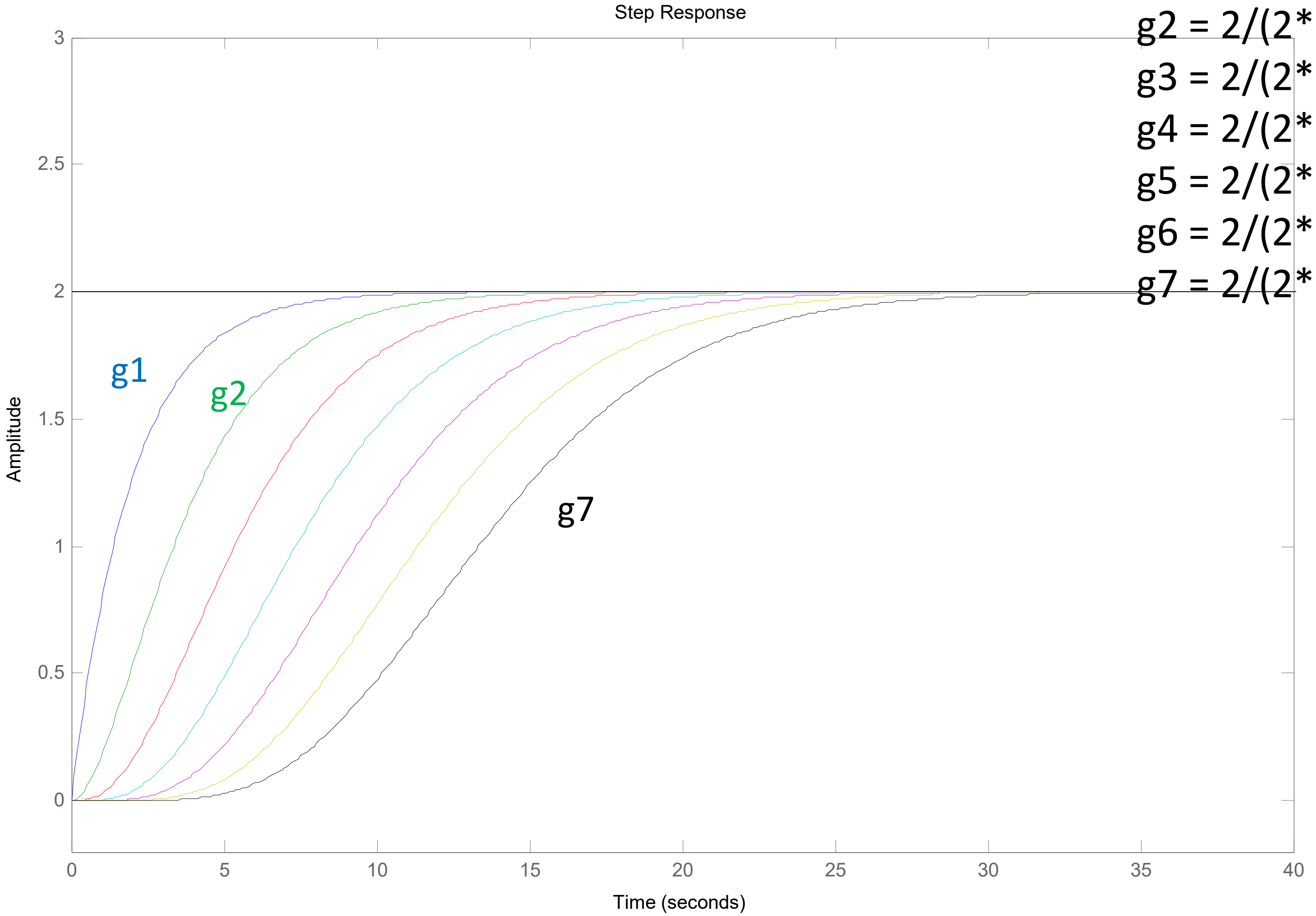
S-shaped: Initial slope=0 for g2a and g2b

Note: Second-order system with tau1 much larger than tau2 can be approximated as first-order plus delay with delay = tau2:



g2b = 1/((0.9*s+1)*(0.1*s+1))
g1b = exp(-0.1*s)/(0.9*s+1)

n identical first-order systems in series



g1 = 2/(2*s+1), step(g1,50)
g2 = 2/(2*s+1)^2, step(g2,50)
g3 = 2/(2*s+1)^3, step(g3,50)
g4 = 2/(2*s+1)^4, step(g4,50)
g5 = 2/(2*s+1)^5, step(g5,50)
g6 = 2/(2*s+1)^6, step(g6,50)
g7 = 2/(2*s+1)^7, step(g7,50)

Note: More poles (relative to zeros) gives flatter initial step response. Proof:

$$\text{Initial value theorem: } \lim_{t\to 0} y^{(n)}(t) = \lim_{s\to\infty} s^n(sy(s)) = \lim_{s\to\infty} s^n g(s) M$$

For system with poles excess= m = n_p-n_z, we get that g(s) $\sim 1/s^m$ when s goes to infinity.

Then the m'th derivative, $y^{(m)}(t)$, is finite (non-zero) for step-response. The other m-1 derivatives of y(t) are zero!

Example G1(s) = 2/(2s+1). m=n_p=1. So first derivative y'(t) (initial slope) is non-zero

Example G7(s) = 2/(2s+1)^7. m=n_p=7. So six first derivatives of y(t) are zero -> Very flat initial response. Almkost like time delay.

Chapter 5

General 2nd order system

$$G(s) = \frac{K}{\tau^2 s^2 + 2\zeta\tau s + 1} = \frac{K}{\tau^2 (s-\lambda_1)(s-\lambda_2)}$$

Roots (poles, eigenvalues): $\lambda_{1,2} = \dfrac{-\zeta \pm \sqrt{\zeta^2 - 1}}{\tau}$

$\zeta > 1$	Overdamped (two real poles)
$\zeta = 1$	Critically damped (two real identical poles)
$\lvert\zeta\rvert < 1$	Underdamped (complex poles; oscillations)
$\zeta < 0$	Unstable

1. Two real poles (overdamped), ^{3} >1: Two first-order in series

$$G(s) = \frac{K}{(\tau_1 s+1)(\tau_2 s+1)} = \frac{K}{\tau_1\tau_2 s^2 + (\tau_1+\tau_2)s + 1}$$

$$\zeta = \frac{\tau_1 + \tau_2}{2\sqrt{\tau_1\tau_2}} \geq 1$$

$$\tau = \sqrt{\tau_1\tau_2}$$

$Two - real - poles:$

$\lambda_1 = -1/\tau_1, \lambda_2 = -1/\tau_2$

$$y(t) = KM\left(1 - \frac{\tau_1 e^{-t/\tau_1} - \tau_2 e^{-t/\tau_2}}{\tau_1 - \tau_2}\right) \quad (5\text{-}47)$$

2. Complex poles, |³|<1

$$G(s) = \frac{K}{\tau^2 s^2 + 2\zeta\tau s + 1} = \frac{K}{\tau^2 (s-\lambda_1)(s-\lambda_2)}$$

$$\boxed{\begin{array}{l}\text{Poles: } \lambda_{1,2} = \frac{-\zeta \pm \sqrt{\zeta^2 - 1}}{\tau} = \sigma \pm i\omega \\ \text{where } \sigma = \frac{-\zeta}{\tau}, i = \sqrt{-1}, \omega = \frac{\sqrt{1-\zeta^2}}{\tau}\end{array}}$$

y(s) = G(s) u(s) with u(s)=M/s (step).

Inverse Laplace

$$y(s) = kM\left(\frac{C_0}{s} + \frac{C_1}{s-\lambda_1} + \frac{C_2}{s-\lambda_2}\right) \rightarrow \quad y(t)/kM = C_0 + C_1 e^{\lambda 1 t} + C_2 e^{\lambda 2 t}$$

Complex poles give oscilllations!

Use Euler's formula for complex parts: $\boxed{e^{i\omega t} = \cos\omega t + i\sin\omega t}$

The complex parts in y(t) cancel!

Finally get for y(t)/kM:

$$21.\ 1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta t/\tau} \sin\left[\sqrt{1-\zeta^2}\, t/\tau + \psi\right]$$

$$\psi = \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}, \quad (0 \le |\zeta| < 1)$$

Note: $\sqrt{1-\zeta^2} = \omega\tau$ where ω is complex part of pole

ω = Complex-part

$$\boxed{\begin{array}{l}\text{Simpler form:} \\ y(t) = kM\left(1 - \frac{1}{\omega\tau} e^{\sigma t} \sin(\omega t + \psi)\right) \\ \psi = \tan^{-1}\left(\frac{\omega}{\sigma}\right)\end{array}}$$

ω/σ = Complex-part/Real-part

Table 3.1 Laplace Transforms for Various Time-Domain Functions[a] (continued)

$f(t)$	$F(s)$
20. $1 + \frac{1}{\tau_2 - \tau_1}(\tau_1 e^{-t/\tau_1} - \tau_2 e^{-t/\tau_2})$ $(\tau_1 \neq \tau_2)$	$\frac{1}{s(\tau_1 s + 1)(\tau_2 s + 1)}$
21. $1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta t/\tau} \sin[\sqrt{1-\zeta^2}\, t/\tau + \psi]$ $\psi = \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}, \quad (0 \leq \lvert\zeta\rvert < 1)$	$\frac{1}{s(\tau^2 s^2 + 2\zeta\tau s + 1)}$
22. $1 - e^{-\zeta t/\tau}[\cos(\sqrt{1-\zeta^2}\, t/\tau) + \frac{\zeta}{\sqrt{1-\zeta^2}} \sin(\sqrt{1-\zeta^2}\, t/\tau)]$ $(0 \leq \lvert\zeta\rvert < 1)$	$\frac{1}{s(\tau^2 s^2 + 2\zeta\tau s + 1)}$
23. $1 + \frac{\tau_3 - \tau_1}{\tau_1 - \tau_2} e^{-t/\tau_1} + \frac{\tau_3 - \tau_2}{\tau_2 - \tau_1} e^{-t/\tau_2}$ $(\tau_1 \neq \tau_2)$	$\frac{\tau_3 s + 1}{s(\tau_1 s + 1)(\tau_2 s + 1)}$
24. $\frac{df}{dt}$	$sF(s) - f(0)$
25. $\frac{d^n f}{dt^n}$	$s^n F(s) - s^{n-1} f(0) - s^{n-2} f^{(1)}(0) - \cdots - s f^{(n-2)}(0) - f^{(n-1)}(0)$
26. $f(t - t_0)S(t - t_0)$	$e^{-t_0 s} F(s)$

[a]Note that $f(t)$ and $F(s)$ are defined for $t \geq 0$ only.

Alternative forms of step response for oscillating 2nd order system (0<zeta<1)

Simpler form:
$y(t) = kM(1 - \frac{1}{\omega\tau} e^{\sigma t} \sin(\omega t + \psi))$
$\psi = \tan^{-1}(\frac{\omega}{\sigma})$

Chapter 5

$$G(s)=\frac{1}{\tau^2 s^2+2\zeta\tau s+1}$$

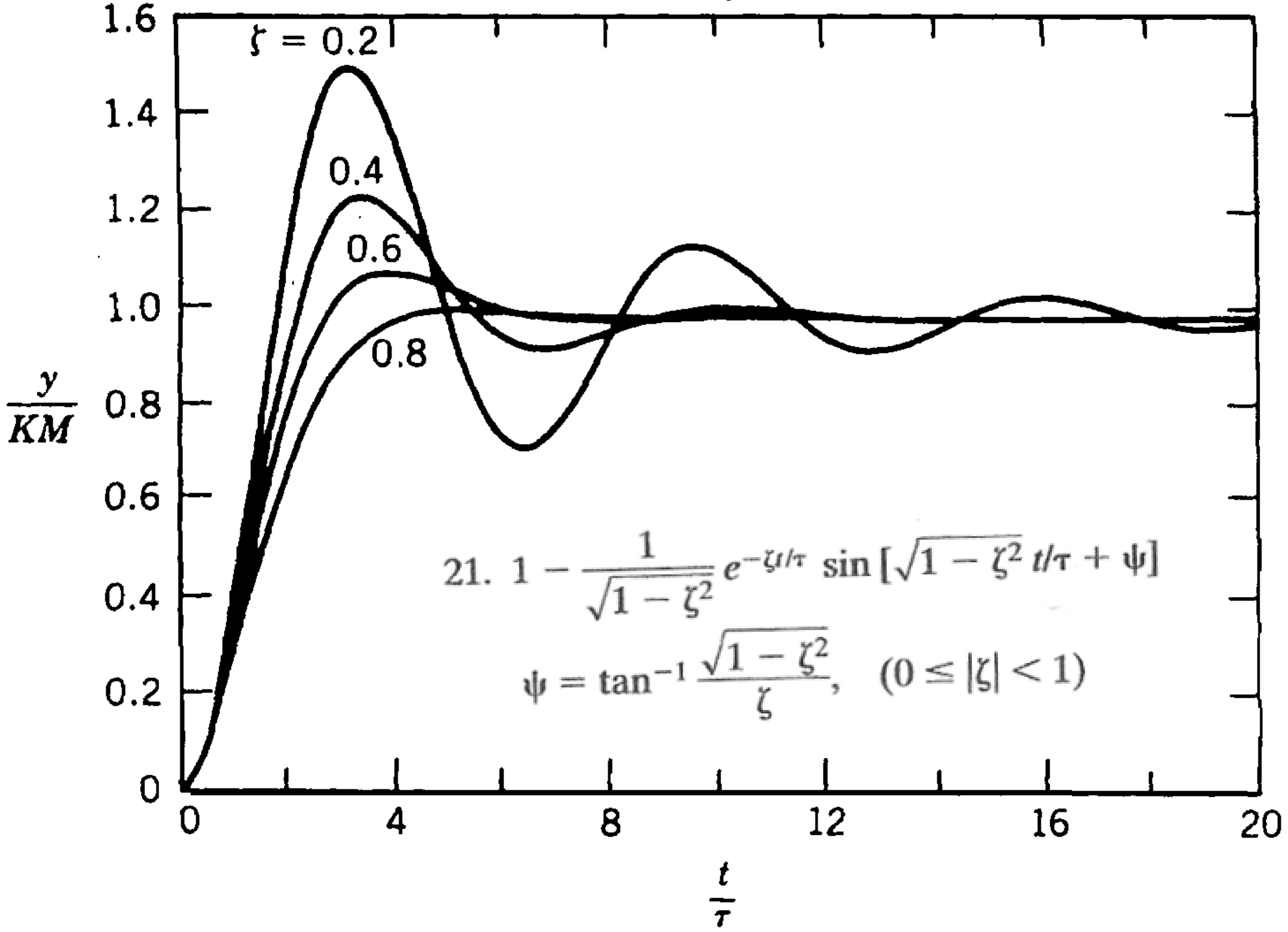
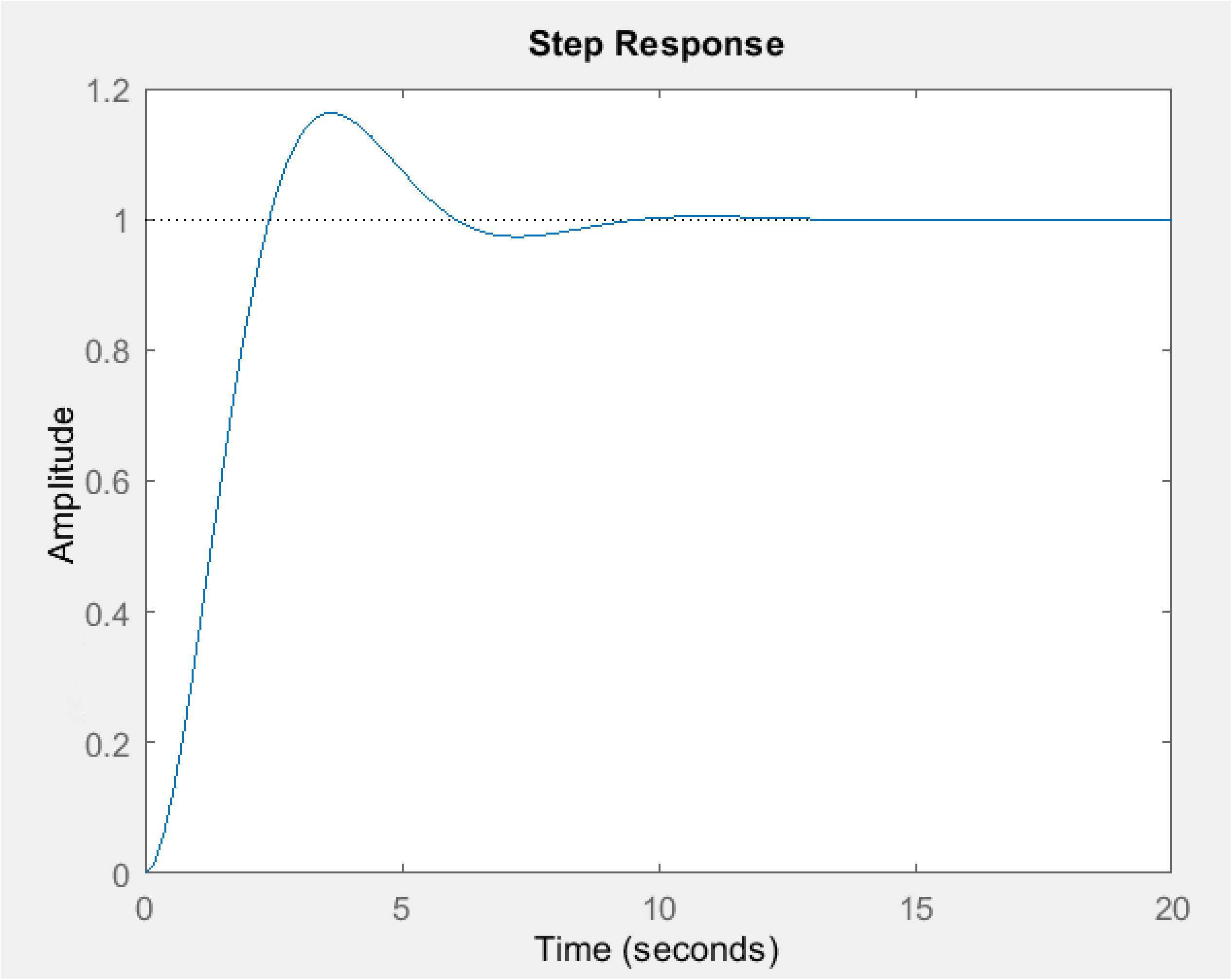


Figure 5.8. Step response of underdamped second-order processes.

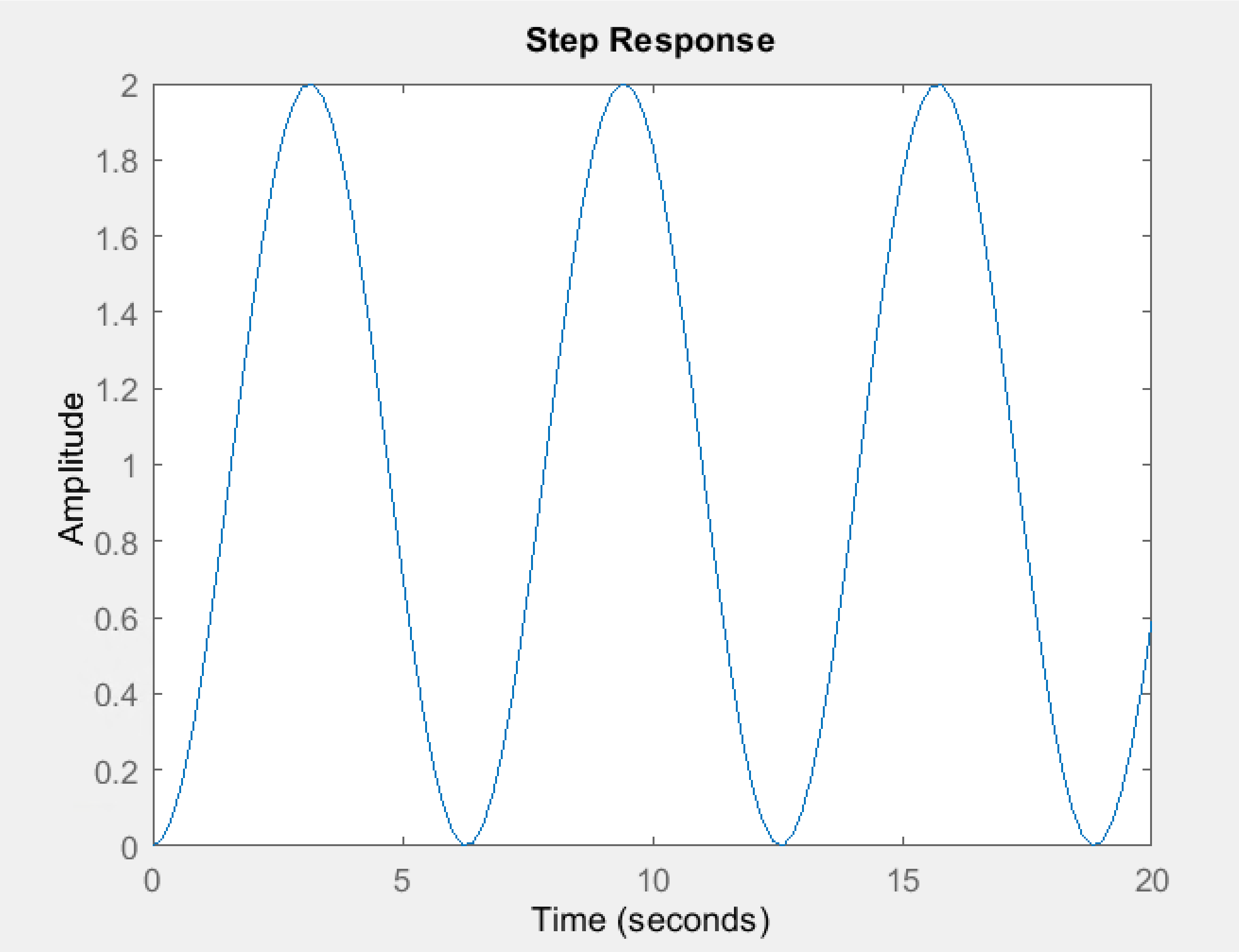
```
s=tf('s')
zeta=0.5, tau=1
g = 1/[(tau*s)^2 + 2*tau*zeta*s + 1]
step(g,20)
```




```
g =
       1
  -----------
  s^2 + s + 1

>> pole(g)

ans =

  -0.5000 + 0.8660i
  -0.5000 - 0.8660i
```

PERSISTENT OSCILLATIONS

(Zero damping. On the limit to instability)




```
s=tf('s')
zeta=0, tau=1
g = 1/[(tau*s)^2 + 2*tau*zeta*s + 1]

g =
         1
      -------
      s^2 + 1

step(g,20)
pole(g)

ans =

      0.0000 + 1.0000i
      0.0000 - 1.0000i
```

Chapter 5

$$G(s) = \frac{K}{\tau^2 s^2 + 2\zeta\tau s + 1}$$

$|\zeta|<1$

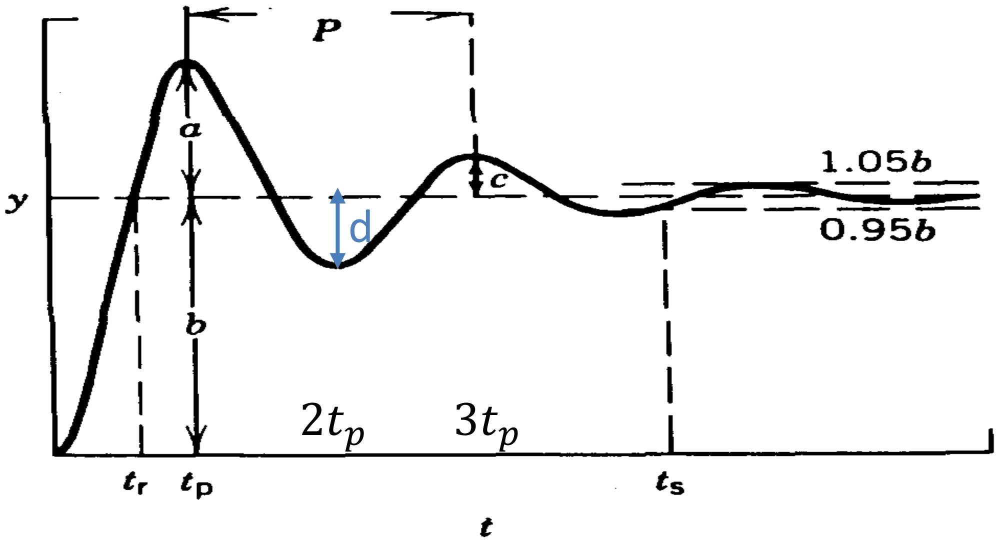


Figure 5.10. Performance characteristics for the step response of an underdamped process.

Poles: $\lambda_{1,2} = \sigma \pm i\omega$

$\sigma = \frac{-\zeta}{\tau}, \quad \omega = \frac{\sqrt{1-\zeta^2}}{\tau}$

Time to first peak: $t_p = \pi\tau/\sqrt{1-\zeta^2}$ (5-50) — $t_p=\pi/\omega$

a/b= Overshoot: $OS = \exp\left(-\pi\zeta/\sqrt{1-\zeta^2}\right)$ (5-51) — $OS=\exp(\pi\sigma/\omega)$

d/b = undershoot

c/a= Decay ratio: $DR = (OS)^2 = \exp\left(-2\pi\zeta/\sqrt{1-\zeta^2}\right)$ (5-52) — $DR = OS^2$

Period: $P = \frac{2\pi\tau}{\sqrt{1-\zeta^2}}$ (5-53) — $P=2t_p=2\pi/\omega$

Example 1

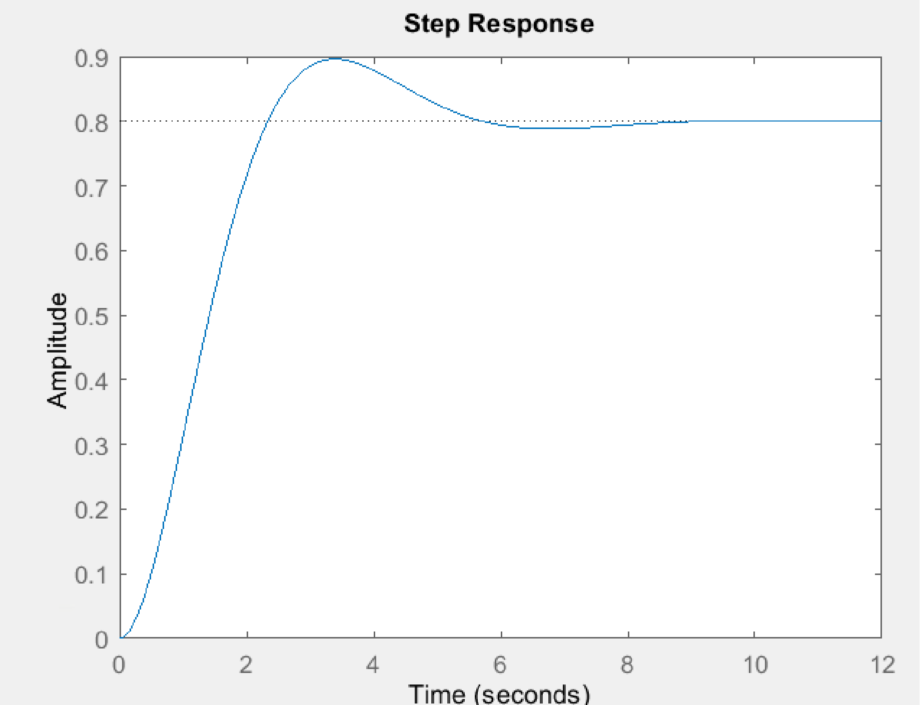
$$T(s) = \frac{4}{(4s+1)(s+1)+4} = \frac{0.8}{0.8s^2+s+1} = \frac{k}{\tau^2 s^2 + 2\tau\zeta s + 1}$$

Get

$$\tau = \sqrt{0.8} = 0.8944s, \quad \zeta = \frac{1}{2 \cdot 0.8944} = 0.559$$

$$\sigma = \frac{-\zeta}{\tau} = -0.625, \ \omega = \frac{\sqrt{1-\zeta^2}}{\tau} = 0.927$$

Time to first peak: $t_p = \pi/\omega = 3.14/0.927 = 3.39$ s
Overshoot: OS $= \exp(\pi\sigma/\omega) = exp(-2.12) = 0.120$
Period: $P = 2\pi/\omega = 2t_p = 6.78$ s
Time to second peak: $3t_p = 10.17$ s



COMPLEX POLES IN PRACTISE

Underdamped (Oscillating) second-order systems (|³|<1)

$$G(s) = \frac{K}{\tau^2 s^2 + 2\zeta\tau s + 1}$$

Corresponds to complex poles, $\lambda_{1,2} = \sigma \pm i\omega$

Process systems:
Oscillations are usually caused by (too) aggressive control
U-tube is an exception (see example next slide)

Example 1: P-control (controller gain K_c) of second-order process, g(s) = k/(¿$_1$ s+1)(¿$_2$ s+1)

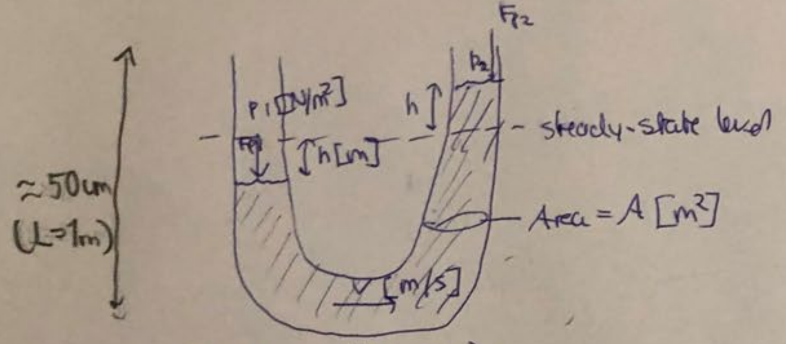
- Oscillates (³<1) if K_ck is <u>large</u>

But there also cases where we need «aggressive» control to avoid oscillations:

Example 3: PI-control of integrating process, g(s)=k'/s
- Need control to stabilize
- Oscillates (³<1) if K_ck' is <u>small (!)</u>

29/9-23

Example naturally oscillation process: U-tube



Newtons 2nd law (F = ma)

$$F_{p1} - F_{p2} - F_g - F_{friction} = m\frac{dv}{dt}$$

Here

$$F_{p1} = p_1 A$$
$$F_{p2} = p_2 A$$
$$F_g = \rho g \cdot 2h \cdot A$$
$$F_f = k_f v \quad \text{(laminar flow)}$$

Also note that

$$v = \frac{dh}{dt}$$

So get

$$\underbrace{(p_1 - p_2)}_{\Delta p(t)} A - \rho g 2A h(t) - k\frac{dh(t)}{dt} = m\frac{d^2h}{dt}$$

Note: The system is linear

Laplace

$$ms^2 h(s) + k_f s h(s) + 2\rho g A h(s) = A\Delta p(s)$$

$$h(s) = \underbrace{\frac{A}{ms^2 + k_f s + 2\rho g A}}_{g(s)} \Delta p(s)$$

Oscillates for

$$B^2 - 4AC < 0 \Rightarrow k_f^2 - 8\rho g m A < 0$$

$$\Rightarrow \boxed{k_f < \sqrt{8\rho g m A}}$$

So oscillates if friction is small.

On standard form

30/9-23

$$\underbrace{\frac{m}{2\rho g A}}_{\tau^2} s^2 + \underbrace{\frac{k_f}{2\rho g A}}_{2\tau\zeta} s + 1$$

$m = \rho V = \rho L A$ (L = total length of fluid)

So $\tau = \sqrt{\frac{m}{2\rho g A}} = \sqrt{\frac{L}{2g}} \Rightarrow \zeta = \frac{k_f/2\rho g A}{\sqrt{\frac{m}{2\rho g A}}} = \frac{k_f}{\sqrt{8\rho g m A}} = \frac{k_f}{\sqrt{8gL}\cdot\rho A}$

(oscillates for $\zeta < 1 \Rightarrow k_f < \sqrt{8\rho g m A}$ as found above)

Friction force:

$F_f = \Delta p_f A$ where $\Delta p_f = 4f\frac{L}{D}\rho\frac{v^2}{2}$, $f = \frac{16}{Re}$ (laminar), $Re = \frac{\rho v D}{\mu}$
(pipe flow) $A = \frac{\pi}{4}D^2$

$$\Rightarrow F_f = \underbrace{\boxed{8\pi\mu L}}_{k_f} v$$

Numerical example

D = 1cm = 0.01m
L = 1m
$\mu = 10^{-3}\ \frac{kg}{m\cdot s}$ (water)
$\rho = 10^3\ kg/m^3$ (-"-)
$g = 10\ m/s^2$

$A = \frac{\pi}{4}D^2 = 0.78\cdot10^{-4}\ m^2$

Get $\boxed{\tau = \sqrt{\frac{L}{2g}}} = \sqrt{\frac{1}{2\cdot10}} = 0.22\ s$ (fast)) $P \approx 2\pi\tau = 1.40s$

$$\boxed{\zeta = \frac{k_f}{\rho A\sqrt{8gL}}} = \frac{8\pi\mu L}{\rho A\sqrt{8gL}} = \boxed{\pi\sqrt{\frac{8L}{g}}\frac{\mu}{\rho A}} = 3.14\sqrt{\frac{8}{10}}\cdot\frac{10^{-3}}{10^3\cdot0.78\cdot10^{-4}}$$

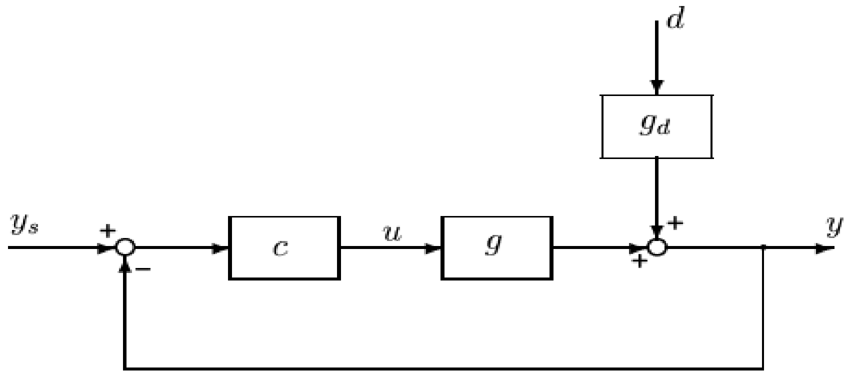
$= 0.036$ (oscillates!!!)

To get rid of the oscillations we need to have a contraction (smaller A) or a viscous fluid (larger μ).

For example, reducing D by a factor 10 (from 1cm to 1mm) reducing A by a factor 100 and we get $\zeta = 3.6$ (no oscillations) >1

2024: Start week 6 (only 2 lectures this week)

Example 1. Setpoint response for P-control of 2nd order process



g(s) = k/[(4s+1)(s+1)]
P-controller: c(s) = K_c

Task: Derive closed-loop transfer function T(s) for setpoint change (can set d=0).

Note: same example as before

$g(s) = k/[(4s+1)(s+1)]$, $g_m=1$

P-controller: $c(s) = K_c$. Data: $K_c k = 1$

Example 1

$$T(s) = \frac{4}{(4s+1)(s+1)+4} = \frac{0.8}{0.8s^2+s+1} = \frac{0.8}{\tau^2 s^2 + 2\tau\zeta s + 1}$$

Get

$$\tau = \sqrt{0.8} = 0.8944s, \quad \zeta = \frac{1}{2 \cdot 0.8944} = 0.559$$

$$\sigma = \frac{-\zeta}{\tau} = -0.625, \quad \omega = \frac{\sqrt{1-\zeta^2}}{\tau} = 0.927$$

Time to first peak: $t_p = \pi/\omega = 3.14/0.927 = 3.39$ s

Overshoot: $\mathrm{OS} = \exp(\pi\sigma/\omega) = exp(-2.12) = 0.120$

Period: $P = 2\pi/\omega = 2t_p = 6.78$ s

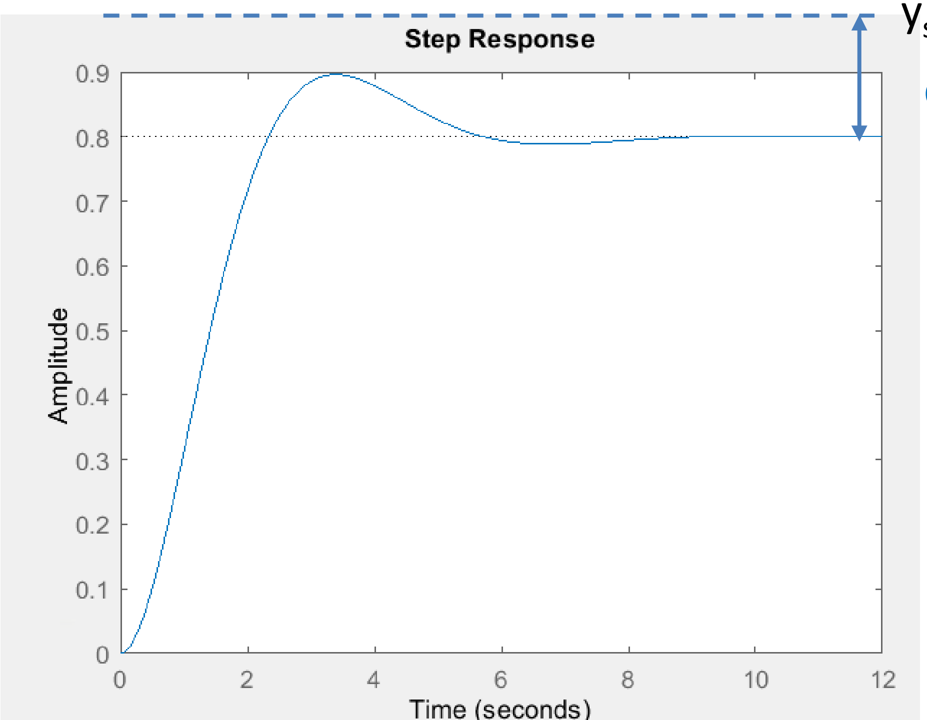
Time to second peak: $3t_p = 10.17$ s

Can also find ω from poles

$0.8\, s^2 + s + 1 = 0$

Solution:

$s = -0.625 \pm 0.927i$



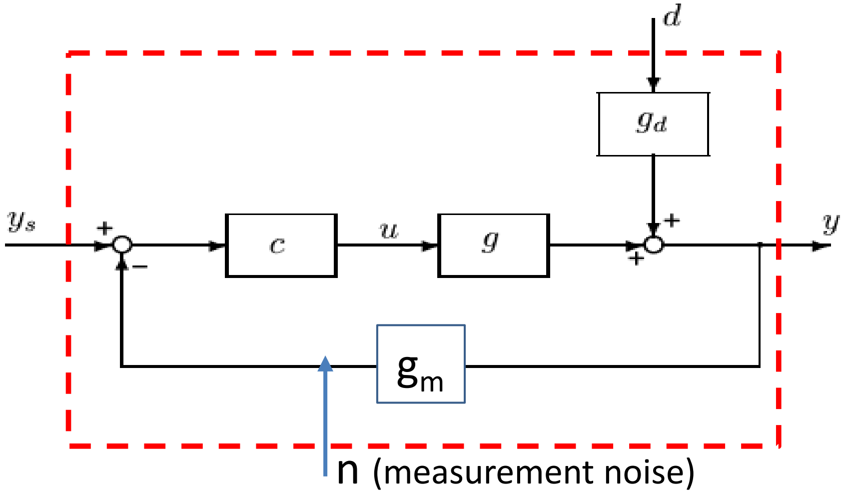
$y_s=1$

Offset = $1 - kK_c/(1+kK_c) = 1/(1+kK_c)$

$= 1/5 = 0.2$

Note: Larger value of K_c gives less offset but more oscillations. Oscillations (complex poles) for $K_c>0.5625$.

Closed-loop transfer function

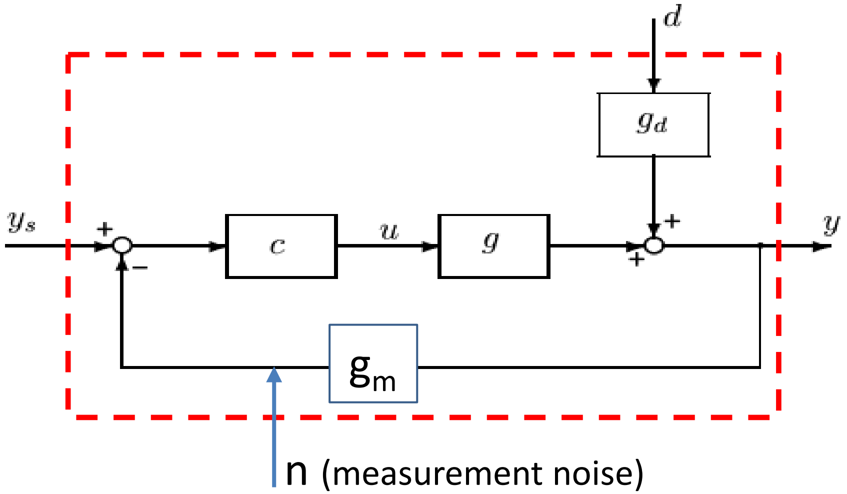


(1) Process: $y = g(s)\ u + g_d(s)\ d$
(2) Controller: $u = c(s)\ (y_s - y_m)$
(3) Measurement: $y_m = g_m(s)\ y + n$

Closed-loop response: Want to find effect of y_s , d an n on output y.
Task: Eliminate u and y_m to find

$$y = T(s)\ y_s + T_d(s)\ d + T_n(s)\ n$$

Closed-loop transfer functions



- Closed-loop output response: $y = T\, y_s + T_d\, d + T_n\, n$
- Introduce «loop»= $L = g\, c\, g_m$

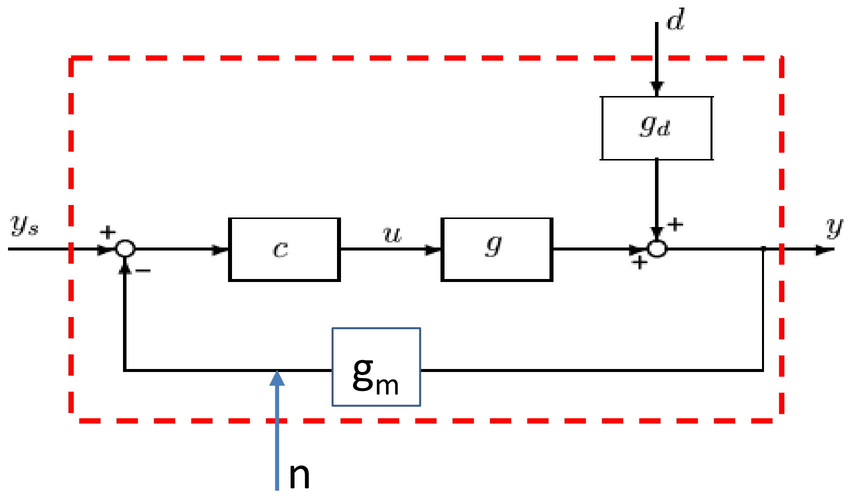
  $T(s) = gc/(1+L)$

  $T_d(s) = g_d/(1+L)$

  $T_n = -T$
- General rule for negative feedback:
  - Transfer function = «direct(s)» / (1 + «loop(s)»)
- Example: What is response from y_s, d and n to u?
  - $u = T_{us}\, y_s + T_{ud}\, d + T_{un}\, n$
    - Note that $T_{ud} = -T$ for input disturbance (g_d=g) and g_m=1
      - which is interesting since with SIMC-rule we choose desired T, so we also «have control» over input change for disturbances

$$T_{ud} = \frac{-g_d g_m c}{1+L}$$

Sensitivity function S



loop = $g\ c\ g_m$

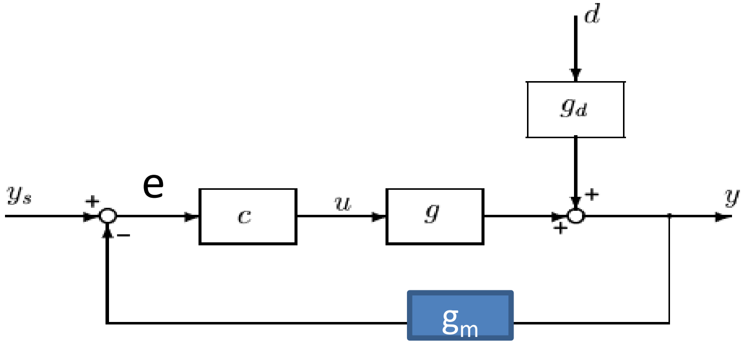
S gives effect of feedback

- T_{cl} = Direct*S where S=1/(1+loop)
    - No control: S=1 (S=I for multivariable case)
    - Want |S| small to have small control error |e|:

      $e = S\,(y_s - g_m\, g_d\, d)$
    - Perfect control (infinite c): S=0 (Achievable at steady state with I-action)

Steady-state offset with P-control

(k=process gain, K_c= controller gain)



Sensitivity fuction: $S(s) = 1/(1+L)$ where $L(s)$=loop = $g\ c\ g_m$.
S is transfer function from y_s to control error e (since «Direct»=1): $e = S(s)\ y_s$
Steady-state offset to step change in setpoint: $e = S(0)\ y_s$ where

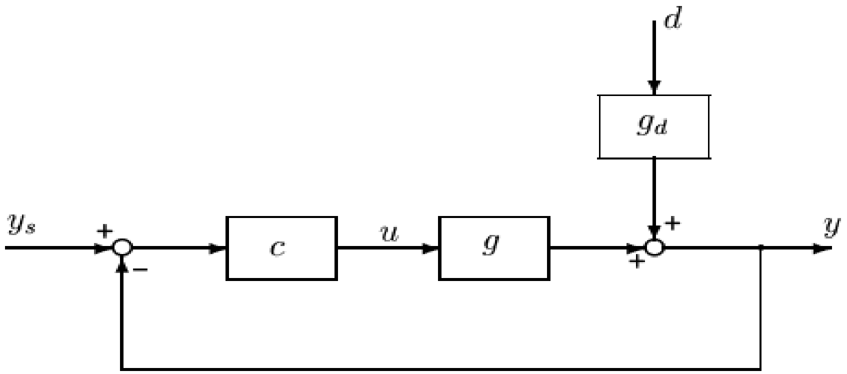
$S(0)= 1/(1+\text{loop}(0))= 1/(1+K_c k)$

since $g(0)=k$, $c(0)=K_c$, $g_m(0)=1$

Example P-control. $k=1$, $K_c = 4$. Relative Steady-state offset e/y_s is $S(0)=1/(1+K_c k) = 1/5= 0.20$ (20%)

Note with I-action (PI-control): Get $c(0)=\infty$, loop(0)= ∞, so $S(0)=0$ and get no steady-state offet.

Example 2. PI-control of 1st order process



Example.

$g(s) = g_d(s) = 2/(3s+1)$

SIMC PI-controller, $\tau_c=0.5$ (so we are speeding up the response relative to $\tau=3$):

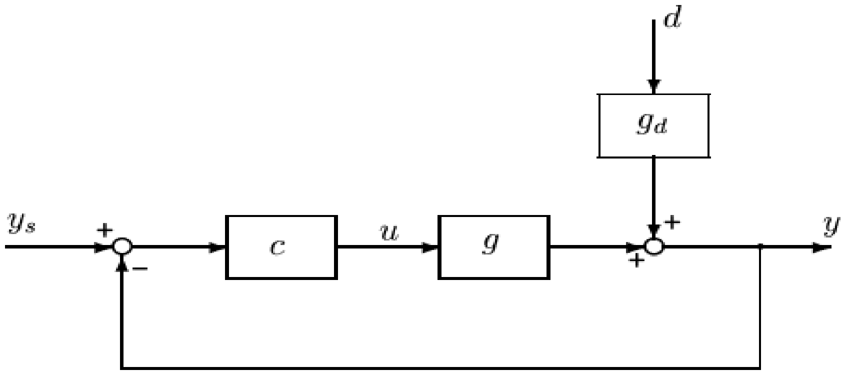
$c(s) = K_c(1 + \frac{1}{\tau_I s})$, $K_c = 3$, $\tau_I = 2$

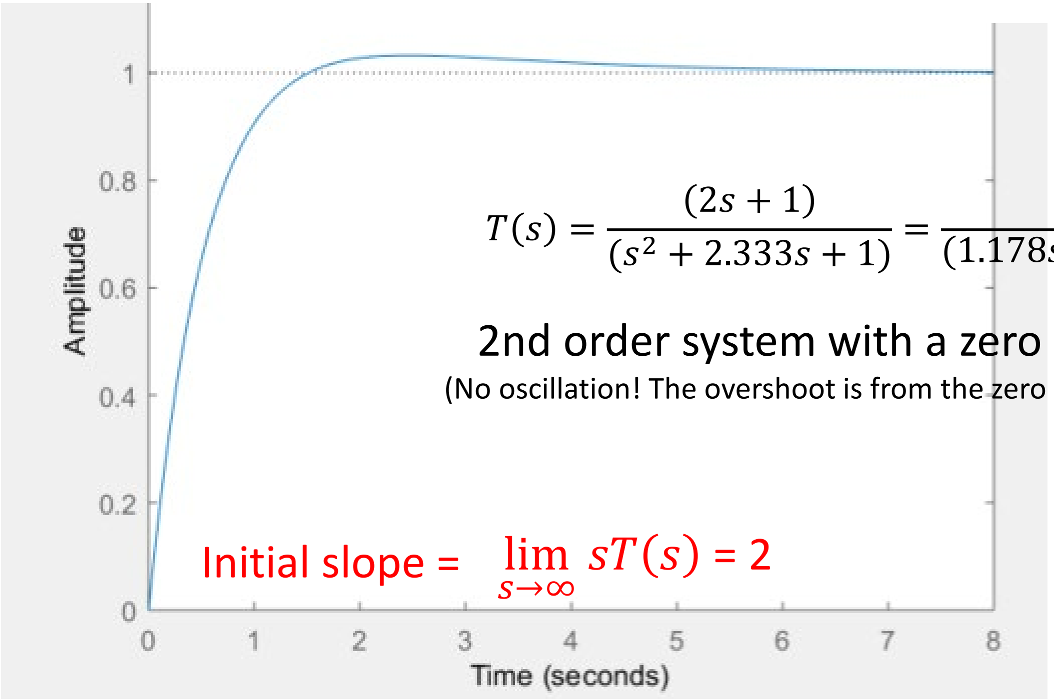
-> $c(s) = 3\left(1 + \frac{1}{2s}\right) = 3\,\frac{2s+1}{2s}$

Example 2. PI-control of 1st order process

Setpoint response. y_s=1, d=0

$$y = T(s)\, y_s$$





$$T(s) = \frac{(2s+1)}{(s^2+2.333s+1)} = \frac{(2s+1)}{(1.178s+1)(0.566s+1)}$$

2nd order system with a zero

(No oscillation! The overshoot is from the zero polynomial n(s)=2s+1)

Initial slope = $\lim_{s\to\infty} sT(s)$ = 2

Note: Reach 63% approximately at t=τ_c=0.5

```
% Matlab
s=tf('s')
g = 2/(3*s+1)
c = 3*(1 + 1/(2*s))
T = g*c/(1+g*c)
step(T)
```

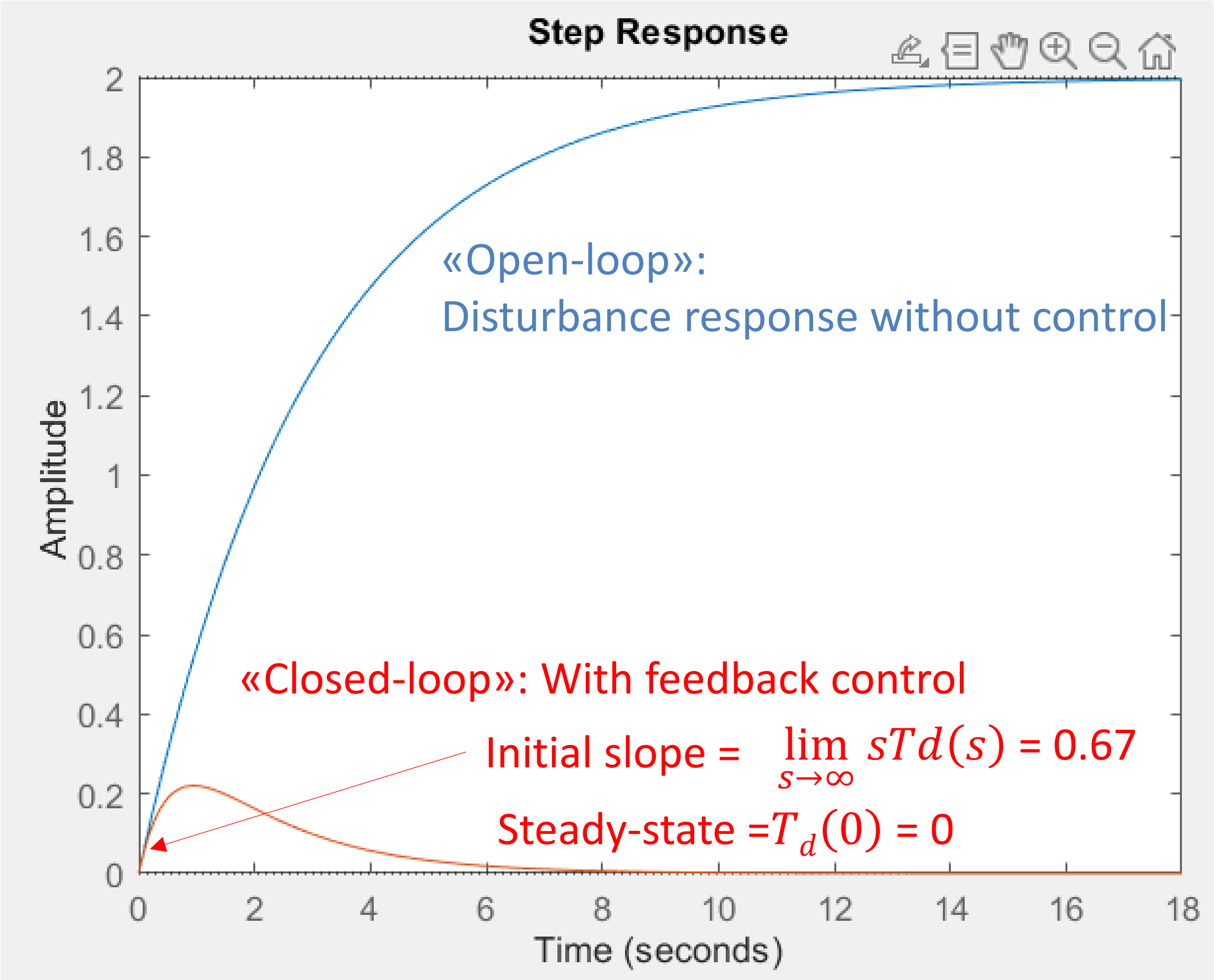
Input Disturbance response (g_d=g)

g = gd =

```
    2
  -------
  3 s + 1
```

Td =

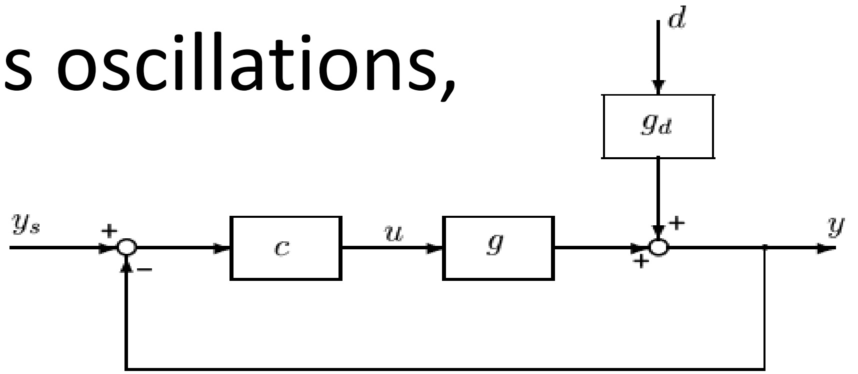
```
  0.6667 s
  ----------------
  s^2 + 2.333 s + 1
```



IMPORTANT:
The initial disturbance response (with or without feedback control) is always the same

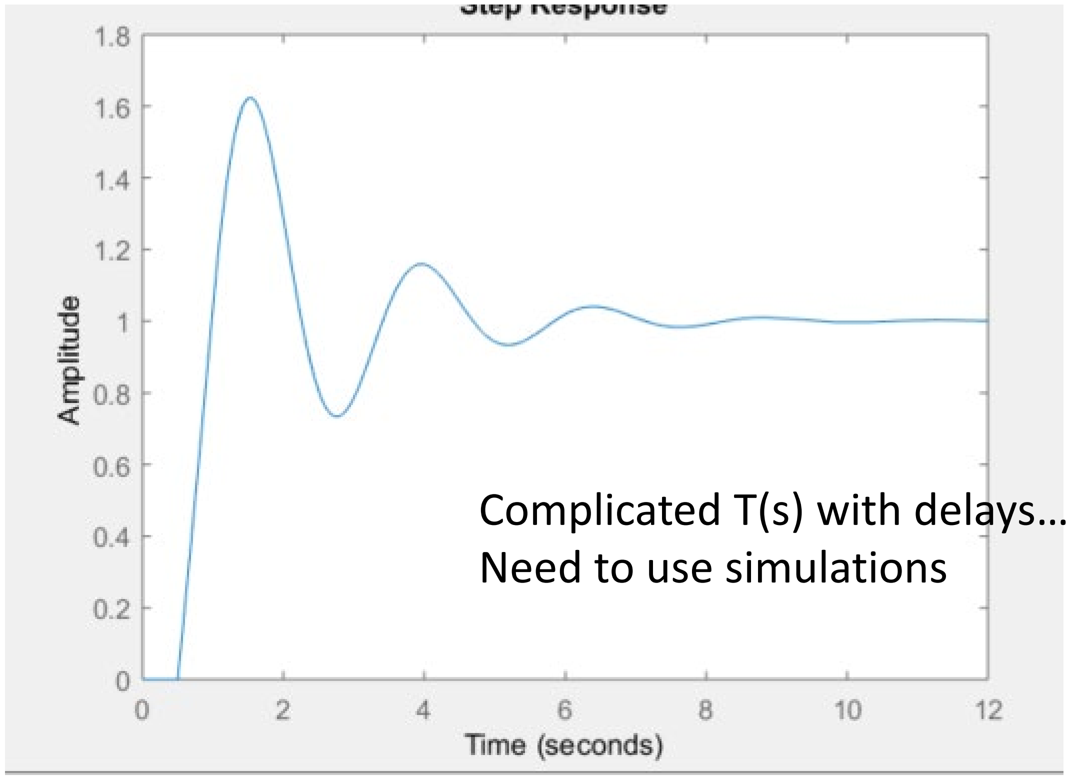
```
% Matlab
s=tf('s')
g = 2/(3*s+1)
c = 3*(1 + 1/(2*s))
Td=g/(1+g*c)
step(g,Td)
```

Comment: Adding delay gives oscillations, $\theta = 0.5$ s



Setpoint response. y_s=1, d=0

```
>> delay = exp(-0.5*s)
delay =
  exp(-0.5*s) * (1)
>> L =g*c*delay

L =
                    12 s + 6
  exp(-0.5*s) * -----------
                    6 s^2 + 2 s

 >> T2 = L/(1+L)
 Internal delays (seconds): 0.5  0.5

>> step(T2)
```

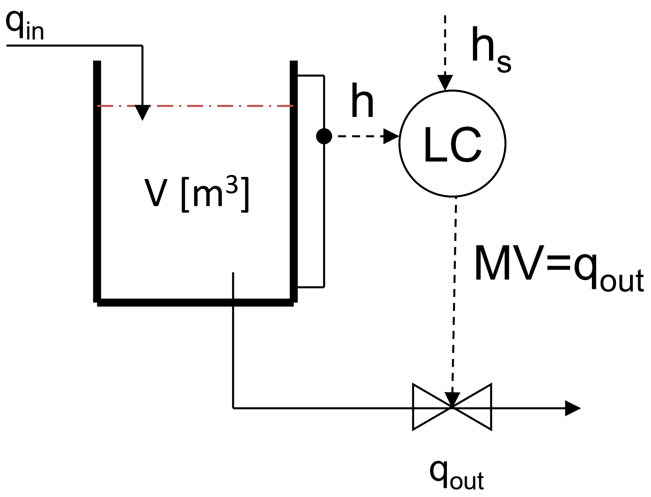


Unstable with $\theta = 1$ s

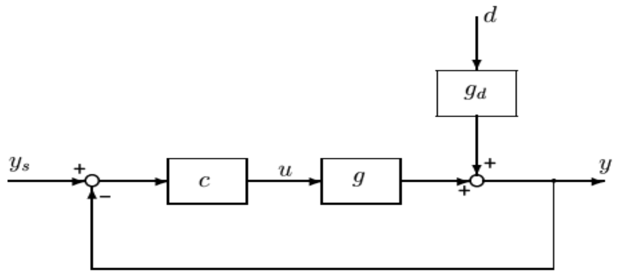
Comment: The SIMC-controller for the process $g(s) = 2e^{-0.5s}/(3s+1)$ with $\tau_c=\theta=0.5$ has $K_c=1.5$ and $\tau_I=3$.

Example 3. PI-control of integrating process (level)

FLOWSHEET:



BLOCK DIAGRAM:



y = Δh
u = -Δq_{out}
d = Δq_{in}

$$g(s) = gd(s) = \frac{k'}{s}$$

$$c(s) = K_c(1 + \frac{1}{\tau_I s})$$

Mass balance with constant density (V=Ah):

$$dV/dt = q_{in} - q_{out}$$

Deviation variables + linearize (well, it's already linear!)

$$A\, d\Delta h/dt = \Delta q_{in}(t) - \Delta q_{out}(t)$$

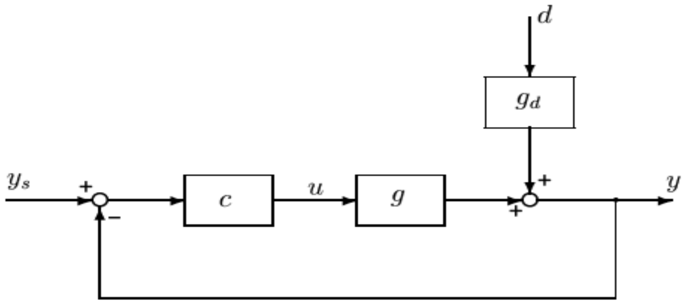
Laplace

$$\Delta h(s) = \frac{\Delta q_{in}(s) - \Delta q_{out}(s)}{As} = (k'/s)\,(u+d) \qquad k' = 1/A$$

Task: Derive condition to avoid «slow» ocillations that may occur when K_c is too small

*Yes, this may seem a bit strange, but for PI-control of integrating process you may get oscillatons when Kc is too small!
In addition, you may of course get the more common «fast» oscillations if Kc is too large because of «overreaction» with time delay.

Integrating process with PI-control:



$$G(s) = \frac{k'}{s}$$

$$C(s) = K_c(1 + \frac{1}{\tau_I s})$$

General rule to avoid slow oscillations ($\zeta \geq 1$) :

$$k' K_C \tau_I \geq 4$$

Need large controller gain and/or large integral time (!)

Proof:

$G(s) = k\frac{e^{-\theta s}}{\tau_1 s+1} \approx \frac{k'}{s}$ where $k' = \frac{k}{\tau_1}$; $C(s) = K_c\left(1 + \frac{1}{\tau_I s}\right)$

Closed-loop poles:

$1 + GC = 0 \Rightarrow 1 + \frac{k'}{s}K_c\left(1 + \frac{1}{\tau_I s}\right) = 0 \Rightarrow \tau_I s^2 + k' K_c \tau_I s + k' K_c = 0$

To avoid oscillations we must not have complex poles:

$B^2 - 4AC \geq 0 \Rightarrow k'^2 K_c^2 \tau_I^2 - 4k' K_c \tau_I \geq 0 \Rightarrow \boxed{k' K_c \tau_I \geq 4} \Rightarrow \tau_I \geq \frac{4}{k' K_c}$

Closed-loop responses

Closed-loop response to disturbance d at input and setpoint change

$$y = \frac{g}{1+gc} d + \frac{gc}{1+gc} y_s$$

PI-control of integrator:

$$g(s) = \frac{1}{s}; \quad c(s) = K_c \frac{\tau_I s+1}{\tau_I s}$$

Get

$$y = \frac{\tau_I s}{\tau_I s^2 + K_c \tau_I s + K_c} d + \frac{K_c(\tau_I s+1)}{\tau_I s^2 + K_c \tau_I s + K_c} y_s$$

Note: This is not a good tuning, Will get slow oscillations since $k' K_c \tau_I = 1*0.25*1=0.25 < 4$

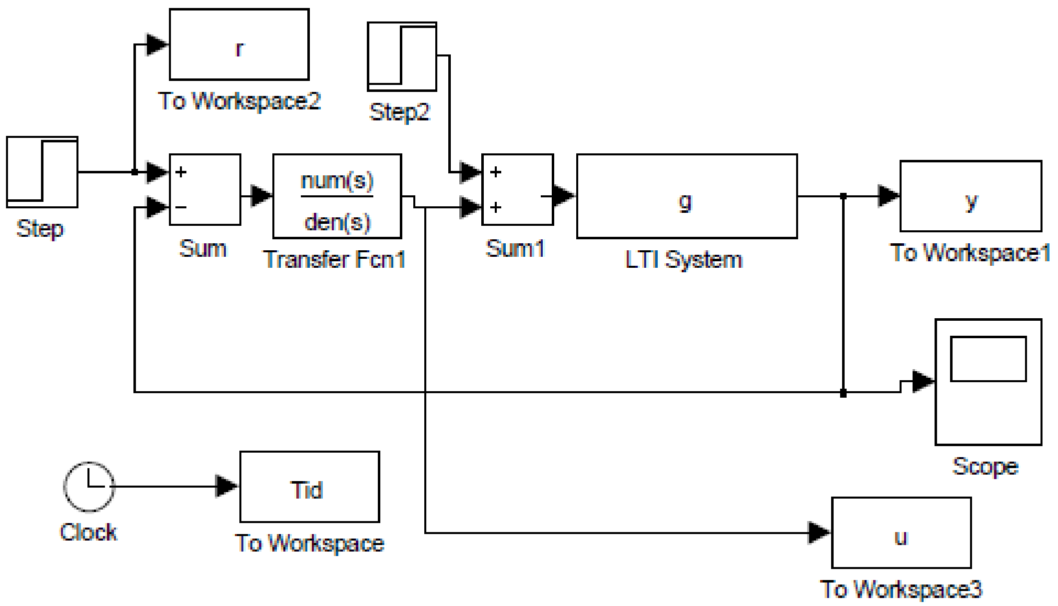
With $\tau_I = 1, K_c = 0.25$:

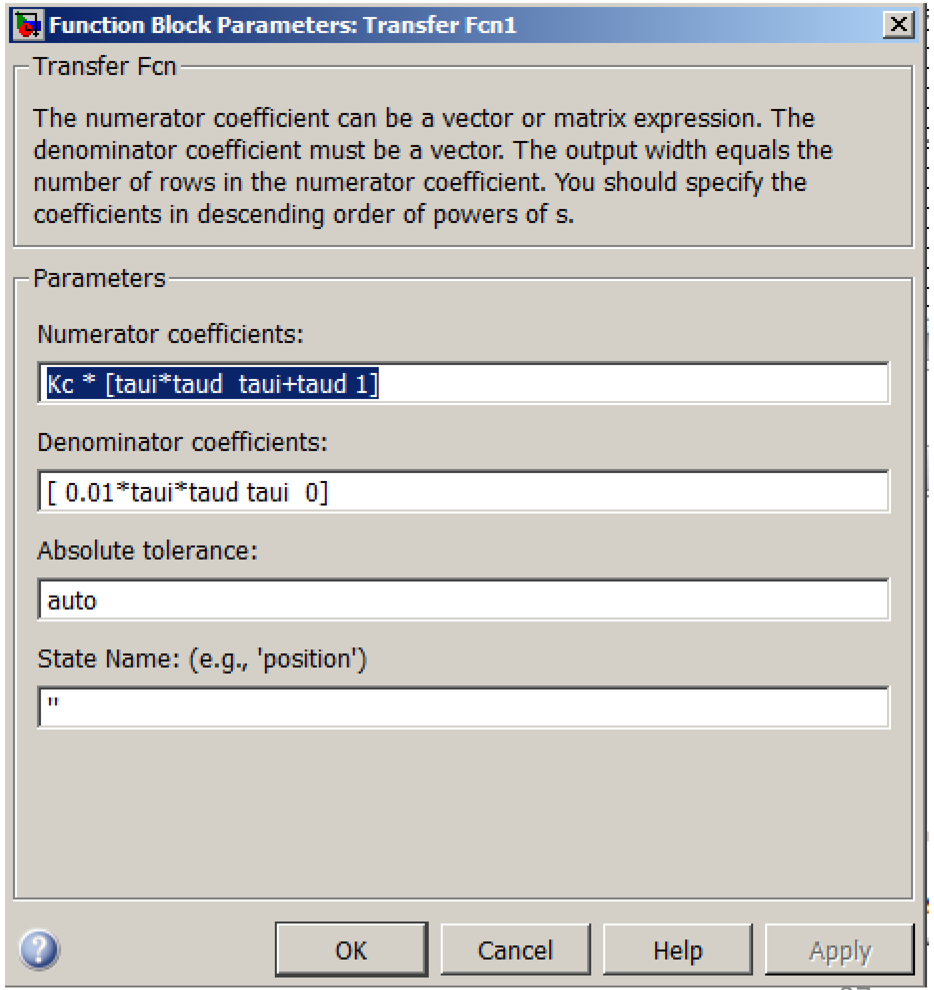
$$y = \frac{s}{s^2+0.25s+0.25} d + \frac{0.25(s+1)}{s^2+0.25s+0.25} y_s = \underbrace{\frac{4s}{4s^2+s+1}}_{T_d = \; h(s)} d + \underbrace{\frac{(s+1)}{4s^2+s+1}}_{T(s)} y_s$$

Notes:

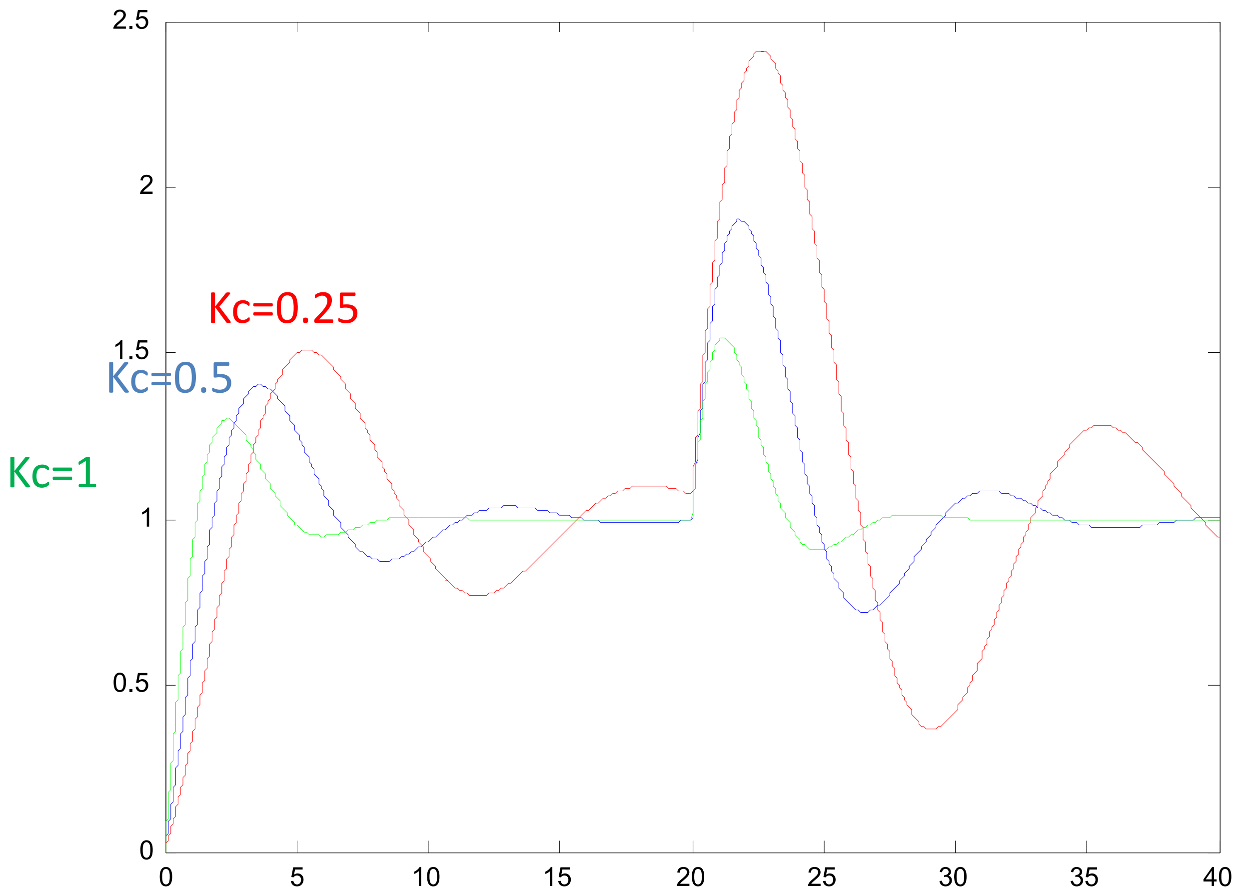
- Steady-state gain $h(0)$ for disturbance transfer function $h(s)$ is zero (because controller has integral action)
- Steady-state gain $T(0)$ for setpoint transfer function $T(s)$ is 1 (because controller has integral action)
- Denominator is on form $\tau^2 s^2 + 2\tau\zeta s + 1$ with $\tau = 2$ and $\zeta = 0.25 < 1$, so there will be oscillations with period $P \approx 2\pi\tau$
- Initial response ($t \to 0$) to disturbance is the same as with no control ($h(s) = \frac{g}{1+gc} \to g(s)$ when $s \to \infty$ since $g(s)c(s) \to 0$ (which is the case for all real systems))

Simulink, tunepid4





PI-control of integrator (level control). G = 1/s, taui=1. **VARY Kc**




```
%tunepid4
s=tf('s')
theta=0
g=(1/s)*exp(-theta*s) % integrating
taud=0
taui=1
Kc=0.5 % oscillations (Kc*k'*taui = 0.5 < 4)
sim tunepid4; plot(Tid,y); hold on %

Kc=0.25 % more oscillations (Kc*k'*taui = 0.25 < 4)
sim tunepid4; plot(Tid,y,'red');

Kc=1   % less oscillations (Kc*k'*taui = 1)
sim tunepid4; plot(Tid,y,'green');
hold off
```

Input disturbance at t=20

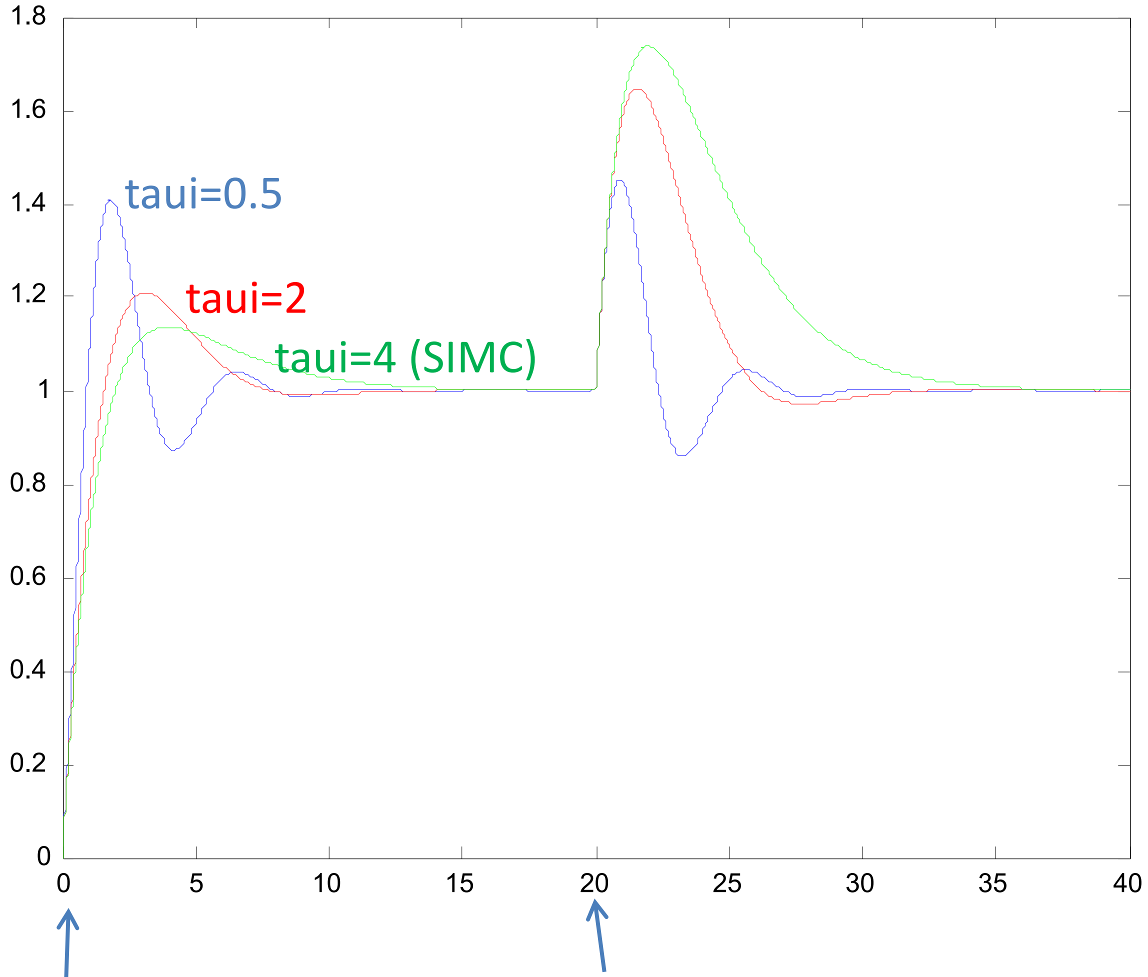
Setpoint change at t=0

Note: Need higher controller gain to reduce "slow" oscillations!

$$Avoid\ slow\ oscillations\!: \ k' K_C \tau_I \geq 4$$

So would need to increase Kc to 4 in this case (SIMC-rule)

PI-control of integrator (level control). G = 1/s, Kc=1. **VARY taui**



Setpoint change at t=0

Input disturbance at t=20

Note: Need larger integral time reduce "slow" oscillations

$Avoid\ slow\ oscillations:\ k' K_C \tau_I \geq 4$

So need to use taui=4 to have no oscillations (SIMC-rule).

Get T(s)=(4s+1)/(2s+1)^2

Poles and zeros

g(s) = n(s)/d(s).

Example.

$$g(s) = \frac{12s + 6}{30s^2 + 33s + 3}$$

Standard forms:

1. Time constant form

$$g(s) = k\frac{(T_1 s + 1)\cdots}{(\tau_1 s + 1)(\tau_2 s + 1)\cdots}$$

$$g(s) = 2\frac{2s + 1}{(10s + 1)(s + 1)}$$

2. Pole-zero form,

p=pole, z=zero

(more general, for unstable and complex poles/zeros)

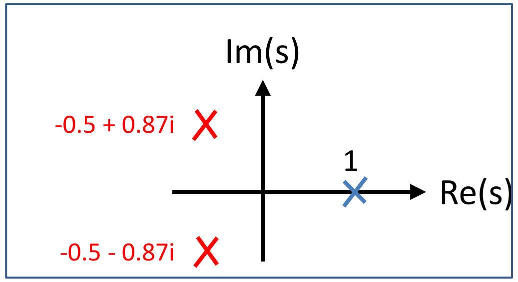
$$g(s) = c\frac{(s - z_1)\cdots}{(s - p_1)(s - p_2)\cdots}$$

$$g(s) = \frac{12}{30}\frac{s + 0.5}{(s + 0.1)(s + 1)}$$

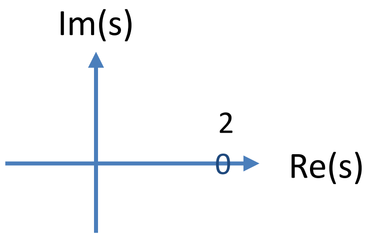
$$z_1 = -1/T_1 = -1/2 = -0.5$$
$$p_1 = -1/\tau_1 = -1/10 = -0.1;$$
$$p_2 = -1/\tau_2 = -1/1 = -1$$

Poles and zeros

- Transfer functions *G(s)* of linear, time-invariant systems with time delay are ratios of two polynomials in *s* (Laplace variable)
  - G(s) = n(s)/d(s)
- Polynomials have roots.
  root in denominator, d(s)=0: *G(s)* $\rightarrow \infty$ "pole" (x)
  root in numerator, n(s)=0: *G(s)* $\rightarrow$ 0 "zero" (0)
- Effect on dynamics:
  - Poles determine stability and fast or slow dynamics
    - d(s)=(τs+1). «Stable» LHP-poles slow down the response
      - slower for large τ
    - Poles in right half plane (RHP): Unstable .
      - Example: g(s)=1/(s-1). Has RHP-pole at s=1
    - Complex poles (=eigenvalues of A): Oscillations with frequency ω
      - Example: g(s) = 1/(s^2 + s + 1). Solve d(s)= s^2 + s + 1 = 0. Get poles s1 = -0.5 + 0.87*i, s2 = -0.5 – 0.87*i, ω=0.87

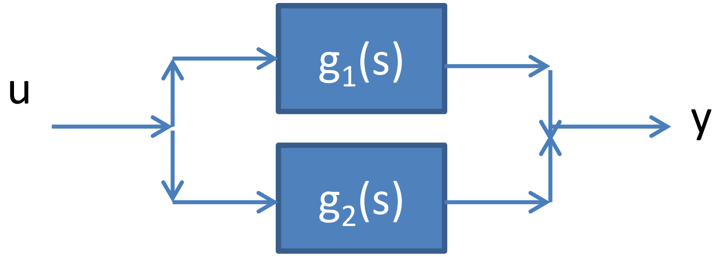


  - Zeros are responsible for shape of response
    - n(s)=(Ts+1). Zeros in left half plane (LHP): «Lift» the response
      - give overshoot for large T.
    - n(s)=(-Ts+1). Zeros in right half plane (RHP): always give inverse response
      - Inverse response makes problems for feedback control
      - Example: g(s)=(-0.5s+1) / (10s^2+11s+1). Has RHP-zero at s=2



Zeros

- Zeros are common in practise
- Occur when there are several «paths» to the output.
- **RHP zero: «competing effects where slow wins (has largest gain)»**



- Example 1

$$g_1(s) = \frac{2}{10s+1}, \quad g_2(s) = \frac{0.3}{s+1}$$
$$g(s) = g_1 + g_2 = \frac{2(s+1)+0.3(10s+1)}{(10s+1)(s+1)} = 2.3\frac{2.17s+1}{(10s+1)(s+1)}$$

All coefficients positive: LHP zero

- Example 2

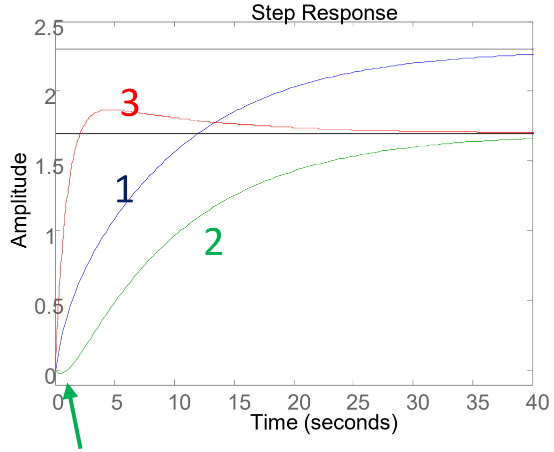
$$g_1(s) = \frac{2}{10s+1}, \quad g_2(s) = -\frac{0.3}{s+1}$$
$$g(s) = g_1 + g_2 = \frac{2(s+1)-0.3(10s+1)}{(10s+1)(s+1)} = 1.7\frac{-0.59s+1}{(10s+1)(s+1)}$$

Sign change: RHP zero) Inverse response

- Example 3

$$g_1(s) = -\frac{0.3}{10s+1}, \quad g_2(s) = \frac{2}{s+1}$$
$$g(s) = g_1 + g_2 = \frac{2(s+1)-0.3(10s+1)}{(10s+1)(s+1)} = 1.7\frac{11.3s+1}{(10s+1)(s+1)}$$

Note; Overshoot since 11.3>10
(overshoot: competing effects where fast wins)



Example 2: RHP-zero with «time constant» -0.59: Similar to delay of 0.59.

Start here week 7

Zeros

$$G(s) = \frac{K(\tau_a s + 1)}{(\tau_1 s + 1)(\tau_2 s + 1)} \qquad (5\text{-}14)$$

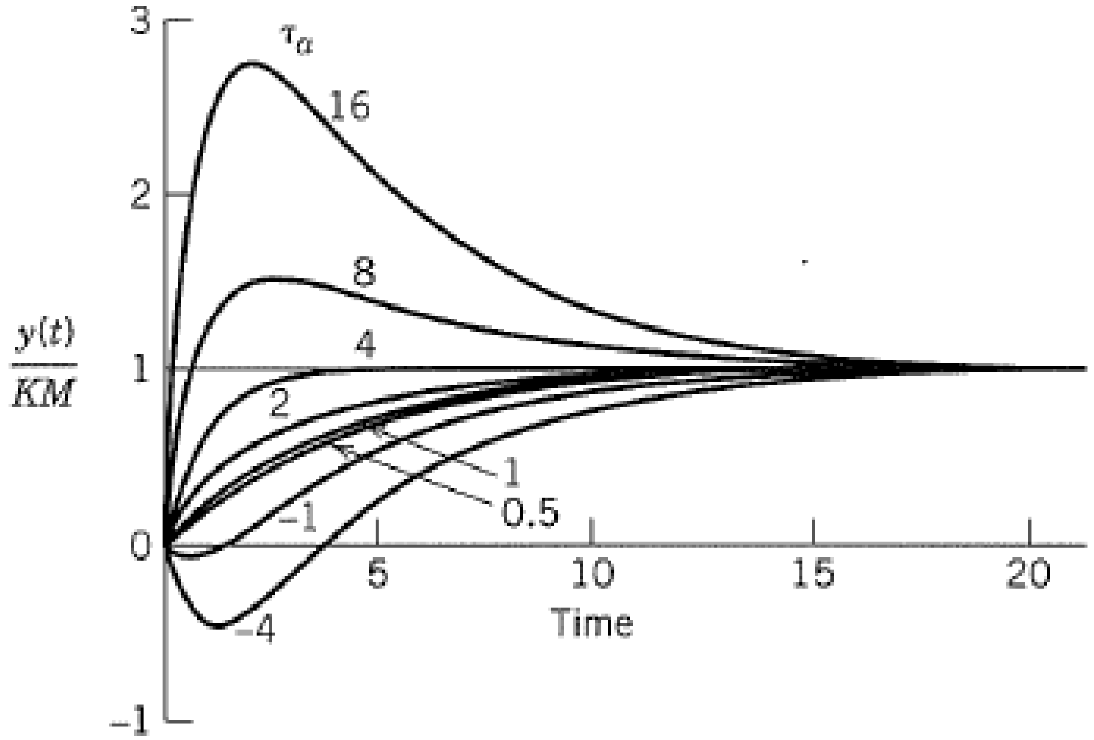


Figure 5.3 Step response of an overdamped second-order system (Eq. 5-14) for different values of τ_a ($\tau_1 = 4$, $\tau_2 = 1$).

Summary poles and zeros

- G(s) = n(s) / d(s)=k'(s-z_1) / (s-p_1)(s-p_2)..
- Example: G(s) = 4 (3s-1)/(s^2+s-2),

  Get: k'=12, z_1=1/3, p_1=-2 p_2=1

- Poles p (=eigenvalues of A-matrix)
  - Determine speed of response, exp(p*t)
  - Negative sign in d(s)) p_2 in RHP: unstable, exp(p_2*t) ! 1 (NEED control)
  - Pole p complex: oscillating response
- Zeros z
  - Determine shape of response
  - Negative sign in n(s)) z_1 in RHP: inverse response (BAD for control)
  - LHP-zero may give overshoot

Approximation of time delay

- Time delay is a bit difficult because it's «infinite order»
  - So cannot directly use with g(s)=n(s)/d(s) (where n and d are polynomials)
  - or with «state-space» form (dx/dt=Ax+Bu)
- Taylor expansion of e^x ($x=-\theta s$)

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

- Approximation of delay as n(s)/d(s):

$$e^{-\theta s} = 1 - \theta s + \cdots \approx -\theta s + 1$$

RHP-zero approximation (we use this for SIMC)

$$e^{-\theta s} = \frac{1}{e^{\theta s}} \approx \frac{1}{\theta s + 1}$$

1st order approximation

$$e^{-\theta s} = \frac{e^{-\frac{\theta}{2}s}}{e^{-\frac{\theta}{2}s}} \approx \frac{-\frac{\theta}{2}s + 1}{\frac{\theta}{2}s + 1}$$

Pade approximation (better)

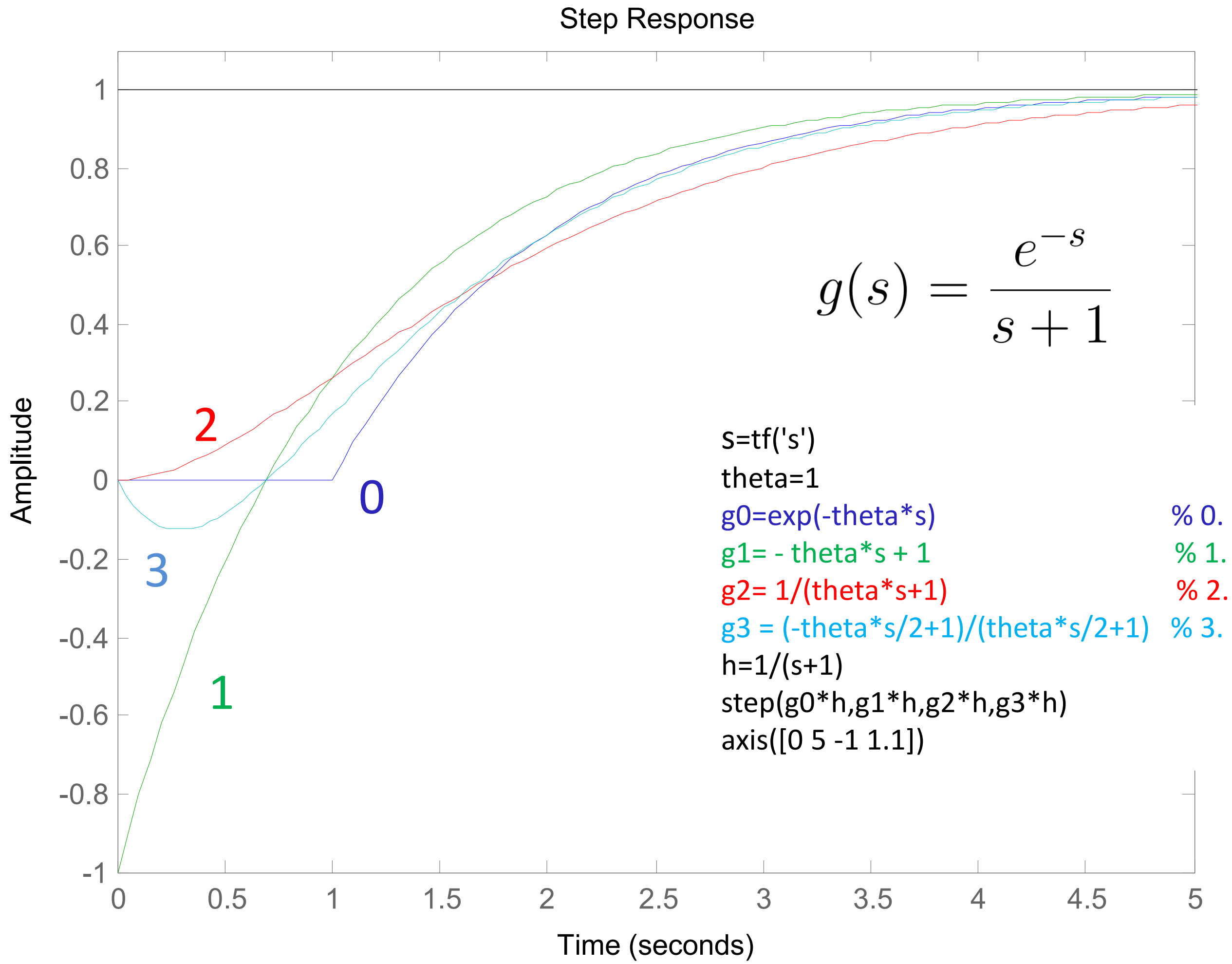
$$e^{-\theta s} \approx \frac{(-\frac{\theta}{2n}s + 1)^n}{(\frac{\theta}{2n}s + 1)^n}$$

n'th order Pade approximation (even better)

“Going the other way”

Simple approximations of time delay

Example: Step response of first-order system plus delay



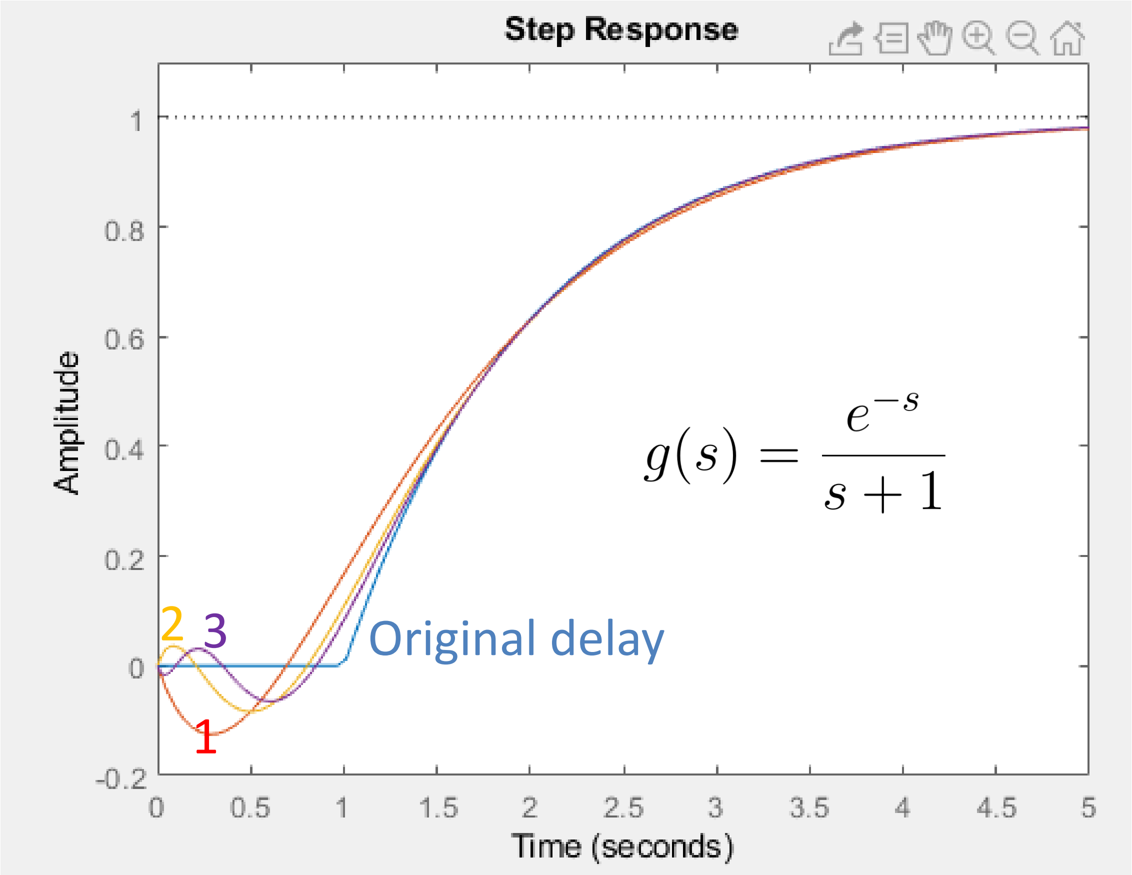
$$g(s) = \frac{e^{-s}}{s+1}$$

```
s=tf('s')
theta=1
g0=exp(-theta*s)                        % 0. Original time delay.
g1= - theta*s + 1                       % 1. As RHP-zero.
g2= 1/(theta*s+1)                       % 2. As First order
g3 = (-theta*s/2+1)/(theta*s/2+1)       % 3. Combined: Pade-approx
h=1/(s+1)
step(g0*h,g1*h,g2*h,g3*h)
axis([0 5 -1 1.1])
```

n'th order Pade approximation

- Accurate for large n

$$e^{-\theta s} \approx \frac{(-\frac{\theta}{2n}s+1)^n}{(\frac{\theta}{2n}s+1)^n}$$



Note: Number of RHP-zeros
= number of 0-crossings of step response

```
s=tf('s')
theta=1
g0 =exp(-theta*s)                          % Original time delay
g1 = (-theta*s/2+1)/(theta*s/2+1)          % 1st-order Pade-approximation
g2 = (-theta*s/4+1)^2/(theta*s/4+1)^2   % 2nd-order Pade-approximation
g3 = (-theta*s/6+1)^3/(theta*s/6+1)^3   % 3rd-order Pade-approximation
h=1/(s+1)
step(g0*h,g1*h,g2*h,g3*h)
axis([0 5 -0.2 1.1])
```

Why use Pade?
To get model on state space form, dx/dt=Ax+Bu

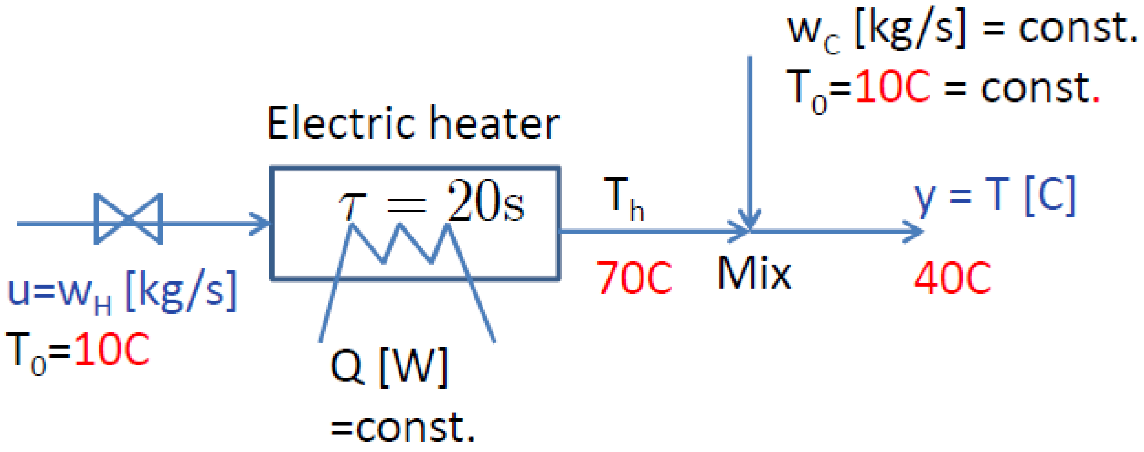
Approximations of transfer functions

- Going the «other way»
- Want to approximate g(s)=n(s)/d(s) as first-order plus delay
  - «Skogestad half rule» to find effective delay. IMPORTANT!
  - see slides SiS6 for SIMC-rule

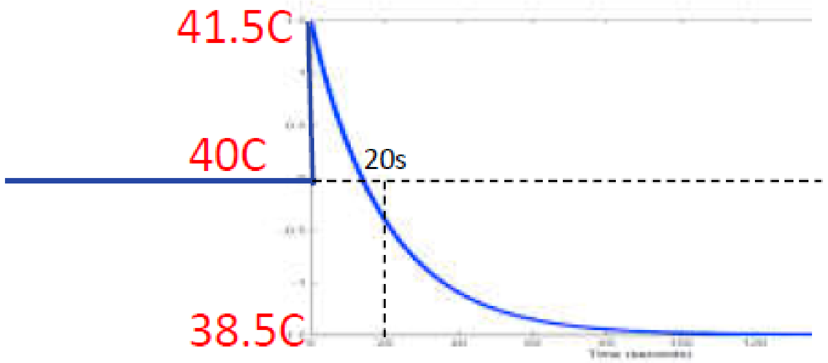
Extra: Examples of dynamic model structures

How do we get zeros?

RHP-zero (inverse response)



Response in y=T to a 10% step increase in u=w$_H$ =0.1:

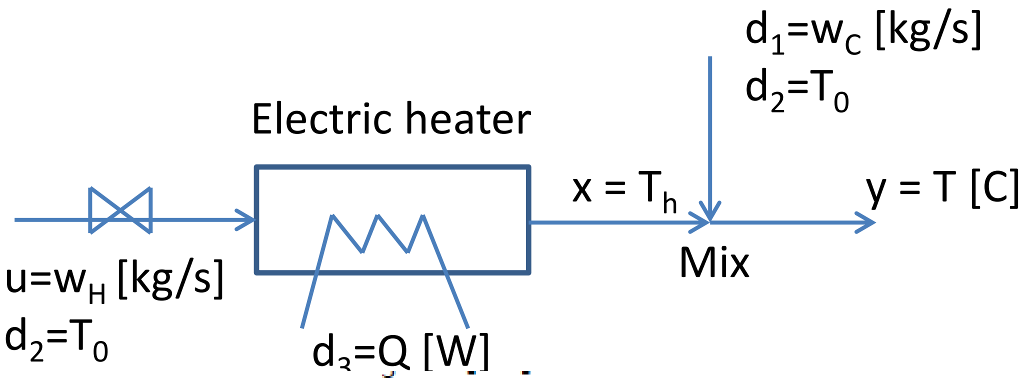


Two effects: 1) Direct effect of mixing: $g_1(s)=15$

2) Indirect effect of changed T$_h$: $g_2(s) = -30/(20s+1)$

$$g(s) = g_1 + g_2 = -15\frac{-20s+1}{20s+1}$$

Model derivation



1. Model. Assume:

   Mass m [kg] in heater constant

   c_P constant

Energy balance heater + mixer:

$$\frac{d(mc_P T_h)}{dt} = w_h c_P(T_0 - T_h) + Q$$

$$T = \frac{w_h T_h + w_c T_c}{w_c + w_h}$$

2. Linearize:

$y = \Delta T, x = \Delta T_h, u = \Delta w_h$

$\tau \frac{dx}{dt} = -x + ku$

$y = Cx + Du$

$k = \frac{T_o^* - T_h^*}{w_h^*}$

$\tau = m/w_h^*$

$C = \frac{w_h^*}{w_c^* + w_h^*}$

$D = \frac{T_h^* - T^*}{w_c + w_h^*}$

3. Nominal steady-state data:

$T_0 = 10C, T_h = 70C, T = 40C$

$w_h = w_c = 1kg/s, m = 20kg$

Gives:

$k = \frac{T_o^* - T_h^*}{w_h^*} = \frac{10-70}{1} = -60$

$\tau = m/w_h^* = 20/1 = 20$

$C = \frac{w_h^*}{w_c^* + w_h^*} = 0.5$

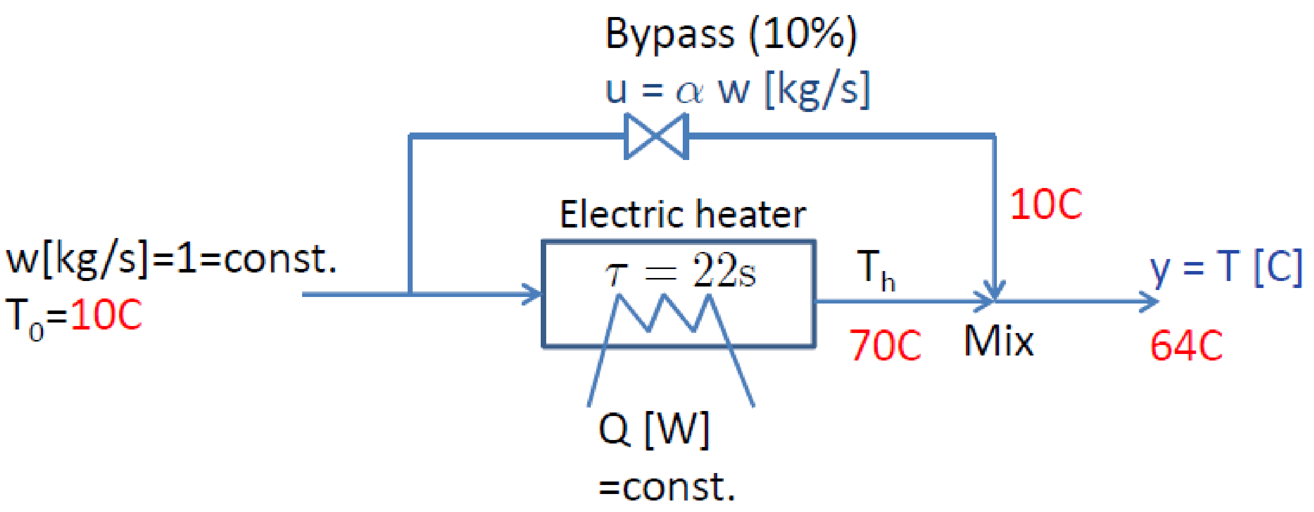
$D = \frac{T_h^* - T^*}{w_c + w_h^*} = \frac{70-40}{2} = 15$

4. Transfer function:

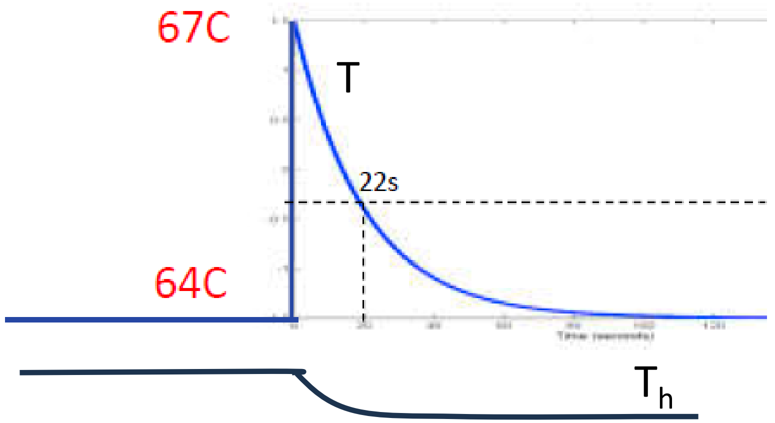
$y(s) = G(s)u(s)$

$$G(s) = C\frac{k}{\tau s+1} + D$$
$$= 0.5\frac{-60}{20s+1} + 15$$
$$= -15\frac{-20s+1}{20s+1}$$

Zero at 0 (no steady-state effect)



Response in y=T to a step decrease in bypass fraction from 0.1 to 0.05:

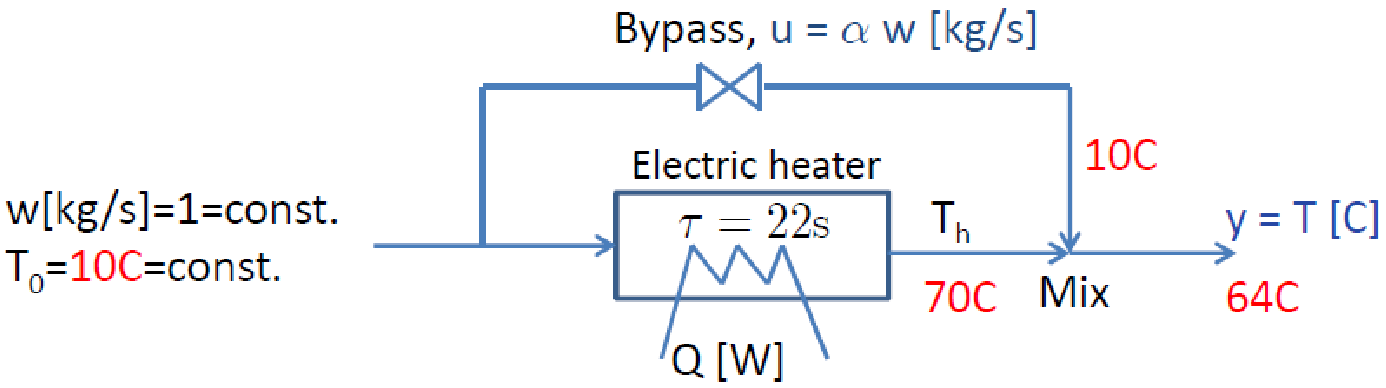


Two effects: 1) Direct effect of mixing: $g_1(s)=-60$

2) Indirect effect of changed T_h: $g_2(s) = 60/(22s+1)$

$$g(s) = g_1 + g_2 = -60\frac{22s}{22s+1}$$

Model derivation



1. Model. Assume:

   Mass m [kg] in heater constant

   c_P constant

Energy balance heater + mixer:

$$\frac{d(mc_P T_h)}{dt} = (1-\alpha)wc_P(T_0 - T_h) + Q$$

$$T = (1-\alpha)T_h + \alpha T_c$$

2. Linearize:

$$y = \Delta T, x = \Delta T, u = \alpha$$
$$\tau\frac{dx}{dt} = -x + ku$$
$$y = Cx + Du$$
$$k = -\frac{T_o^* - T_h^*}{(1-\alpha^*)}$$
$$\tau = m/w_h^*$$
$$C = (1-\alpha^*)$$
$$D = (T_o^* - T_h^*)$$

3. Nominal steady-state data:

$$T_0 = 10C, T_h = 70C, T = 64C$$
$$w = 1kg/s, \alpha = 0.1, m = 20kg$$

Gives:

$$k = -\frac{T_o^* - T_h^*}{(1-\alpha^*)} = -\frac{10-70}{0.9} = 66.67$$
$$\tau = m/w_h^* = 20/0.9 = 22$$
$$C = (1-\alpha^*) = 0.9$$
$$D = (T_o^* - T_h^*) = -60$$

4. Transfer function:

$$y(s) = G(s)u(s)$$
$$G(s) = C\frac{k}{\tau s+1} + D$$
$$= 0.9\frac{66.67}{22s+1} - 60$$
$$= 60(\frac{1}{22s+1} - 1) = -60\frac{22s}{22s+1}$$