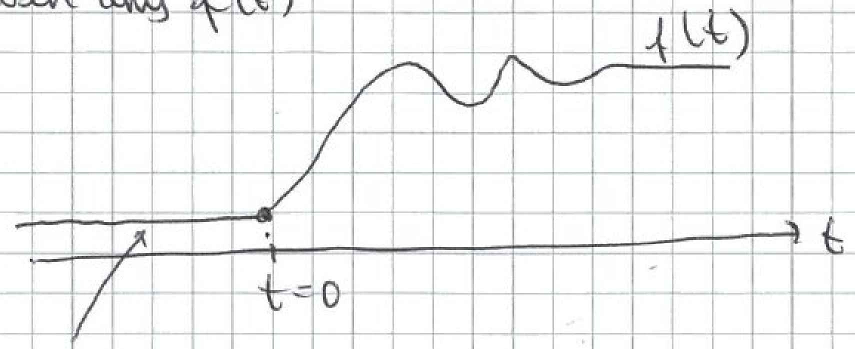
DEFINITION LAPLACE TRANSFORM

Laplace transforms

Given any $f(t)$



Usually at steady state for $t<0$

Laplace transform of $f(t)$

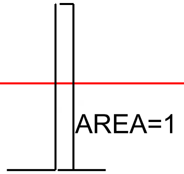
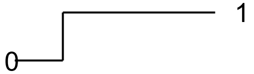
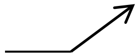
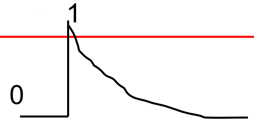
$$\boxed{F(s)} = \mathcal{L}(f(t)) \overset{\text{def.}}{=} \underbrace{\int_0^\infty f(t)\, e^{-st}\, dt}$$

I usually misuse notation and write $f(s)$

"$f(t)$ weighted with e^{-st} and then integrated from $t=0$ to ∞ so that s becomes new independent variable instead of time t"

s [time^{-1}]: Complex number (new indep. variable instead of t)

Table 3.1 Laplace Transforms for Various Time-Domain Functions

$f(t)$			$F(s)$
1. $\delta(t)$	(unit impulse)		1
2. $S(t)$	(unit step)		$\frac{1}{s}$
3. t	(ramp)		$\frac{1}{s^2}$
4. t^{n-1}			$\frac{(n-1)!}{s^n}$
5. $f(t) = e^{at}$			$\frac{1}{s-a}$
6. $\frac{1}{\tau} e^{-t/\tau}$			$\frac{1}{\tau s + 1}$
7. $\frac{t^{n-1}e^{-bt}}{(n-1)!}$	$(n > 0)$		$\frac{1}{(s+b)^n}$
8. $\frac{1}{\tau^n (n-1)!} t^{n-1} e^{-t/\tau}$			$\frac{1}{(\tau s + 1)^n}$
9. $\frac{1}{b_1 - b_2}(e^{-b_2 t} - e^{-b_1 t})$			$\frac{1}{(s+b_1)(s+b_2)}$

Rule exponential (e^{-at}):
Opposite sign in Laplace

Table 3.1 Laplace Transforms for Various Time-Domain Functions

$f(t)$		$F(s)$
10. $\frac{1}{\tau_1 - \tau_2}(e^{-t/\tau_1} - e^{-t/\tau_2})$		$\frac{1}{(\tau_1 s + 1)(\tau_2 s + 1)}$
11. $\frac{b_3 - b_1}{b_2 - b_1} e^{-b_1 t} + \frac{b_3 - b_2}{b_1 - b_2} e^{-b_2 t}$		$\frac{s + b_3}{(s + b_1)(s + b_2)}$
12. $\frac{1}{\tau_1}\frac{\tau_1 - \tau_3}{\tau_1 - \tau_2} e^{-t/\tau_1} + \frac{1}{\tau_2}\frac{\tau_2 - \tau_3}{\tau_2 - \tau_1} e^{-t/\tau_2}$		$\frac{\tau_3 s + 1}{(\tau_1 s + 1)(\tau_2 s + 1)}$
13. $1 - e^{-t/\tau}$		$\frac{1}{s(\tau s + 1)}$
14. $\sin \omega t$		$\frac{\omega}{s^2 + \omega^2}$
15. $\cos \omega t$		$\frac{s}{s^2 + \omega^2}$
16. $\sin(\omega t + \phi)$		$\frac{\omega \cos \phi + s \sin \phi}{s^2 + \omega^2}$
17. $e^{-bt} \sin \omega t$	b, ω real	$\frac{\omega}{(s + b)^2 + \omega^2}$
18. $e^{-bt} \cos \omega t$		$\frac{s + b}{(s + b)^2 + \omega^2}$
19. $\frac{1}{\tau\sqrt{1 - \zeta^2}} e^{-\zeta t/\tau} \sin\left(\sqrt{1 - \zeta^2}\, t/\tau\right)$ $(0 \leq \lvert\zeta\rvert < 1)$		$\frac{1}{\tau^2 s^2 + 2\zeta\tau s + 1}$

Note: This is step response of first-order system

Table 3.1 Laplace Transforms for Various Time-Domain Functions[a] (continued)

	$f(t)$	$F(s)$
20.	$1 + \frac{1}{\tau_2 - \tau_1}(\tau_1 e^{-t/\tau_1} - \tau_2 e^{-t/\tau_2})$ $(\tau_1 \neq \tau_2)$	$\frac{1}{s(\tau_1 s + 1)(\tau_2 s + 1)}$
21.	$1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta t/\tau} \sin[\sqrt{1-\zeta^2}\, t/\tau + \psi]$ $\psi = \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}, \quad (0 \leq \lvert\zeta\rvert < 1)$	$\frac{1}{s(\tau^2 s^2 + 2\zeta\tau s + 1)}$
22.	$1 - e^{-\zeta t/\tau}[\cos(\sqrt{1-\zeta^2}\; t/\tau) + \frac{\zeta}{\sqrt{1-\zeta^2}} \sin(\sqrt{1-\zeta^2}\, t/\tau)]$ $(0 \leq \lvert\zeta\rvert < 1)$	$\frac{1}{s(\tau^2 s^2 + 2\zeta\tau s + 1)}$
23.	$1 + \frac{\tau_3 - \tau_1}{\tau_1 - \tau_2} e^{-t/\tau_1} + \frac{\tau_3 - \tau_2}{\tau_2 - \tau_1} e^{-t/\tau_2}$ $(\tau_1 \neq \tau_2)$	$\frac{\tau_3 s + 1}{s(\tau_1 s + 1)(\tau_2 s + 1)}$
24.	$\frac{df}{dt}$	$sF(s) - f(0)$
25.	$\frac{d^n f}{dt^n}$	$s^n F(s) - s^{n-1} f(0) - s^{n-2} f^{(1)}(0) - \cdots - s f^{(n-2)}(0) - f^{(n-1)}(0)$
26.	$f(t - t_0) S(t - t_0)$	$e^{-t_0 s} F(s)$

[a]Note that $f(t)$ and $F(s)$ are defined for $t \geq 0$ only.

Alternative forms of step response for 2nd order system

Laplace of derivatives

Transfer function for time delay μ is $e^{-\mu s}$

Laplace Transform

Definition*: $F(s) = L(f(t)) = \int_0^\infty f(t)e^{-st}dt$

Usually f(t) is in deviation variables so f(t=0) = 0

Examples

1. f(t)=$\delta(t)$ $(impulse)$. $f(s) = 1$

2. f(t)=a (step change)

a/s = Laplace of step a

$$f(s)=\int_0^\infty ae^{-st}dt = -\frac{a}{s}\left[e^{-st}\right]\Big|_0^\infty = 0-\left[-\frac{a}{s}\right]=\frac{a}{s}$$

5. $f(t)=e^{-bt}$

$$f(s)=\int_0^\infty e^{-bt}e^{-st}dt = \int_0^\infty e^{-(b+s)t}dt = \frac{1}{b+s}\left[-e^{-(b+s)t}\right]\Big|_0^\infty = \frac{1}{s+b}$$

*Will often misuse notation and write f(s) instead of F(s)

Most important property for us.
Laplace of derivative.

$$L\left(\frac{df}{dt}\right) = \int_0^\infty \frac{df}{dt} e^{-st} dt = sL(f) - f(0) = s\,F(s) - f(0)$$

Differentiation: replaced by multiplication with s
Integration: replaced by multiplication with 1/s

Proof: $$\int_0^\infty \frac{df}{dt} e^{-st} dt = \int_0^\infty e^{-st} df$$

Set $v=e^{-st}$ and $du=df$ and use integration by parts

Other properties of Laplace transform:

A. Final value theorem

$$\boxed{y(t=\infty) = \lim_{s\to 0} sY(s)}$$

"steady-state value"

Example: $$Y(s) = \frac{1}{\tau s + 1}\frac{a}{s}$$

$$y(\infty) = \lim_{s\to 0} \frac{a}{\tau s + 1} = a$$

Comment (relevant to step responses):
If Y(s) = g(s)/s then y(t=∞)=g(0)

B. Initial value theorem

$$\boxed{\lim_{t\to 0} y(t) = \lim_{s\to\infty} sY(s)}$$

Example

$$For \quad Y(s) = \frac{4s+2}{s(s+1)}$$

$$y(0) = 4$$

by initial value theorem
(multiply Y(s) by s and set s=1)

$$y(\infty) = 2$$

by final value theorem
(multiply Y(s) by s and set s=0)

C. Initial slope property

$$\boxed{\lim_{t\to 0} y'(t) = \lim_{s\to\infty} s^2 Y(s)}$$

Comment on notation

	Sigurd	Book (Seborg)
Time signal	$y(t)$	$y(t)$
Steady-state value	y^*	$\bar{y}$
Deviation	$\Delta y = y - y^*$	$y' = y - \bar{y}$
Laplace of $\Delta y(t)$	$y(s)$	$Y(s)$

$$f(t) \underset{\text{Inverse, } \mathcal{L}^{-1}}{\overset{\text{Laplace}}{\rightleftharpoons}} f(s)$$

1. Complex contour integration
2. Split into sum of simple terms where $f(t)$ is known (partial fraction expansion)

Partial fraction expansion of f(s)

Consider a Laplace transform

$$f(s) = n(s)/d(s)$$

Assume the order of the polynomial d(s) is higher or equal to order of polynomial n(s).

We first factorize d(s) as a product of terms, for example,

$$d(s)= k(\tau_1 * s+1)(\tau_2 * s+1)...$$

where τ_i are the time constants

- Note that $s=-1/\tau_i$ are the roots in the polynomial d(s)=0.

We then write the "partial fraction expansion" of f(s):

$$f(s) = \alpha_0 + \alpha_1/(\tau_1 * s+1) + \alpha_2/(\tau_2 * s+1) + ...$$

- The constants (α's) may be determined in many ways.
- One simple method is to multiply by both sides by $(\tau_1 * s+1)$ and then set $s=-1/\tau_1$. This gives α_1. Then we find α_2, etc.
- α_0 may be found by letting $s=\infty$.
- Note that $\alpha_0=0$ except for systems with the same number as zeros as poles.

Example. f(s) = (-0.25s+1)/(10s-1)

This a bit unusual since the system is unstable (note the negative sign in d(s)), but it does not matter. We write

$$f(s) = (-0.25s+1)/(10s-1) = \alpha_0 + \alpha_1/(10s-1) \quad (*)$$

- To determine α_1: Multiply on both sides of (*) by 10s-1 and set s=0.1: $(-0.25s+1) = -0.025+1 = 0.975 = \alpha_1$
- To determine α_0: Let $s=\infty$ on both sides of (*): $-0.25/10 = \alpha_0$ -> $\alpha_0=-0.025$

Conclusion: f(s) = -0.025 + 0.975/(10s-1)

Example. $f(s) = \dfrac{k}{s(\tau s+1)}$

Use partial fraction expansion of f(s) to find f(t).

(Without using No. 13 in table)

Solution.

Partial fraction expansion into known terms:

$$f(s) = \frac{\alpha_1}{s} + \frac{\alpha_2}{\tau s+1}$$

Inverse Laplace of each term

$$f(t) = \alpha_1 + \frac{\alpha_2}{\tau} e^{-t/\tau}$$

Transfer function of first-order plus delay (FOD) process

- Dynamic model in time domain: $\tau \frac{dy(t)}{dt} = -y(t) + k\, u(t-\theta)$
- Laplace transform: $\tau\, sy(s) = -y(s) + k\, e^{-\theta s}\, u(s)$
- Rearrange : $y(s) = G(s)u(s)$ where
- $$\boxed{G(s) = k\frac{e^{-\theta s}}{\tau s+1}}$$
- $G(s)$ - Transfer function! Independent of what u(t) is (step, sinus, etc…)

Examples transfer functions

- First-order (concentration in tank)
- Integrating
- First-order with delay
- PID
- Higher order

See lectures

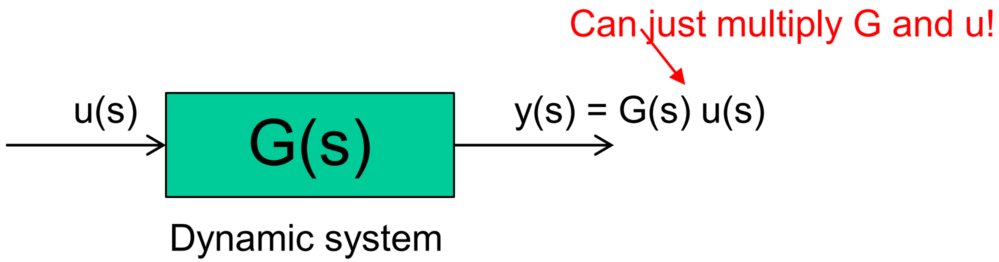
Use of Laplace Transforms in control

1. Standard notation in dynamics and control for linear systems (shorthand notation)
    - Independent variable: Change from t (time) to s (complex variable; inverse time)
    - Just a mathematical change in variables: Like going from x to y=log(x)

2. Converts differential equations to algebraic operations

3. Advantageous for block diagram analysis.

   Transfer function, G(s):



Note: Here I use capital letters for the transfer function G(s), but we may sometimes use lower case, g(s)

General procedure in this course

1. Nonlinear dynamic model
2. Steady state model: Use to find missing data
3. Introduce deviation variables and linearize*
4. Laplace of both sides of linear model (t !   s)
5. Algebra !   Transfer function, G(s)
6. Block diagram
7. Controller design

*Note: We will only use Laplace for linear systems!