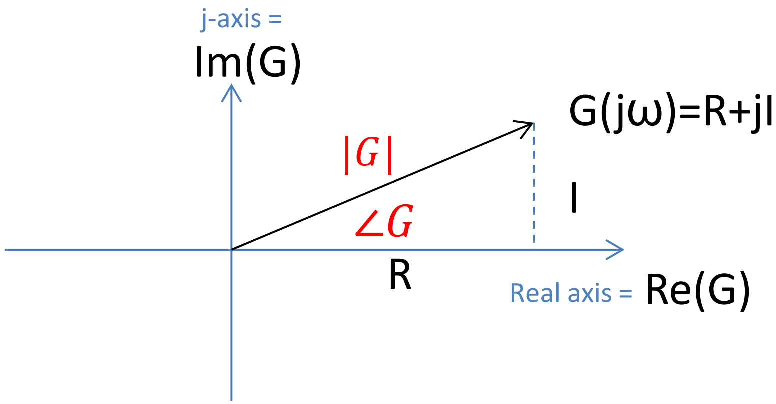
Frequency analysis

Important for understanding stability and robustness of feedback systems

Mathematics. Complex number G, $j^2=-1$



$\mathrm{G} = R + j\,I$

$Im(G)=R, Re(G)=I$

$|G| = \sqrt{R^2 + I^2}$

$\phi = \angle G = \arctan\frac{I}{R}$

Polar form

Examples:

$G = 2. \quad |G| = 2, \quad \angle G = \arctan 0 = 0° = 0\,rad$

$G = 3j. \quad |G| = 3, \quad \angle G = \arctan\frac{3}{0} = \arctan\infty = 90° = \frac{\pi}{2}\,rad = 1.57\,rad$

$G = 2 + 3j.\, |G| = \sqrt{2^2 + 3^2} = \sqrt{13} = 3.61, \quad \angle G = \arctan\frac{3}{2} = 56.3° = 0.983\,rad$

$G = \frac{1}{3j} = -\frac{1}{3}j\,, \quad |G| = \frac{1}{3} = 0.33, \quad \angle G = -90° = -\frac{\pi}{2}\,rad$

Euler's formula

Polar form:

$$G = R + jI = |G|(cos\angle G + jsin\angle G) = |G|e^{j\angle G}$$

Note: $e^{j\pi} = -1$

May also use i instead of j, $i^2=-1$. (i is common in mathematics; j is more common in control)

arctan = arctg = atan = $\tan^{-1}$

From degrees to radians: Multiply by $\frac{\pi}{180°}$

Polar form

Multiply complex numbers:

Multiply magnitudes and add phases

$$G = G_1 \cdot G_2 \cdot G_3$$
$$|G| = |G_1| \cdot |G_2| \cdot |G_3|$$
$$\angle G = \angle G_1 + \angle G_2 + \angle G_3$$

$$Similar - for - ratio:$$
$$G = \frac{G_1}{G_2}$$
$$|G| = |G_1|/|G_2|$$
$$\angle G = \angle G_1 - \angle G_2$$

$G_1 = 2 + 4j,$
$G_2 = -3 + 3j$
$|G_1| = \sqrt{2^2 + 4^2} = \sqrt{20} = 4.47, \angle G_1 = \mathrm{atan}\left(\frac{4}{2}\right) = 63.3° = 1.107\, rad$
$|G_2| = \sqrt{3^2 + 3^2} = \sqrt{18} = 4.24, \angle G_2 = \mathrm{atan}\left(\frac{-3}{3}\right) = -45° = -0.785\, rad$

$G = G_1 \cdot G_2,$
$|G| = 4.47 \cdot\ 4.24 =\ 18.953,\qquad \angle G = 63.3° + (-45°) =\ 18.3° = 0.322\, rad$
$\mathrm{G} =\ 18.953e^{j0.322}$

$G = \frac{G_1}{G_2}\,,\qquad |G| = \frac{4.47}{4.24} = 1.054,\qquad \angle G = 63.3° - (-45°) = 108.3° = 1.89\, rad$

Force linear system with sinusoidal input: $u(t) = u_0 \sin \omega t$

Output has same frequency: $y(t) = y_0 \sin(\omega t + \varphi)$

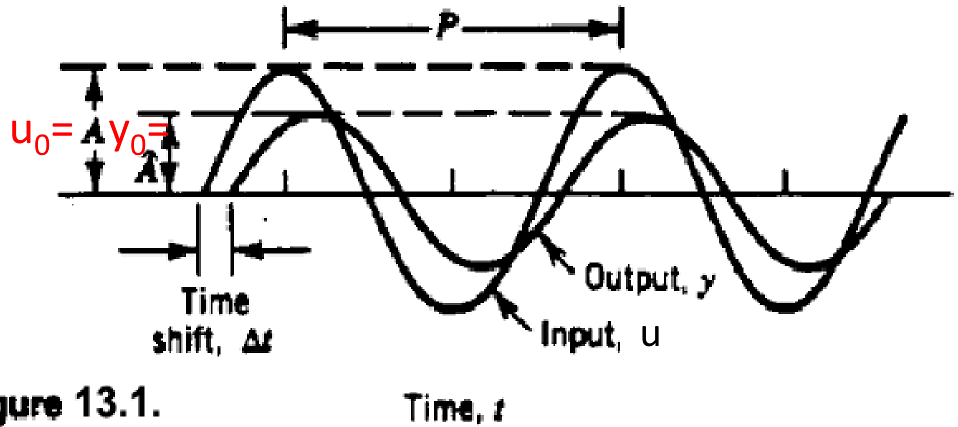


Figure 13.1.

Attenuation and time shift between input and output sine waves The phase angle ϕ of the output signal is given by $\phi = -\Delta t/P \times 360$, where Δt is the time (period) shift and P is the period of oscillation.

Period: P[s]

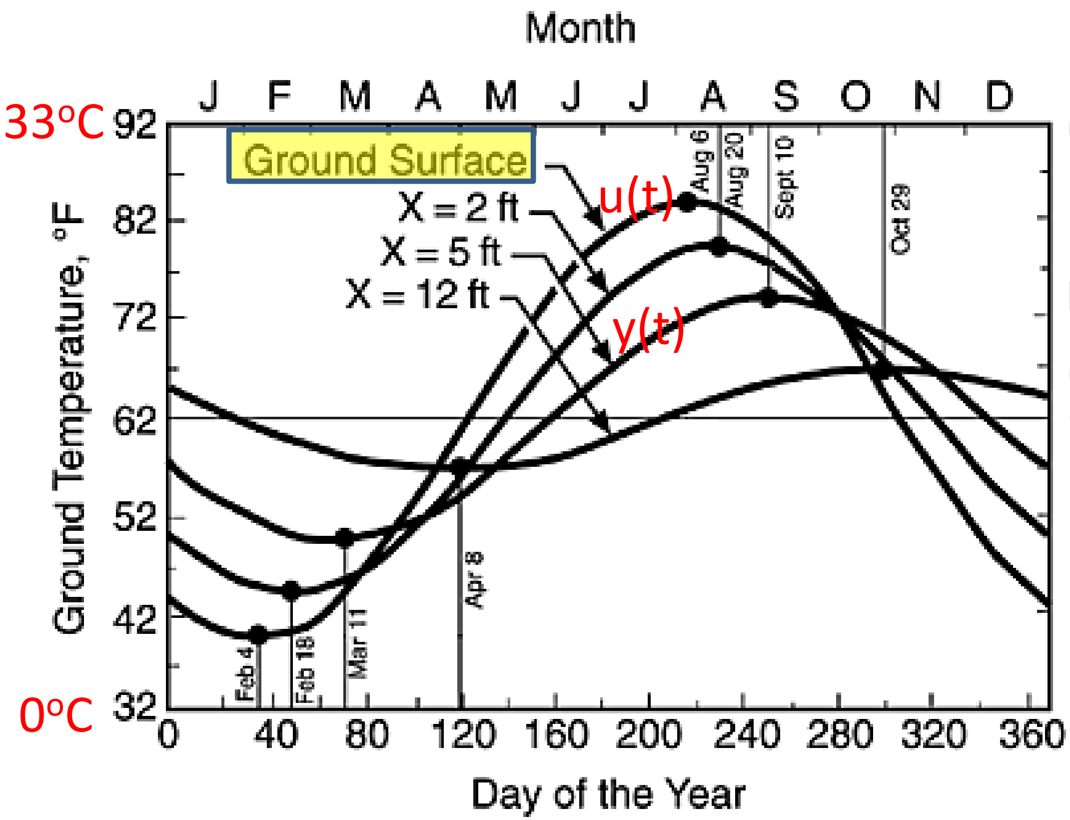
Frequency: ω [rad/s] = $2\pi / P$

Phase shift: φ [rad] $= -\frac{\Delta t}{P} 2\pi = -\Delta t \cdot \omega$

Amplitude ratio (gain): AR = y_0/u_0

- We have assumed deviation variables, otherwise we need to add an «average» or «bias» to both u(t) and y(t).
- We assume that the input sinusoid is persistent and consider the «steady-state» as $t \to \infty$.
- One period (cycle) = 2π [rad] = 360^o

Example: Ground temperature phase shift



Surface temperature:

$$u(t) = u_{avg} + u_0 \sin(\omega(t-t_0))$$

- u_{avg} = 62°F (= 17C)
- Amplitude u_0 = 62-40=22°F

Frequency of oscillations:

- P = 365d,
- $\omega = \frac{2\pi}{P} = \frac{2\pi}{365} = 0.017$ rad/d,

Ground temperature:

$$y(t) = y_{avg} + y_0 \sin(\omega(t-t_0) + \varphi)$$

- y_{avg} = 62°F (= 17C)
- y_0 and φ depend on depth X

Amplitude (y_0) and Gain decreases with depth X.

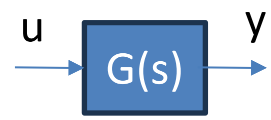
- X=5ft, y_0 = 62-50=12 F,
- Gain = AR = $\frac{y_0}{u_0} = \frac{12}{22} = 0.55$

Time shift and phase shift increases with depth X.

- X=5 ft, Δt = 35 days (=from Feb.4 to Mar. 11 = from Aug. 6 to Sep.10):
- $\varphi = -\Delta t\,\omega$ = -35d * 0.017 rad/d= - 0.602 rad = -34.5°

AR = amplitude ratio = gain = $\frac{y_0}{u_0}$

General: Simple method to find sinusoidal response of system G(s)

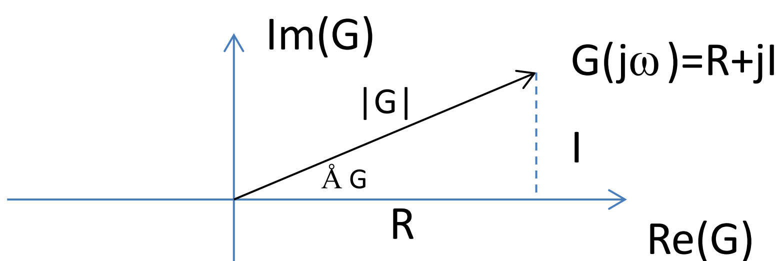


1. Input signal to linear system: $u = u_0 \sin(\omega t)$
2. Steady-state ("persistent", $t \rightarrow \infty$) output signal: $y = y_0 \sin(\omega t + \varphi)$
3. What is $AR = y_0/u_0$ and φ?

Solution (extremely simple!)

1. Find system transfer function, G(s)
2. Let $s=j\omega$ (imaginary number, $j^2=-1$) and evaluate $G(j\omega) = R + j\,I$ (complex number)
3. Then ("believe it or not!")

$AR = \lvert G(j\omega) \rvert$	(magnitude of the complex number)
$\varphi = \angle\, G(j\omega)$	(phase of the complex number)



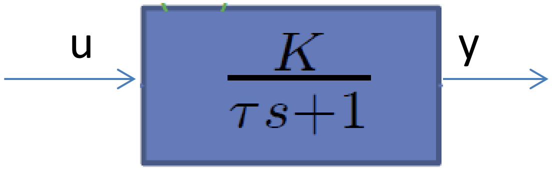
Proof: $y(s) = G(s)u(s)$ where $u(s) = \frac{u_0\omega}{s^2+\omega^2} = \frac{u_0\omega}{(s-j\omega)(s+j\omega)}$, etc...

(poles of $G(s)$ "die out" as $t \rightarrow \infty$)

Term $\frac{1}{s-j\omega}$ gives $G(j\omega)$ with partial fraction expansion

Proof, first-order system

$u(t) = A \sin(\omega t)$



As $t \rightarrow \infty$: (see 4-26)

$y(t) = AR * A * \sin(\omega t+\varphi)$

$$AR = \frac{K}{\sqrt{\omega^2\tau^2+1}}$$

$$\phi = -\tan^{-1}(\omega\tau)$$

Note: A is the same as u_0

General (VERY SIMPLE).
Set s=jω in G(s). Then
AR = |G(jω)|
φ = Å G(jω)

4.2.3 Sinusoidal Response

As a final example of the response of first-order processes, consider a sinusoidal input $\boxed{u_{sin}(t) = A \sin \omega t,}$ with transform given by Eq. (4-15): $\boxed{u(s) = A\frac{\omega}{s^2+\omega^2}}$

$$y(s) = \frac{KA\omega}{(\tau_s + 1)(s^2 + \omega^2)} \qquad (4\text{-}23)\ (5\text{-}22)$$

$$= \frac{KA}{\omega^2\tau^2+1}\left(\frac{\omega\tau^2}{\tau s+1} - \frac{s\omega\tau}{s^2+\omega^2} + \frac{\omega}{s^2+\omega^2}\right) \qquad (4\text{-}24)$$

Inversion gives

$$y(t) = \frac{KA}{\omega^2\tau^2 + 1}(\omega\tau e^{-t/\tau} - \omega\tau \cos \omega t + \sin \omega t) \qquad (4\text{-}25)$$

or, by using trigonometric identities,

$$y(t) = \frac{KA\omega\tau}{\omega^2\tau^2+1}e^{-t/\tau} + \frac{KA}{\sqrt{\omega^2\tau^2+1}}\sin(\omega t + \phi) \qquad (4\text{-}26)$$

where

$$\phi = -\tan^{-1}(\omega\tau) \qquad (4\text{-}27)$$

Notice that in both (4-25) and (4-26) the exponential term goes to zero as $t \rightarrow \infty$, leaving a pure sinusoidal response. This property is exploited in Chapter 13 for frequency response analysis.

Example: Gain and phase shift for first-order system:

1. $$G(s) = \frac{k}{\tau s + 1}$$

2. $$G(j\omega) = \frac{k}{1+\tau j\omega} \cdot \frac{1-\tau j\omega}{1-\tau j\omega} \qquad (j^2 = -1)$$

$$G(j\omega) = \underbrace{\frac{k}{1+\omega^2\tau^2}}_{\mathbf{R}} - \underbrace{\frac{k\ \ \omega\tau}{1+\omega^2\tau^2}}_{\mathbf{I}} j$$

3. $$|G| = AR = \sqrt{R^2 + I^2} = \frac{k}{\sqrt{1+\omega^2\tau^2}}$$

$$\phi = \angle G = \arctan\frac{I}{R} = -\arctan(\omega\tau)$$

Gain and phase shift

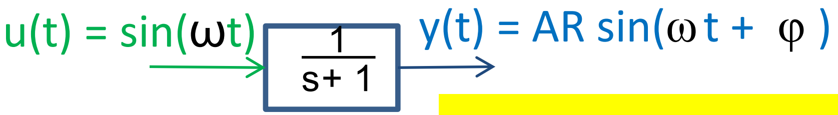
This method is not really recommended

SIMPLER: Use polar form formulas for complex numbers! $G=G_1/G_2$, where $G_1=k$, $G_2=\tau s+1$.

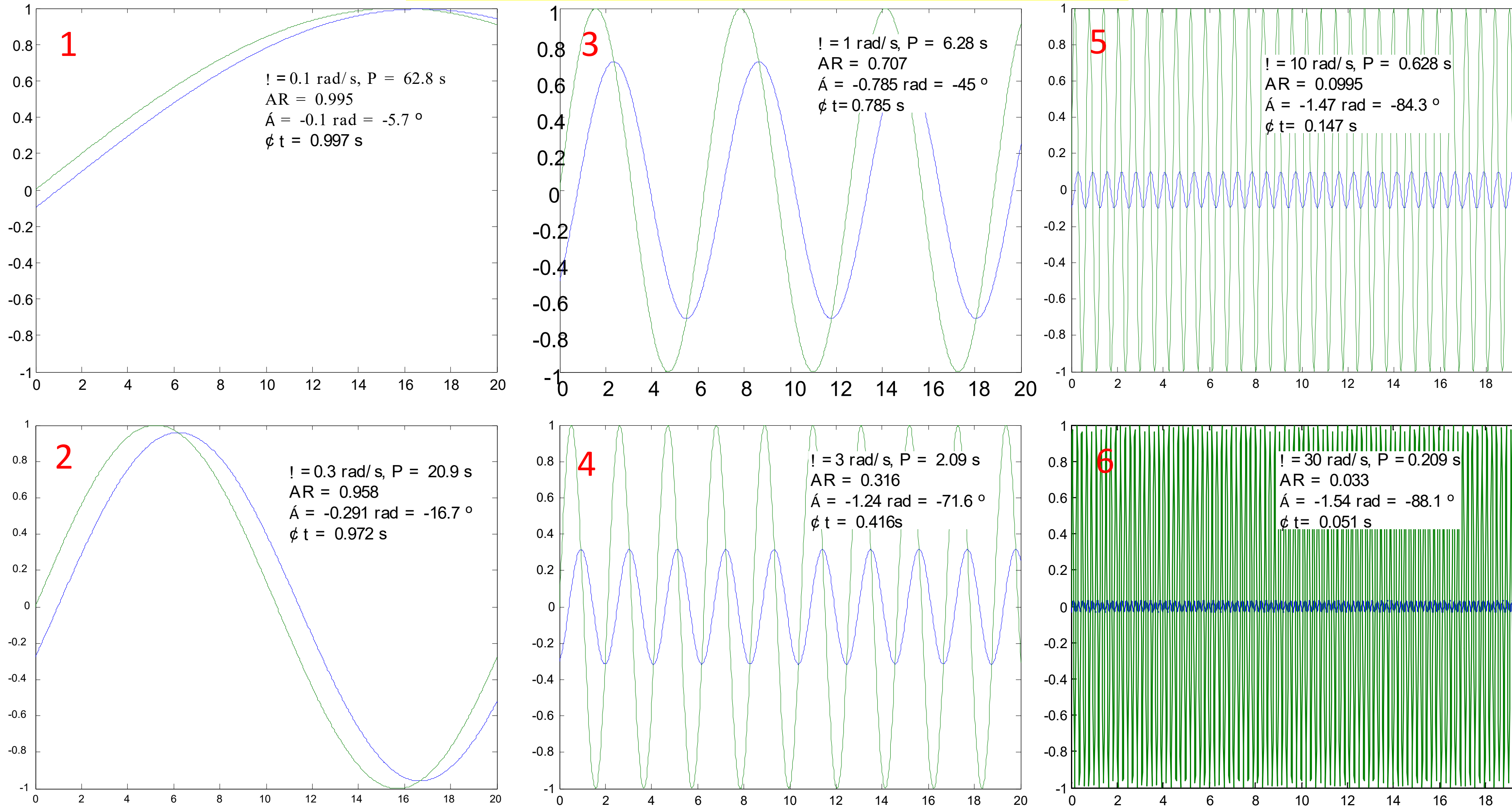
Set $s = j\omega$. *Get*: $|G| = \frac{|G_1|}{|G_2|} = \frac{k}{\sqrt{(\omega\tau)^2+1}}$, $\phi = \angle G = \angle G_1 - \angle G_2 = 0 - \arctan(\omega\tau)$

SINUSOIDAL RESPONSE OF FIRST-ORDER SYSTEM

$k = 1,\ \tau = 1\ [s]$



6 Plots: Increase ω from 0.1 to 30 rad/s




```
w=0.3; tau=1; t = linspace(0,20,1000);
u = sin(w*t);
AR = 1/sqrt((w*tau)^2+1)
phi = - atan(w*tau), phig=phi*180/pi, dt=-phi/w
y = AR*sin(w*t+phi);
plot(t,y,t,u)
```

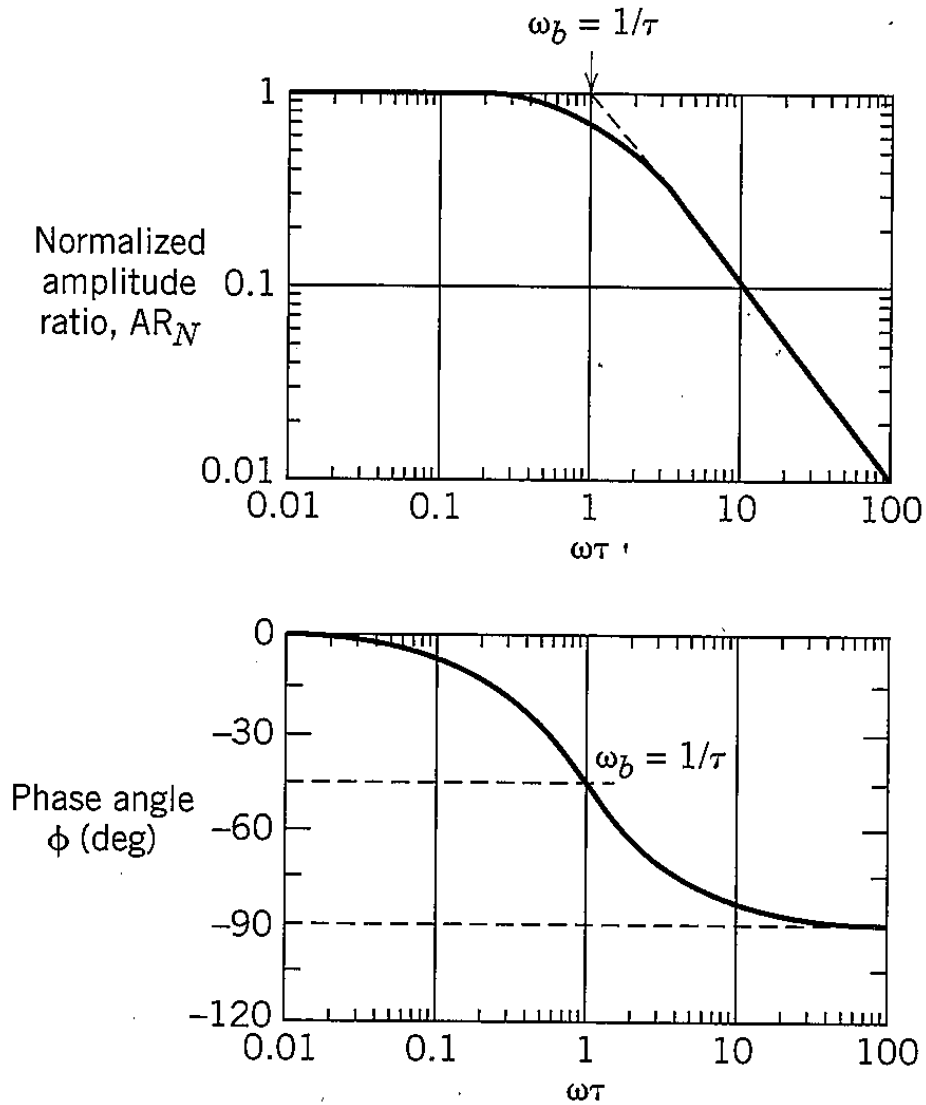


Figure 14.2 Bode diagram for a first-order process

$$AR = |G(j\omega)| = \frac{1}{\sqrt{(\omega\tau)^2+1}}$$

$$\phi = \angle G(j\omega) = -\arctan(\omega\tau)$$

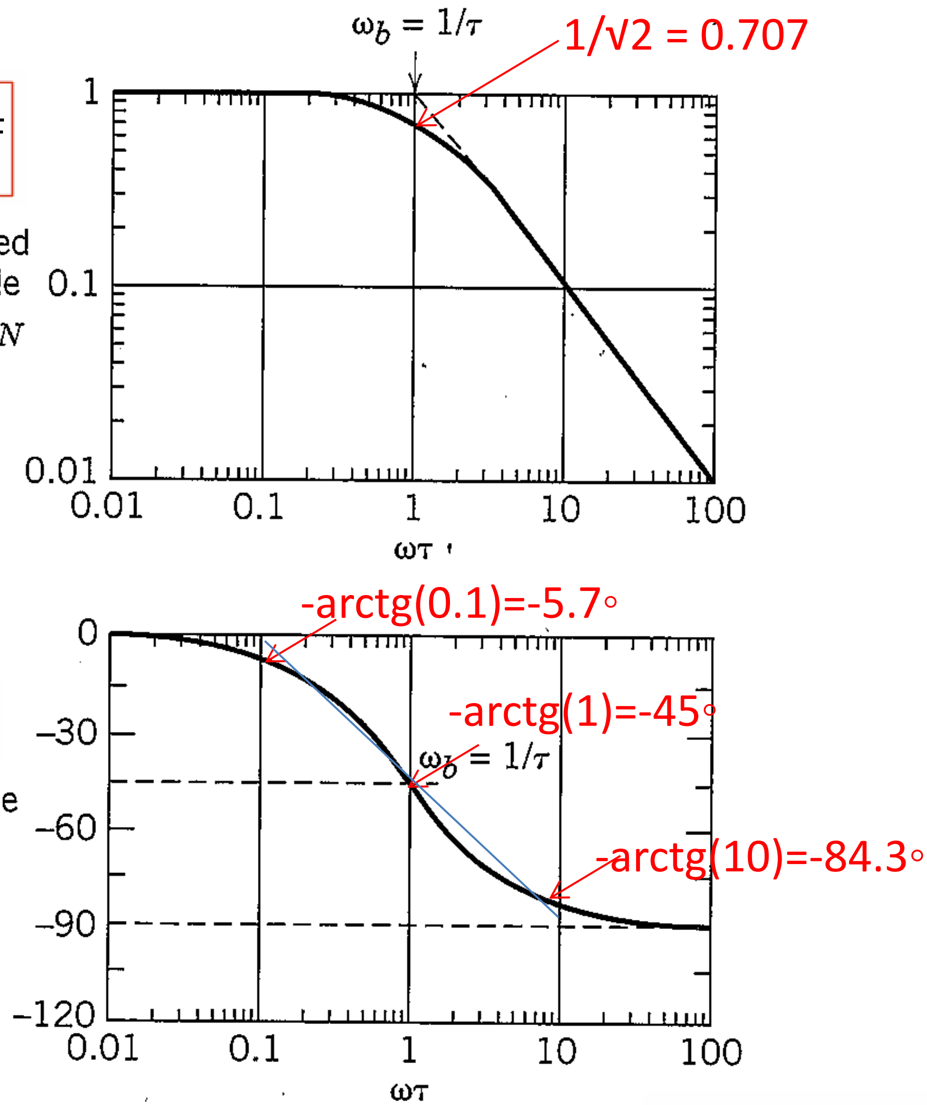


Figure 14.2 Bode diagram for a first-order process

$$G(s) = \frac{1}{\tau s+1}$$

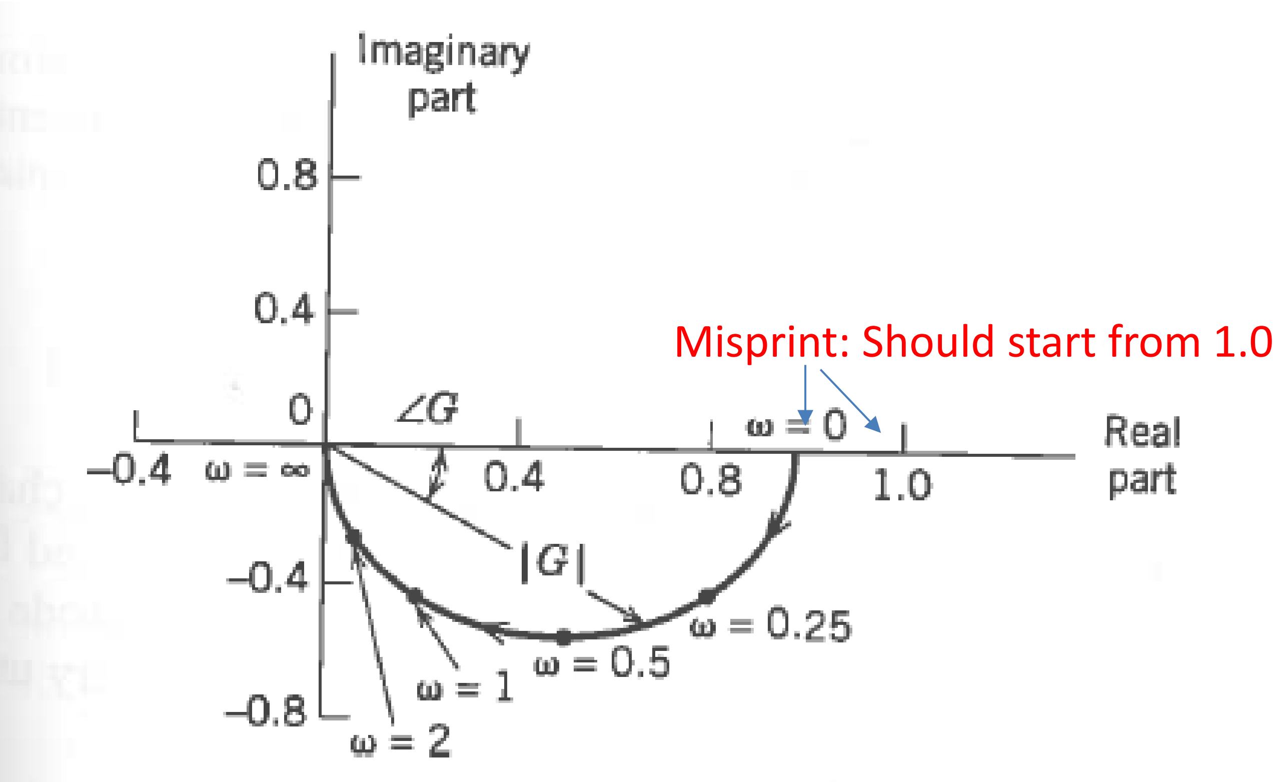


Figure 13.12 The Nyquist diagram for $G(s) = 1/(2s + 1)$ plotting $\mathrm{Re}(G(j\omega))$ and $\mathrm{Im}(G(j\omega))$.

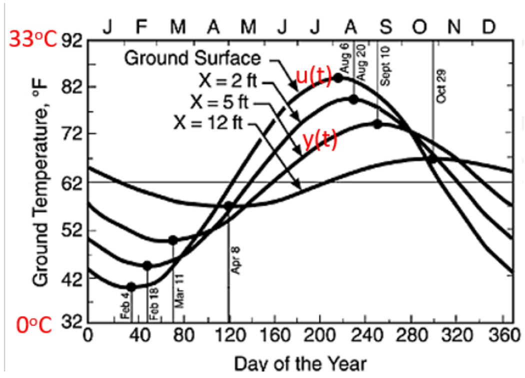
Note: Nyquist plot is not included in last edition

Example: Ground temperature phase shift. X=5ft

What is τ if assume a first-order response from u to y? g(s) = k/(τs+1)

$$\mathrm{AR} = \frac{y_0}{u_0} = \frac{k}{\sqrt{(\omega\tau)^2+1}}$$

$$\phi = \arctan(-\omega\tau)$$



Data: u_0 = A= 22, y_0 = 12, ω=0.017 rad/d, φ = – 35°

Solution:

- We know from physics that the gain k=1. So g(s) = 1/(τs+1)

1. From amplitude data: AR = y_0/u_0 = 0.545.

   Get: $$\tau = \frac{1}{\omega}\sqrt{\frac{1}{\mathrm{AR}^2}-1} = \frac{1}{0.017}\sqrt{\frac{1}{0.545^2}-1} = 90.5d$$

2. From phase shift data. φ = – 35°

   Get: $$\tau = -\frac{1}{\omega}\tan\phi = -\frac{1}{0.017}\tan(-0.568) = 37.4d$$

Conclusion: This system is more complex than first order (no big surprise!)

It's described by partial differential equations and can be approximated by a high-order system with many poles and zeros.

For example, g(s) = (τ_2s+1) / (τ_1s+1) (τ_3s+1) where $\tau_1 > \tau_2 > \tau_3$

Frequency response of time delay

$g=e^{-\theta s}$

Gain = $|g(j\omega)| = 1$

Phase shift = $\varphi = \angle(g(j\omega)) = -\omega\theta$ [rad]

Alternative proof: Time domain

$u(t)$

$y(t)$

General:

$$g(s) = k\ \frac{g_1\ g_2}{g_3\ g_4}\ e^{-\theta s}$$

$$|g| = k\ \frac{|g_1||g_2|}{|g_3|\ |g_4|}$$

$$\angle g| = \angle g_1 + \angle g_2 - \angle g_3 - \angle g_4 - \omega\theta$$

Consider term:

$$g_a = Ts + 1$$

Set $s = j\omega$ and evaluate complex number $g_a(j\omega)$ with magnitude $|g_a|$ and phase $\angle g_a$. Get:

$$|g_a(j\omega)| = \sqrt{\omega^2 T^2 + 1};$$
$$\angle g_a = \arctan \omega T$$

Example 2

$$g(s) = \frac{k(Ts+1)}{(\tau_1 s+1)(\tau_2 s+1)} = \frac{g_1 \; g_2}{g_3 \; g_4}$$

$$g_1 = k$$
$$g_2 = Ts + 1$$
$$g_3 = \tau_1 s + 1$$
$$g_4 = \tau_2 s + 1$$

Solution: $|g(j\omega)| = \frac{|g_1|\cdot|g_2|}{|g_3|\cdot|g_4|}$, $\quad \angle g(j\omega) = \angle g_1 + \angle g_2 - \angle g_3 - \angle g_4$

$$|g_1| = k\,, \qquad \angle g_1 = 0$$
$$|g_2| = \sqrt{1 + (\omega T)^2}\,, \qquad \angle g_2 = \text{atan}\,(\omega T)$$
$$|g_3| = \sqrt{1 + (\omega \tau_1)^2}, \qquad \angle g_3 = \text{atan}\,(\omega \tau_1)$$
$$|g_4| = \sqrt{1 + (\omega \tau_1)^2}, \qquad \angle g_4 = \text{atan}\,(\omega \tau_1)$$

If we also have a time delay $g_5 = \; e^{-\theta s} \; then$

$$|g_5| = 1\,, \angle g_5 = -\omega\theta \; [rad]$$

1. DERIVATIVE

$$g_1(s) = s$$

Frequency response: $g(j\omega) = j\omega = 0 + j\omega$

$$|g_1(j\omega)| = \omega$$
$$\angle g_1(j\omega) = 90^o = \pi/2 \text{ rad (purely complex at all } \omega)$$

Check:

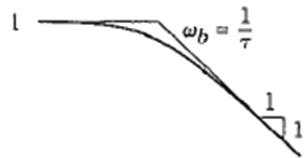
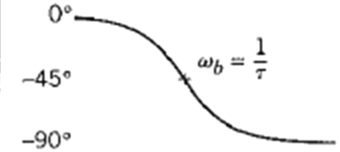
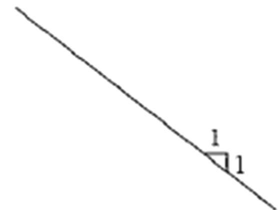
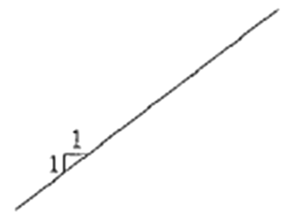
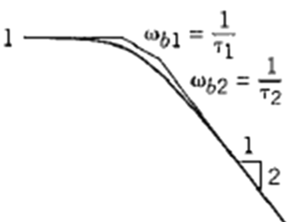
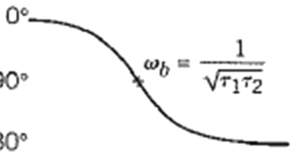
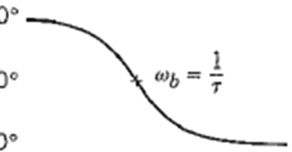
$$u(t) = u_0 sin(\omega t)$$
$$y(t) = u'(t) = u_0 \omega cos(\omega t) = \omega u_0 sin(\omega t + \pi/2)$$ OK!

2. INTEGRATOR

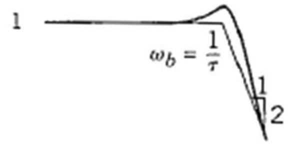
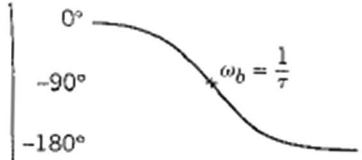
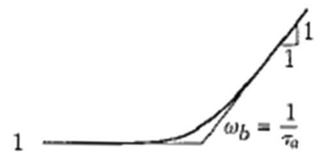
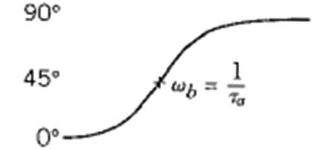
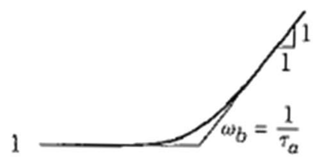
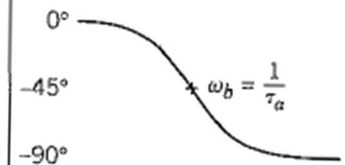
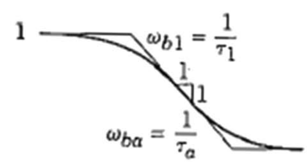
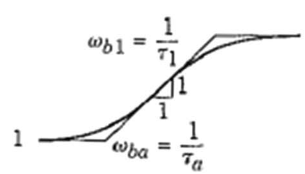
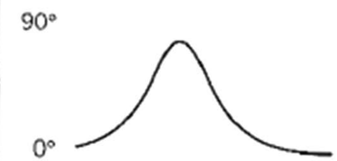
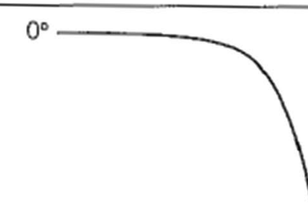
$$g_2(s) = \frac{1}{s} = \frac{1}{g_1}$$

$$|g_2(j\omega)| = \frac{1}{|g_1|} = \frac{1}{\omega}$$
$$\angle g_2(j\omega) = 0^o - \angle g_1 = -90^o = -\pi/2 \text{ rad}$$

Table 13.2 Frequency Response Characteristics of Important Process Transfer Functions

Transfer Function	$G(s)$	$\text{AR} = \lvert G(j\omega)\rvert$	Plot of log AR_N vs. log ω	$\phi = \angle G(j\omega)$	Plot of ϕ vs. log ω
1. First-order	$\dfrac{K}{\tau s + 1}$	$\dfrac{K}{\sqrt{(\omega\tau)^2 + 1}}$		$-\tan^{-1}(\omega\tau)$	
2. Integrator	$\dfrac{K}{s}$	$\dfrac{K}{\omega}$		$-90°$	
3. Derivative	Ks	$K\omega$		$+90°$	
4. Overdamped second-order	$\dfrac{K}{(\tau_1 s + 1)(\tau_2 s + 1)}$	$\dfrac{K}{\sqrt{(\omega\tau_1)^2 + 1}\sqrt{(\omega\tau_2)^2 + 1}}$		$-\tan^{-1}(\omega\tau_1) - \tan^{-1}(\omega\tau_2)$	
5. Critically damped second-order	$\dfrac{K}{(\tau s + 1)^2}$	$\dfrac{K}{(\omega\tau)^2 + 1}$		$-2\tan^{-1}(\omega\tau)$	

Peak goes to infinity when ξ=0

6. Underdamped second-order	$\frac{K}{\tau^2 s^2 + 2\zeta\tau s + 1}$	$\frac{K}{\sqrt{(1-(\omega\tau_1)^2)^2+(2\zeta\omega\tau)^2}}$		$-\tan^{-1}\left[\frac{2\zeta\omega\tau}{1-(\omega\tau)^2}\right]$	
7. Left-half plane (positive) zero	$K(\tau_a s + 1)$	$K\sqrt{(\omega\tau_a)^2+1}$		$+\tan^{-1}(\omega\tau_a)$	 Phase increases for LHP zero
8. Right-half plane (negative) zero	$-\tau_a s + 1$	$K\sqrt{(\omega\tau_a)^2+1}$		$-\tan^{-1}(\omega\tau_a)$	 Oops! Phase drops for RHP zero
9. Lead–lag unit ($\tau_a < \tau_1$)	$K\frac{\tau_a s+1}{\tau_1 s+1}$	$K\frac{\sqrt{(\omega\tau_a)^2+1}}{\sqrt{(\omega\tau_1)^2+1}}$		$+\tan^{-1}(\omega\tau_a) - \tan^{-1}(\omega\tau_1)$	
10. Lead–lag unit ($\tau_a > \tau_1$)	$K\frac{\tau_a s+1}{\tau_1 s+1}$	$K\frac{\sqrt{(\omega\tau_a)^2+1}}{\sqrt{(\omega\tau_1)^2+1}}$		$+\tan^{-1}(\omega\tau_a) - \tan^{-1}(\omega\tau_2)$	
11. Time delay	$Ke^{-\theta s}$	K		$-\omega\theta$	

Bode plot of time delay.

$G = \exp(-\theta s)$

$|G(j\omega)| = 1$

$\angle G(j\omega) = -\omega\theta$

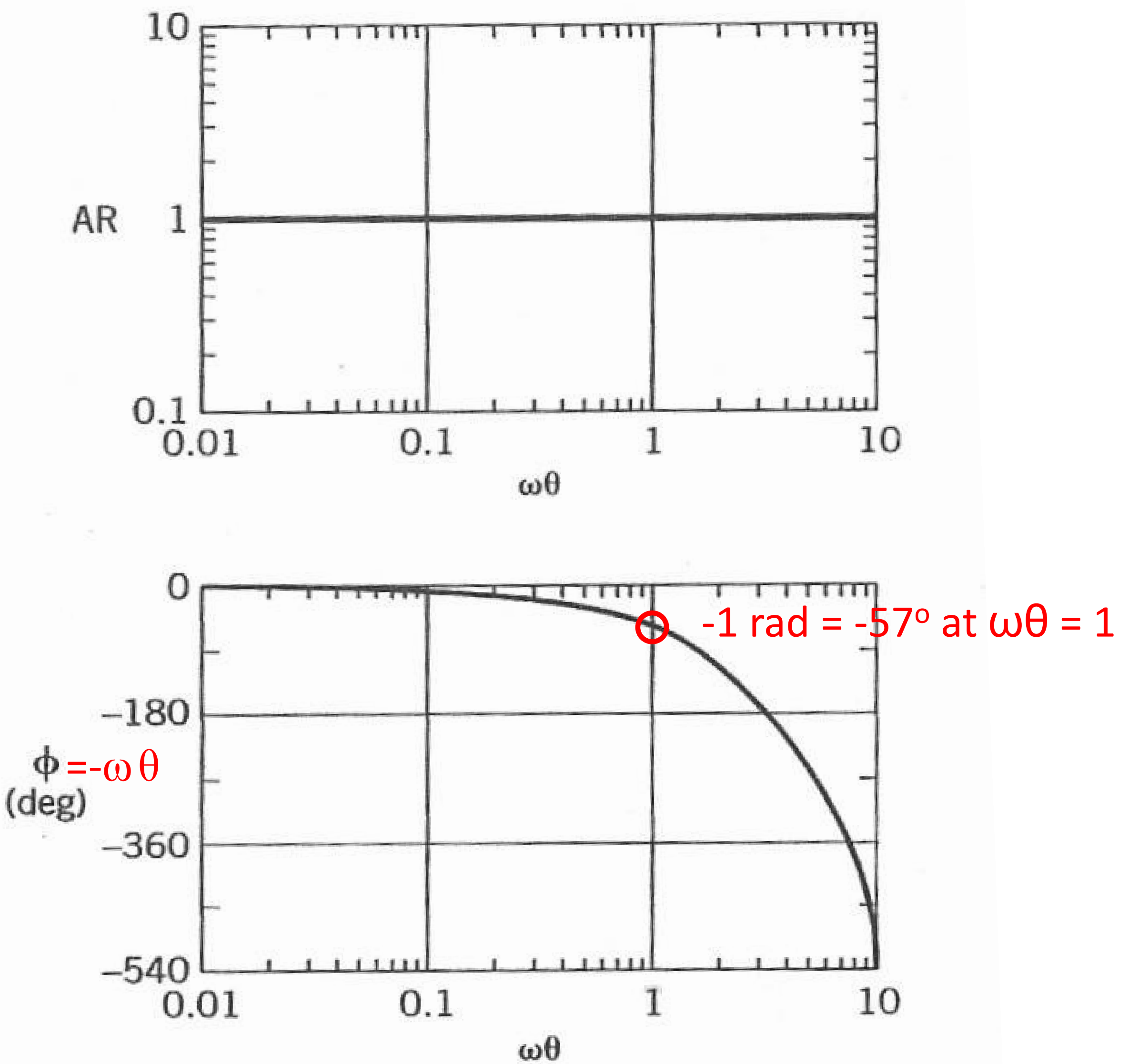


Figure 14.4 Bode diagram for a time delay, $e^{-\theta s}$.

ASYMPTOTES

Frequency response of term (Ts+1): set $s=j\omega$.
Asymptotes:

$(j\omega T + 1) \sim 1$ for $\omega T << 1$ (slope n=0, phase=0)
$(j\omega T + 1) \sim j\omega T$ for $\omega T >> 1$ (slope n=1, phase=90^o)

Gain slope n: $|G| \sim \omega^n$

Rule for asymptotic Bode-plot, $L = k(Ts+1)/(\tau s+1)$….. :

1. Start with low-frequency asymptote ($s \rightarrow 0$)
   (a) If constant ($L(0)=k$):
      Gain=k (slope=0)
      Phase=0^o
   (b) If integrator ($L=k'/s$):
      Gain slope= -1 (on log-log plot). Need one fixed point, for example, gain=1 at $\omega=k'$
      Phase: -90^o.

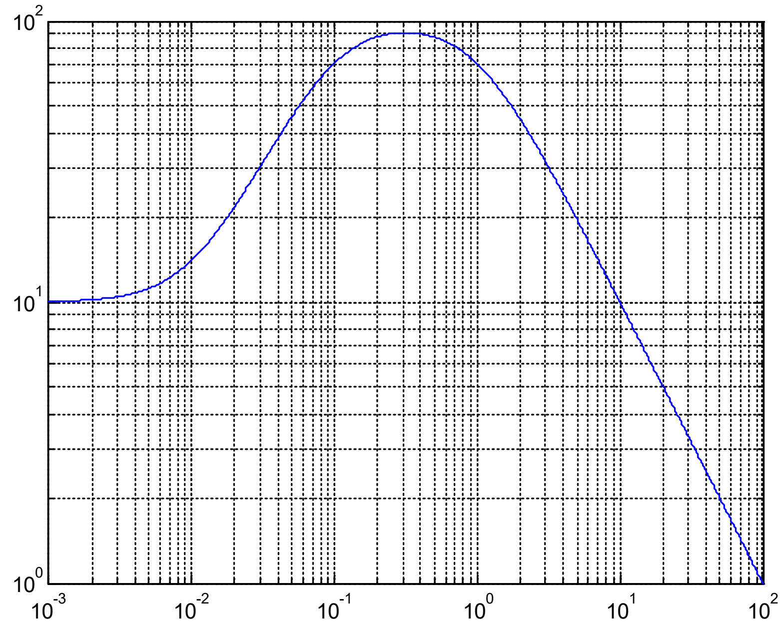
2. Break frequencies (order … from large T and τ … to small T and τ):

	Change in gain slope	Change in phase
$\omega=1/T$ (zero)	+1	$+90^o$ (-90^o if T negative)
$\omega=1/\tau$ (pole)	-1	-90^o ($+90^o$ if τ negative)

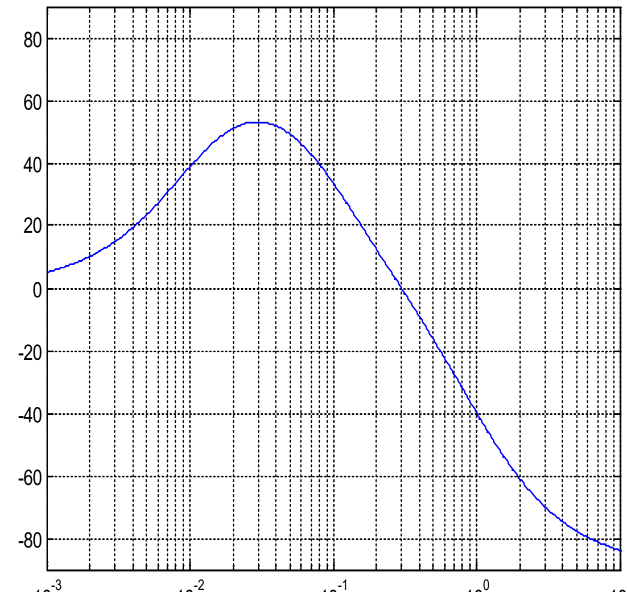
3. Time delay, $e^{-\theta s}$. Gain: no effect, Phase contribution: $-\omega\theta$ [rad] (-1 rad = -57^o at $\omega=1/\theta$)

Example with phase lead (not so common in process control)

$$g(s) = 10\frac{100s+1}{(10s+1)(s+1)}$$



Frequency
[rad/s]



Low-frequency asymptote:

$$G_{c0} = \frac{2}{10s} = \frac{0.2}{s}.$$

"Fixed point":

$$\omega = 0.001 \Rightarrow |G_{c0}| = \frac{0.2}{0.001} = 200$$

PI-controller:

$$G_c(s) = 2\frac{10s+1}{10s}$$

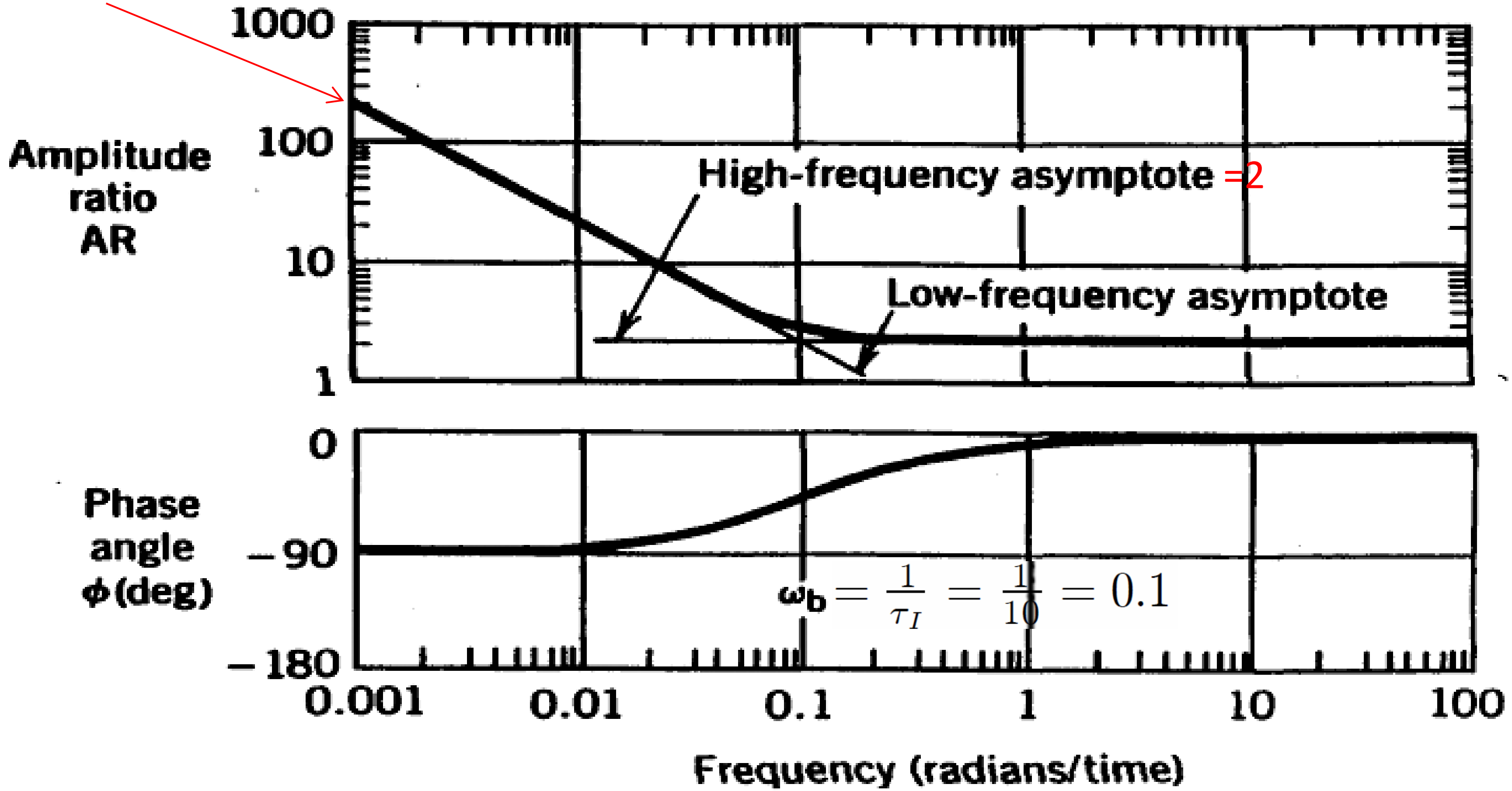
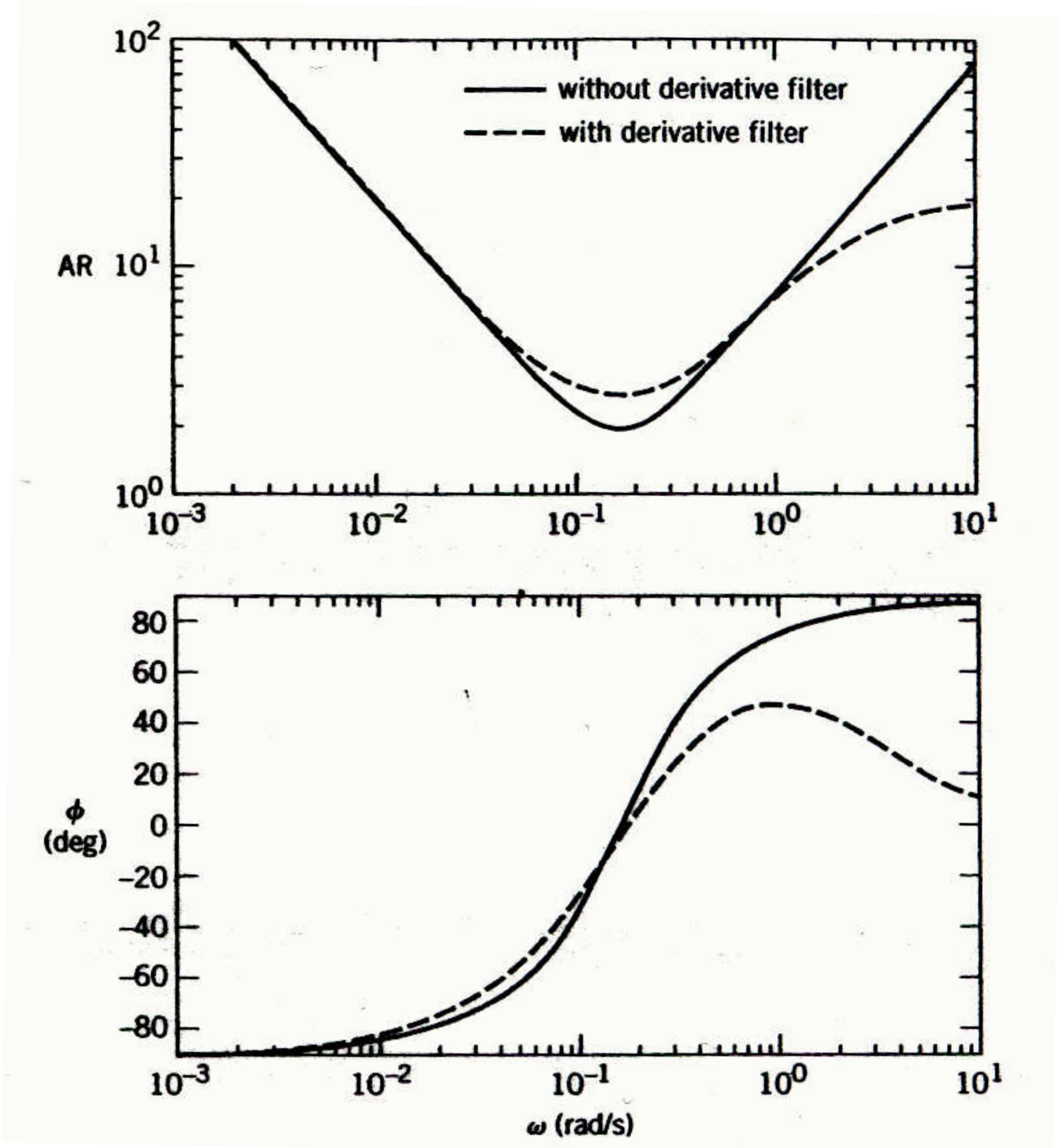


Figure 13.9. Bode plot for PI controller, $G_c(s) = 2\left(1 + \frac{1}{10s}\right)$



Bode plots of ideal parallel PID controller and series PID controller with filter.

Ideal parallel:

$$C(s) = 2\left(1 + \frac{1}{10s} + 4s\right)$$

Series with Derivative Filter:

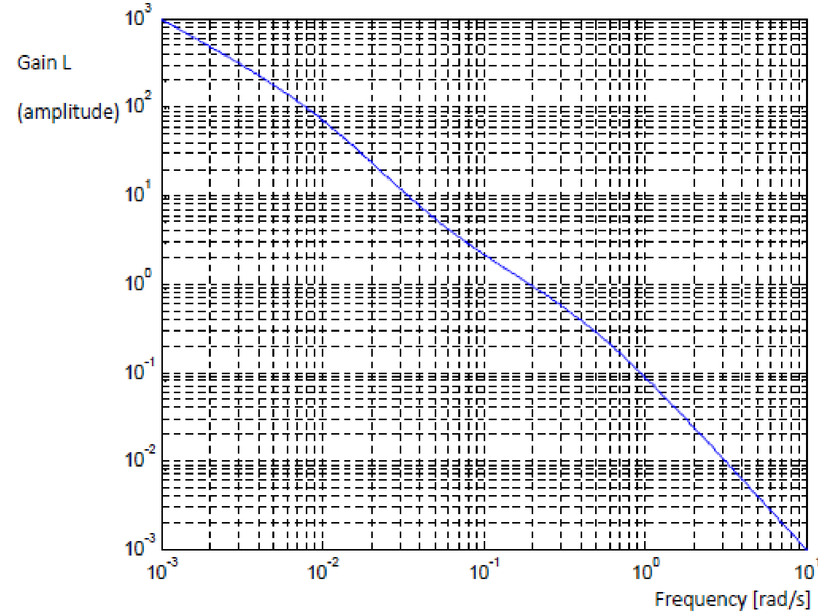
$$C(s) = 2\left(\frac{10s+1}{10s}\right)\left(\frac{4s+1}{0.4s+1}\right)$$

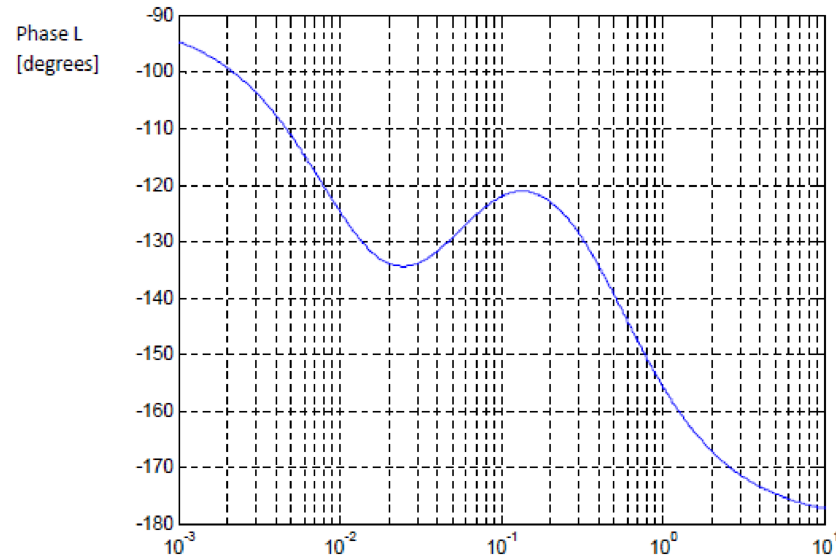
Example: Typical L=GC

EXAMPLE

$$L(s) = \frac{20s+1}{s(100s+1)(2s+1)}$$

L(s)=G(s)C(S):
Loop transfer function for
SIMC PI-control with τ_c=4 for
G(s) = 1/(100s+1)(2s+1)






```
s=tf('s')
L = (20*s+1)/[s*(100*s+1)*(2*s+1)]
figure(3), bode(L) % gives AR in dB
w = logspace(-3,1,1000)
[mag,phase]=bode(L,w)
figure(1), loglog(w,mag(:)), grid on, axis([0.001 10 0.001 1000])
figure(2), semilogx(w,phase(:)), grid on
```

SOLUTION

$$L(s) = \frac{20s+1}{s(100s+1)(2s+1)}$$

L(s): SIMC PI-control with τ_c=4 for g(s) = 1/(100s+1)(2s+1)

Low-frequency asymptote ($s = j\omega \to 0$) is integrator: $L = \frac{1}{j\omega} = -\frac{1}{\omega}j$
Gain $= \frac{1}{\omega}$ (slope -1 on log-log),
Phase = -90^o

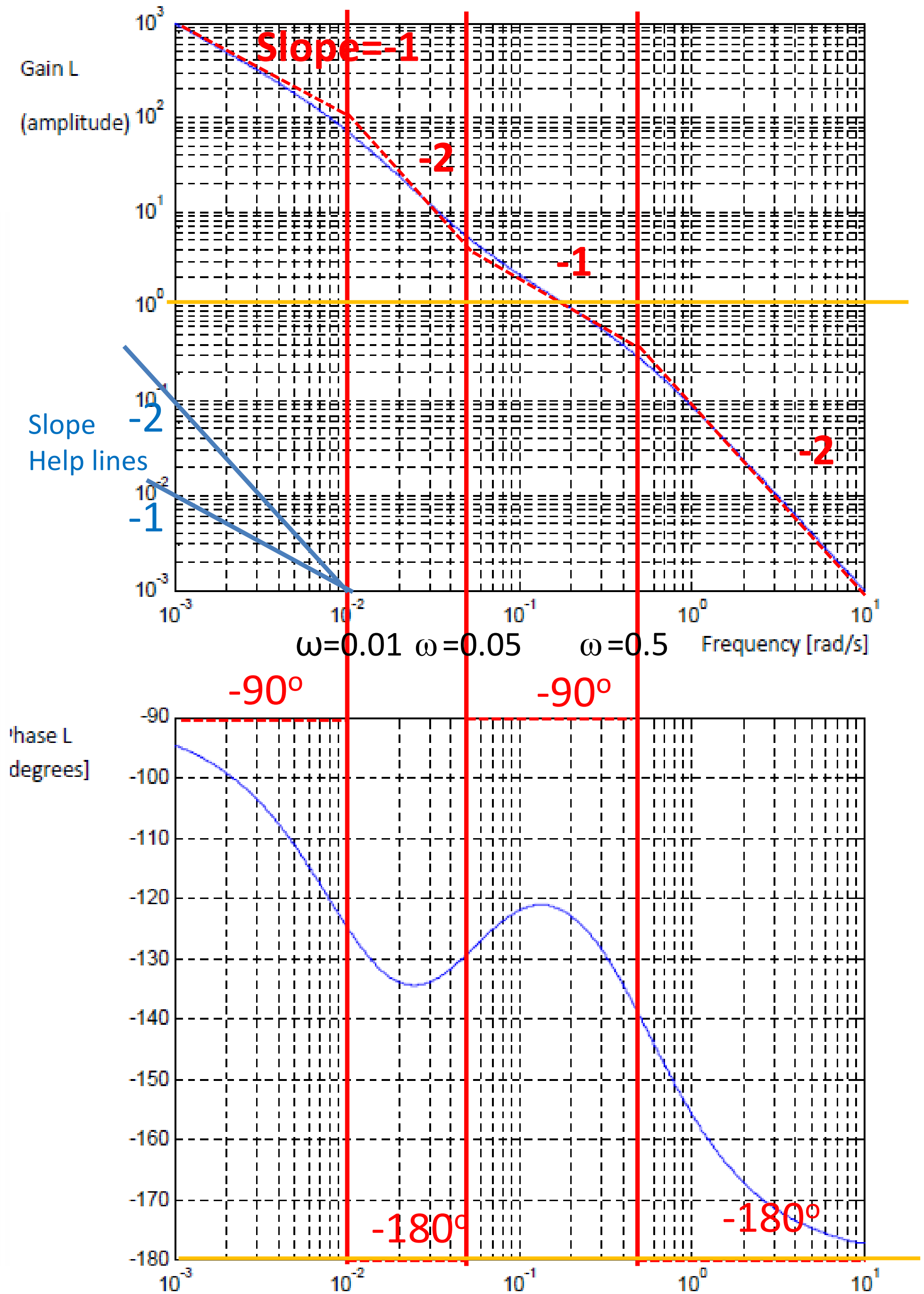
Asymptotes: Start at low frequency, $\omega \to 0$: $|L(j\omega)| = 1/\omega$. So: $|L|=10^3$ at $\omega=10^{-3}$

Break frequencies:
$\omega = 1/100 = 0.01$ (pole), $1/20 = 0.05$ (zero), $1/2 = 0.5$ (pole)

First break frequency (at 0.01) is a pole:
Slope changes by -1 to -2 (log-log)
$\Rightarrow$ gain drops by factor 100 when ω increases by factor 10
Phase drops by -90^o to -180^o
Asymptote $= \frac{1}{100(j\omega)^2} = -\frac{1}{100\omega^2}$

Next break frequency (at 0.05) is a zero:
Slope changes by +1 to -1 (log-log)
Phase increases by +90^o to -90^o
Asymptote $= \frac{20}{100j\omega} = -\frac{1}{5\omega}j$

Final break frequency (at 0.5) is a pole:
Slope changes by -1 to -2 (log-log)
Phase drops by -90^o to -180^o
Asymptote $= \frac{1}{10(j\omega)^2} = -\frac{1}{10\omega^2}$

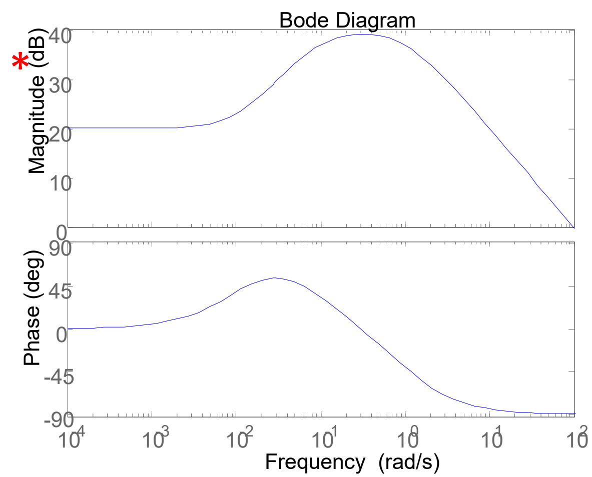


Electrical engineers (and Matlab) use decibel for gain

- $|G|\ [dB] = 20 \log_{10}|G|$

$\|G\|$	$\|G\|$ [dB]
0.1	-20 dB
1	0 dB
2	6 dB
10	20 dB
100	40 dB
1000	60 dB

```
s=tf('s')
g = 10*(100*s+1)/[(10*s+1)*(s+1)]
bode(g) % gives AR in dB*
```



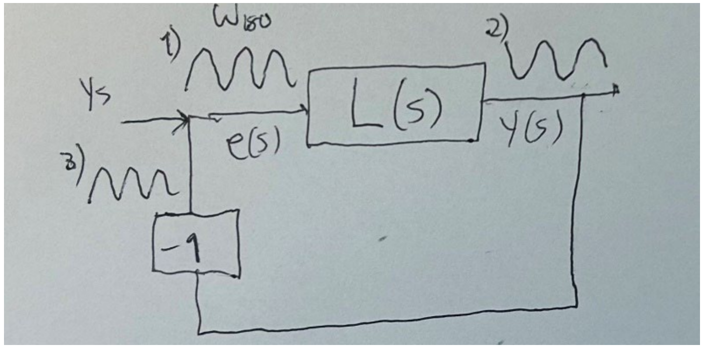
*To change magnitude from dB to abs: Right click + properties + units (absolute, log scale)

Other way: $|G| = 10^{|G|(dB)/20}$

GM=2 is same as GM = 6dB

Bode stability condition

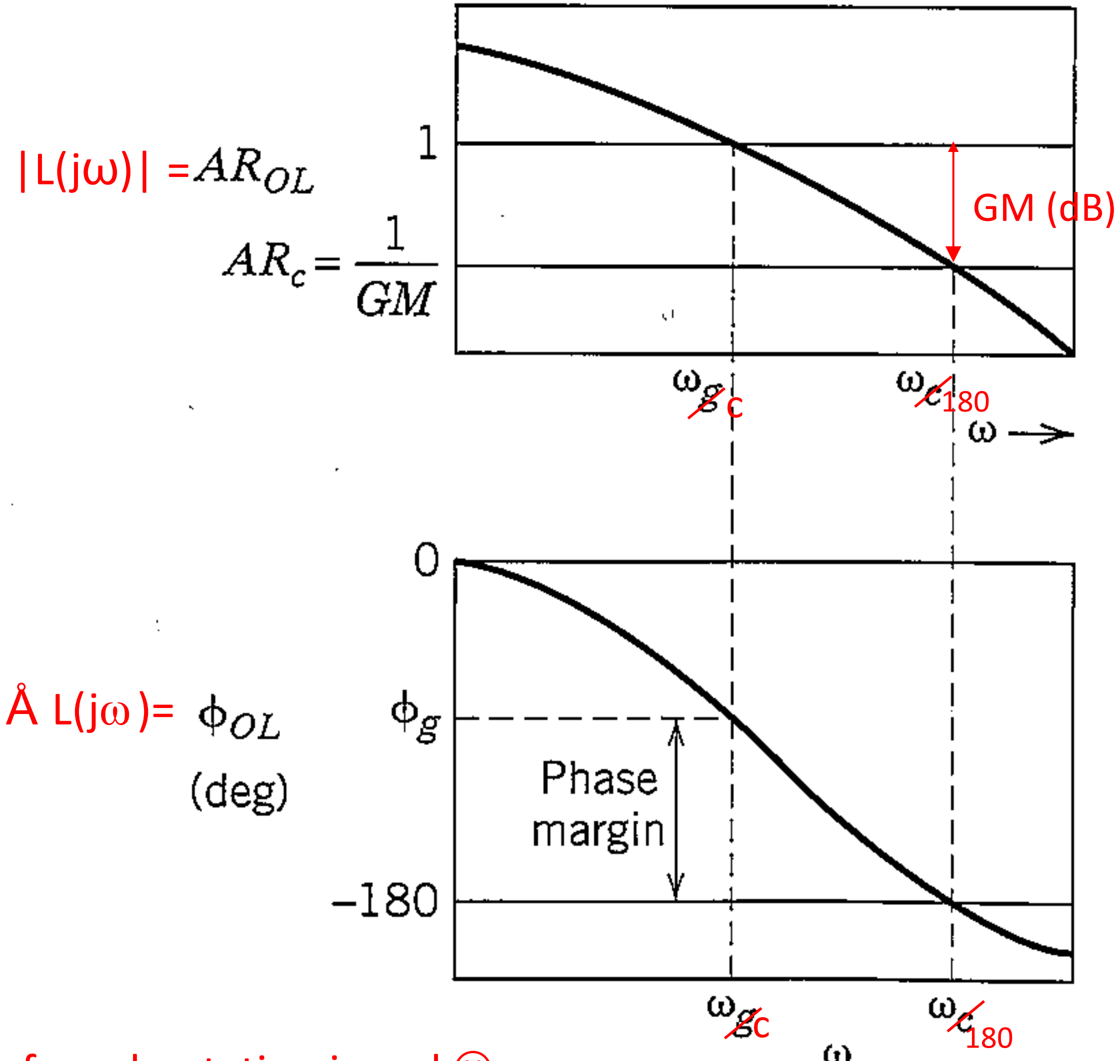
(Closed-loop stability condition from analyzing loop $L(s) = G\ C\ G_m$)



Proof

Starting point: Stability is a system property for linear systems, so if the system is stable for one signal it's stable for all signals.

- Consider a particular signal: Sinusoid with frequency ω_{180} (frequency where frequency shift around loop L(s) is -180° = $-\pi$ rad). $e(t) = \sin(\omega_{180}\, t)$
- With negative feedback, the total phase shift around the loop is -360°, so this sinusoid comes «back in phase»
- If the gain around the loop is less than 1, the sinusoid will die out.
- Conclusion: The closed-loop system is stable if and only if $|L(j\omega)|<1$ at frequency ω_{180}



GM = 1/|L(jω$_{180}$)|

ω_{180} = frequency where phase shift around the loop is -180° = - π rad.

Å L(jω$_{180}$) = -180° = - π rad

PM = Å L(jω$_c$) + 180°

= Å L(jω$_c$) + π [rad]

ω_c = frequency where loop gain is 1.

|L(jω$_c$)| = 1

Sigurd's preferred notation in red ☺

Figure 14.12 Gain and phase margins on a Bode plot

$\boxed{\text{Time delay margin (DM), } \Delta\theta = \text{PM[rad]}/\omega_c}$

Question: For SIMC, is $\omega_c = 1/\tau_c$? No, but it's related. In many cases $\omega_c = 1/(\theta+\tau_c)$

Summary: CLOSED-LOOP STABILITY IN FREQUENCY DOMAIN

- $L = gcg_m$ = loop transfer function with negative feedback
- **Bode's stability condition**: $|L(j\omega_{180})| < 1|$
  - Limitations
    - Open-loop stable (L(s) stable)
    - Phase of L crosses -180° only once
  - Stability margins
    - GM = $1/|L(j\omega_{180})|$, where Å $L(j\omega_{180})|=-180°$
    - PM = Å $L(j\omega_c) + 180°$, where $|L(j\omega_c)|=1$
    - How much delay will "eat up" the PM?

      Answer: PM= $\omega_c \, \Delta\theta$ [rad] $\Rightarrow$ DM = $\Delta\theta$= PM[rad]/ω_c

GM = gain margin (>1 for stability; typicall want >3)
PM = phase margin (>0 for stability; typically want > 50°)
DM = delay margin (>0 for stability; typically want > 2θ)

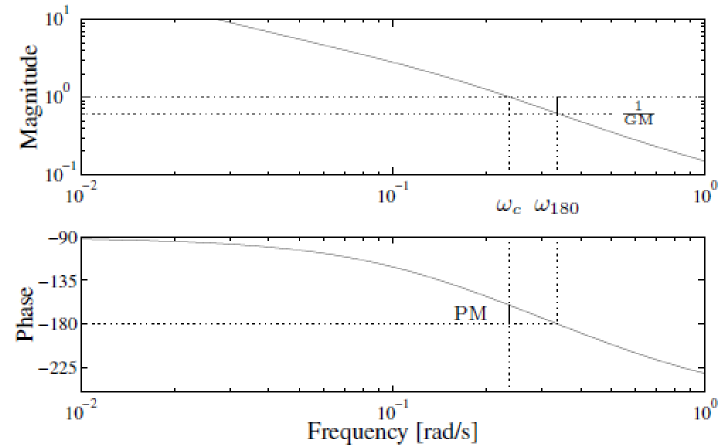
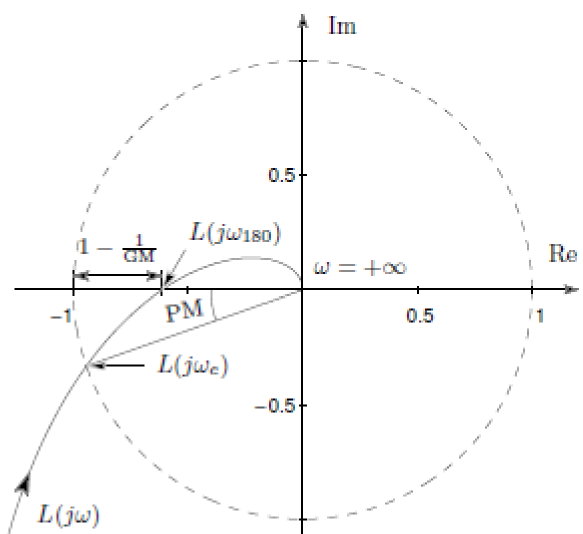


Figure 2.12: Typical Bode plot of $L(j\omega)$ with PM and GM indicated

- The same but more general: **Nyquist stability condition**:

  Locus of $L(j\omega)$ should encircle the (-1)-point P times in the anti-clockwise direction (where P = no. of unstable poles in L).



Stable plant (P=0): Closed-loop stable if L has no encirclements of -1 (=Bode's stability condition)

- Example 1. P-control of delay process, $g(s)=ke^{-\theta s}$. For what K_c is system stable?
- Example 2. I-control of delay process. For what K_I is system stable?

Solution. Stable if and only if

1. P-control: $kK_c < 1$
2. I-control: $kK_I < \frac{\pi}{2}\frac{1}{\theta}$

- Example 2, continued. I-control of delay process
- what is ω_c, ω_{180}, GM, PM and DM & give for SIMC (analytical)

Solution
For any KI:
wc=k' KI, w180=(pi/2)(1/theta).
GM = w180/k'KI = (pi/2)/(k'KI theta),
PM=(pi/2)- k'KI*theta,
DM = PM/wc = (pi/2)/k'KI - theta

SIMC with $\tau_c=\theta$ gives $k'K_I = \frac{1}{2\theta}$, so
wc = $\frac{1}{2\theta}$, w180= = $\frac{\pi}{2\theta}$
GM =π = 3.14.
PM = (pi-1)/2 = 1.07 rad = 61.5^o
DM = (pi-1)*theta = 2.14 theta

General SIMC-PID for 2nd order delay process (with $\tau_I = \tau_1$) gives: L(s) = $\frac{1}{\theta+\tau_c} \frac{e-^{\theta s}}{s}$

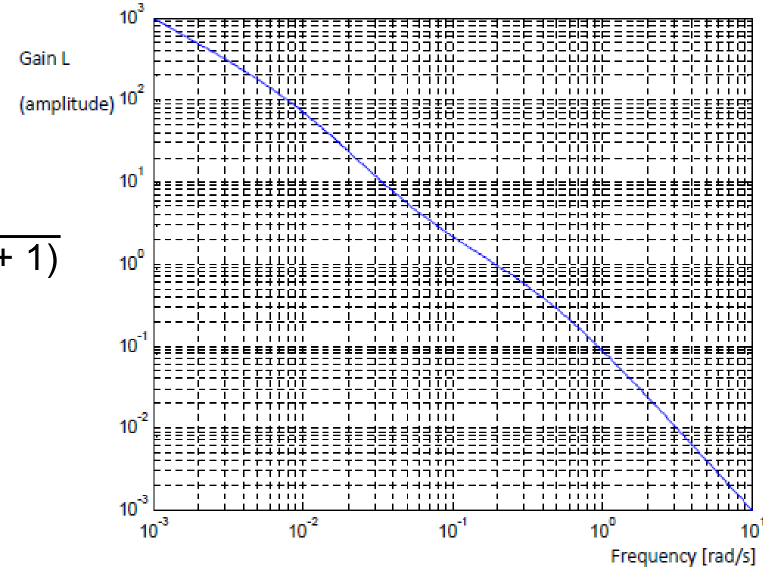
$$GM = \frac{\pi}{2}\left(\frac{\tau_c}{\theta}+1\right)$$ DM = (GM-1)θ

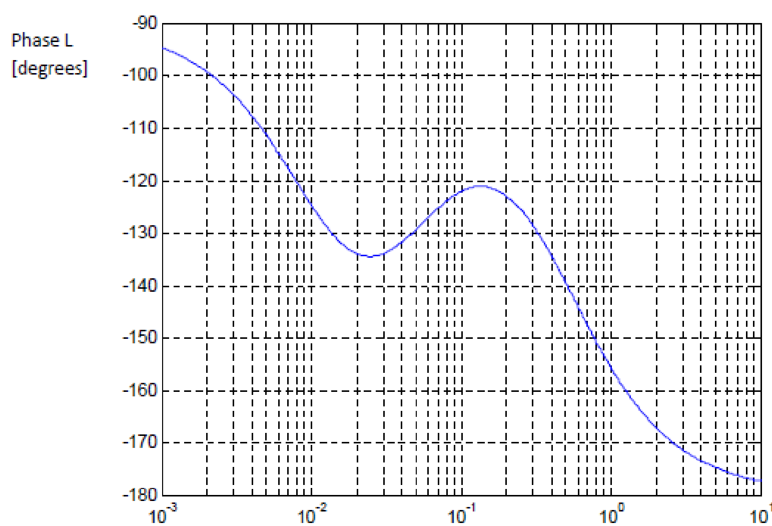
TASK 1: Bode-plot of L(s) = (20s+1)/[s(100s+1)(2s+1)] . Write on the asymptotes
TASK 2: What is GM and PM?
TASK 2: How does the plot change if we add a delay of 2 time units (e^{-2*s})
TASK 3: What is now GM and PM? How much extra time delay can we allow?

EXAMPLE 3

$$L(s) = \frac{20s+1}{s(100s+1)(2s+1)}$$

L(s): SIMC PI-control with τ_c=4 for
g(s) = 1/(100s+1)(2s+1)



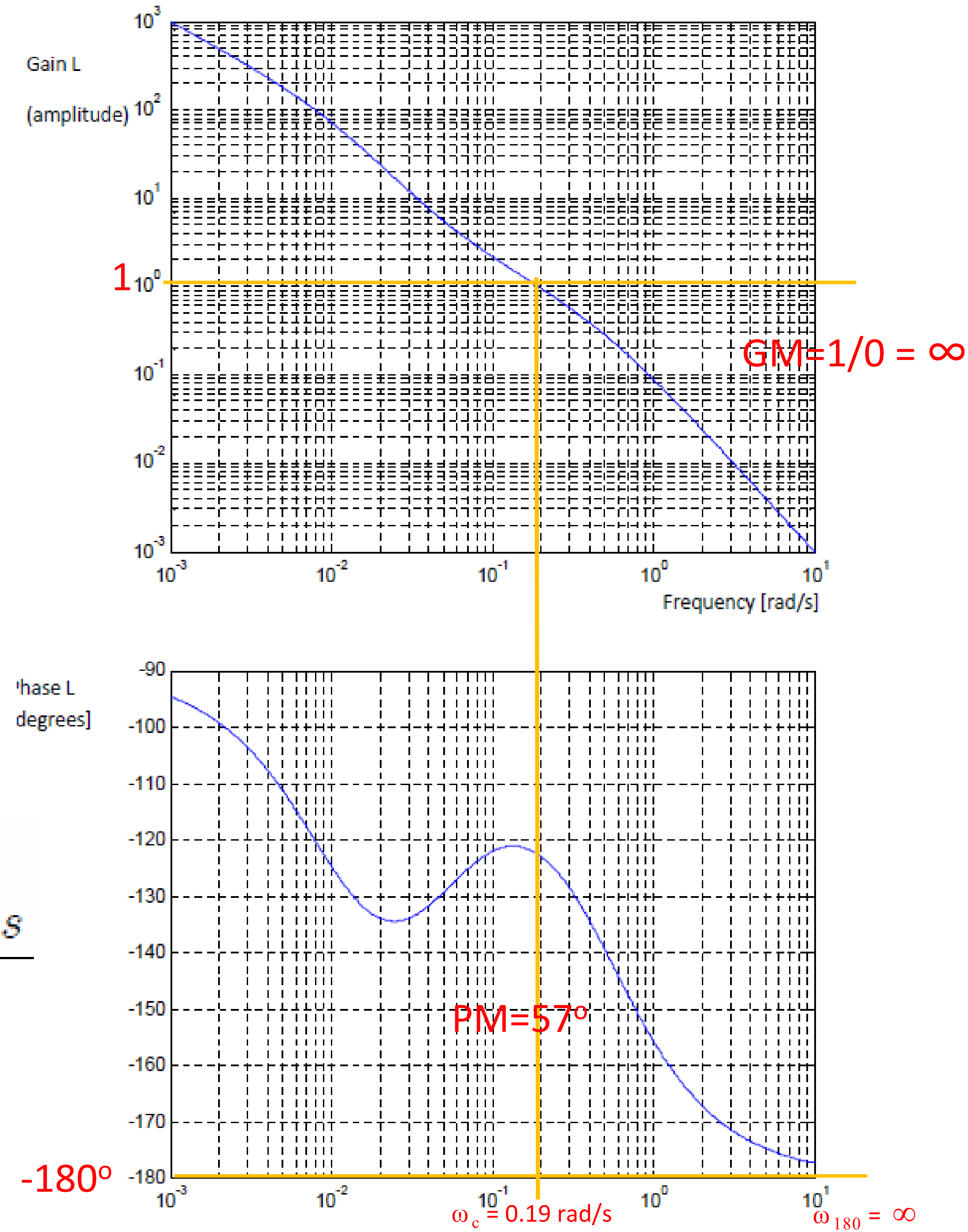



```
s=tf('s')
L = (20*s+1)/[s*(100*s+1)*(2*s+1)]
figure(3), bode(L) % gives AR in dB
w = logspace(-3,1,1000)
[mag,phase]=bode(L,w)
figure(1), loglog(w,mag(:)), grid on, axis([0.001 10 0.001 1000])
figure(2), semilogx(w,phase(:)), grid on
```

SOLUTION

$$L(s) = \frac{20s+1}{s(100s+1)(2s+1)}$$

L(s): SIMC PI-control with τ_c=4 for g(s) = 1/(100s+1)(2s+1)



Time delay margin

$$\Delta\theta = \frac{PM[rad]}{\omega_c[rad/s]} = \frac{1}{0.19} = 5.2s$$

EXAMPLE3': ADD 2 UNITS OF DELAY

$$L = \frac{20s+1}{s(100s+1)(2s+1)} e^{-2s}$$

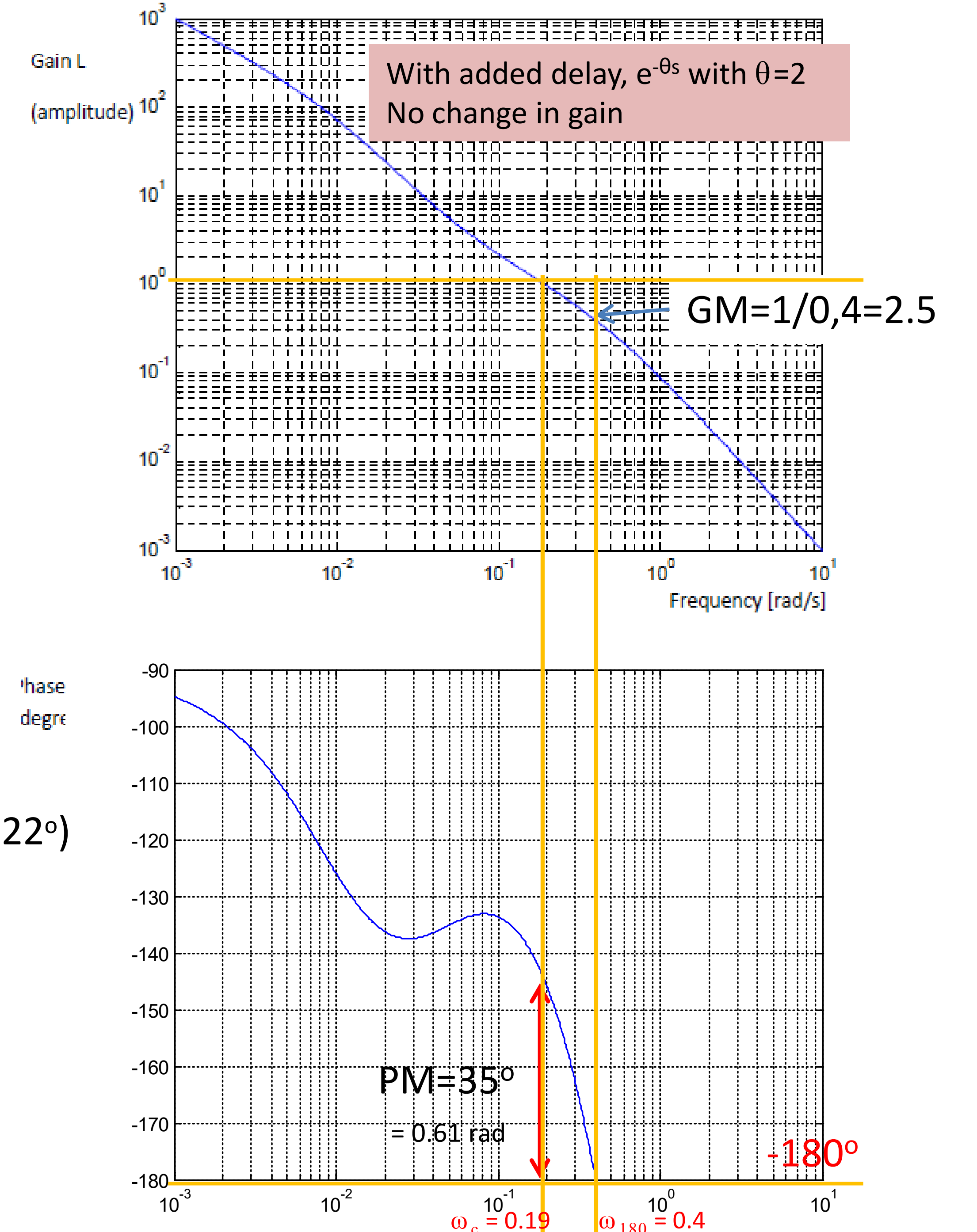
Now phase crosses -180° so
GM is no longer infinity

Phase addition from delay = $-\omega\theta$

At ω_c: $- -\omega_c\theta = 0.19*2 = -0.38$ rad (-22°)

So new PM = 57° (old) – 22° = 35°

$$\text{New time delay margin}$$
$$\Delta\theta = \frac{PM[rad]}{\omega_c[rad/s]} = \frac{0.61}{0.19} = 3.2s$$



Example 4. PI-control of integrating process with delay. Compare ZN and SIMC*

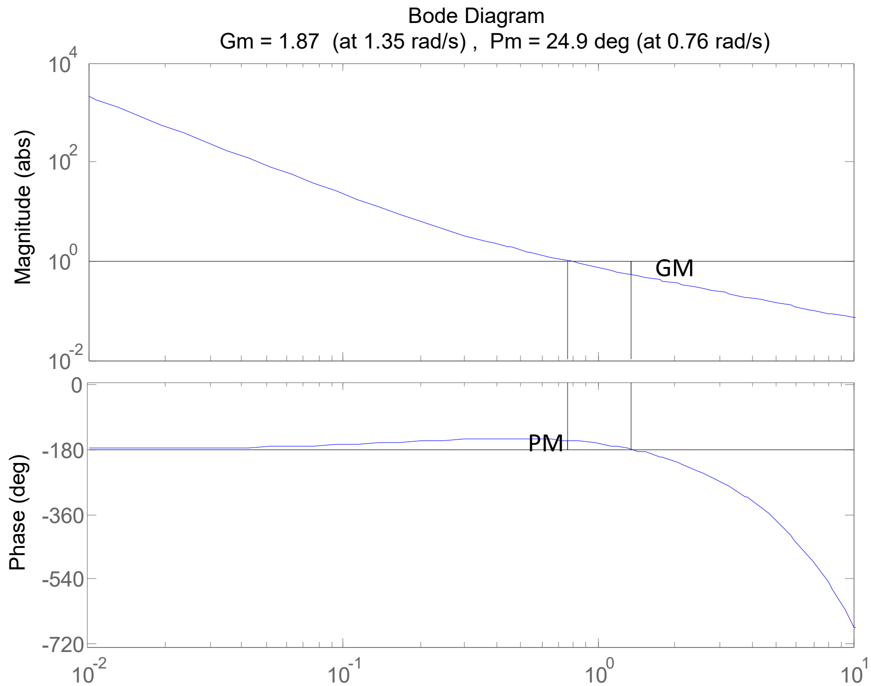
- $g(s) = k' e^{-\theta s}/s$
- ZN: Use P-control and increase K_c until instability.
  - Find: $P_u = 4\theta$ and $K_u = (\pi/2)/(k'\theta)$
  - Derivation: $L(s) = \frac{K_c k' e^{-\theta s}}{s}, \angle L(j\omega) = -\frac{\pi}{2} - \omega\theta[rad], |L(j\omega)| = \frac{K_c k'}{\omega}$,
    - so: $\angle L(j\omega_{180}) = -\frac{\pi}{2} - \omega_{180}\theta = -\pi \Rightarrow \omega_{180} = \frac{\pi}{2\theta} \Rightarrow \mathrm{P_u} = \frac{2\pi}{\omega_{180}} = 4\theta$,
    - and at limit to instability: $|L(j\omega_{180})| = \frac{K_u k'}{\omega_{180}} = 1 \Rightarrow K_u = \frac{\omega_{180}}{k'} = \frac{\pi}{2}\frac{1}{k'\theta}$
- PI-controller, $c(s) = K_c(1+1/(\tau_I s))$

	K_c	τ_I
Ziegler-Nichols	$0.45K_u = 0.707/(k'\theta)$	$P_u/1.2=3.33\theta$
SIMC ($\tau_c=\theta$)	$0.5/(k'\theta)$	8θ

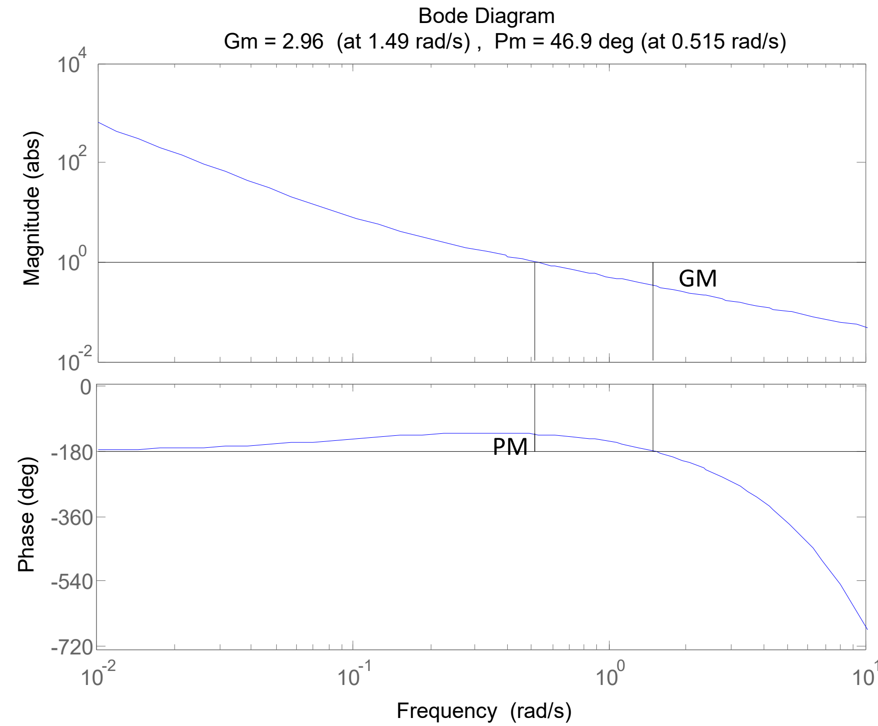
*Task: Compare Bode-plot (L=gc), robustness and simulations (use $k'=1$, $\theta=1$).

$$g(s) = \frac{e^{-s}}{s}, \quad c(s) = K_c \frac{\tau_I s + 1}{\tau_I s}$$

Ziegler-Nichols PI



SIMC-PI



SIMC is a lot more robust:

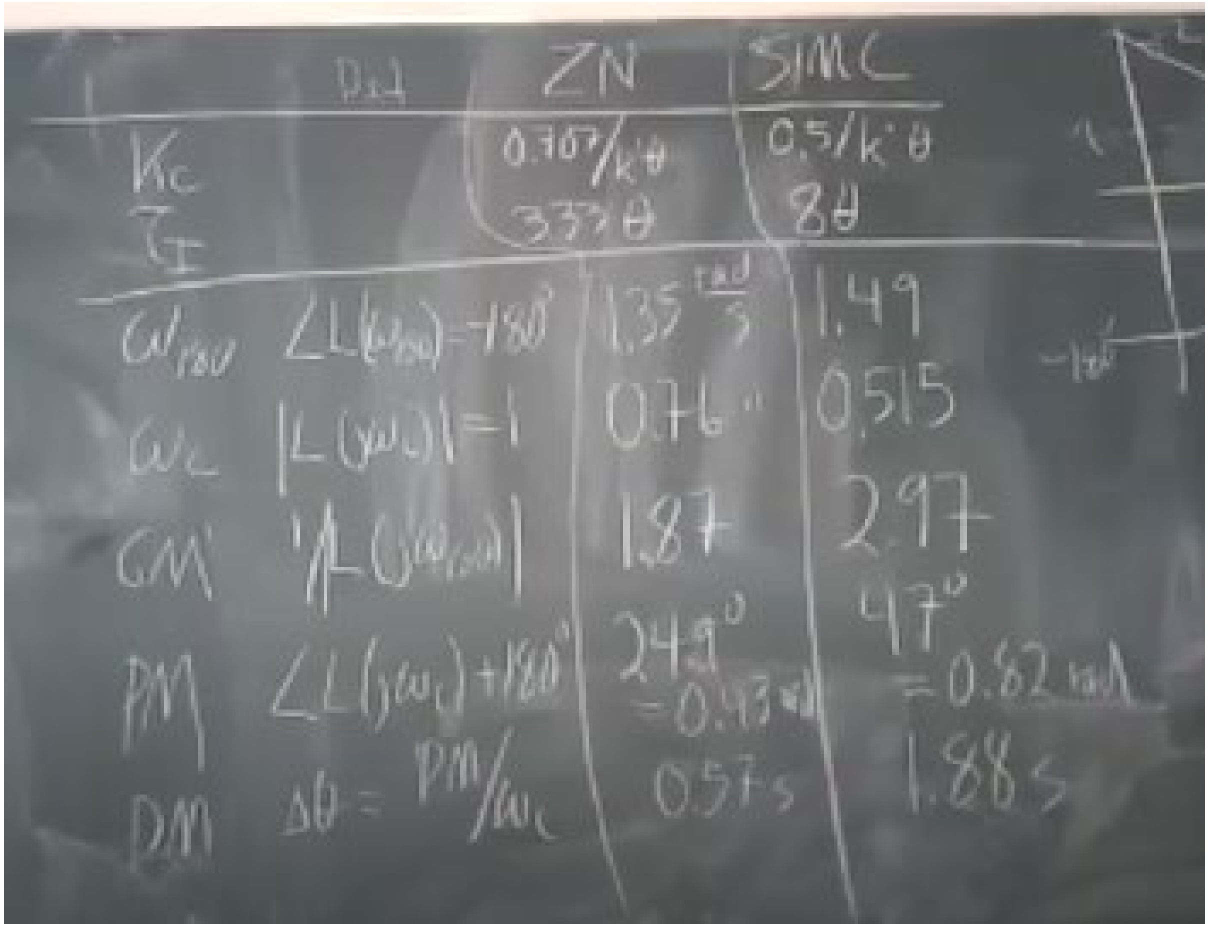
	GM	PM	Delay margin, $\Delta\theta$
Ziegler-Nichols	1.87	24.9°	0.57 s
SIMC ($\tau_c=\theta$)	2.97	46.9°	1.88 s

```
s=tf('s')
g = exp(-s)/s
Kc=0.707, taui=3.33
c = Kc*(1+1/(taui*s))
L1 = g*c
figure(1), margin(L1) % Bode-plot with margins
% To change magnitude from dB to abs: Right click + properties + units
Kc=0.5, taui=8
c = Kc*(1+1/(taui*s))
L2 = g*c
figure(2), margin(L2)
```

$\Delta\theta$= PM[rad]/ω_c

ZN: $\Delta\theta$ = 24.9*(3.14/180)/0.76 = 0.572s

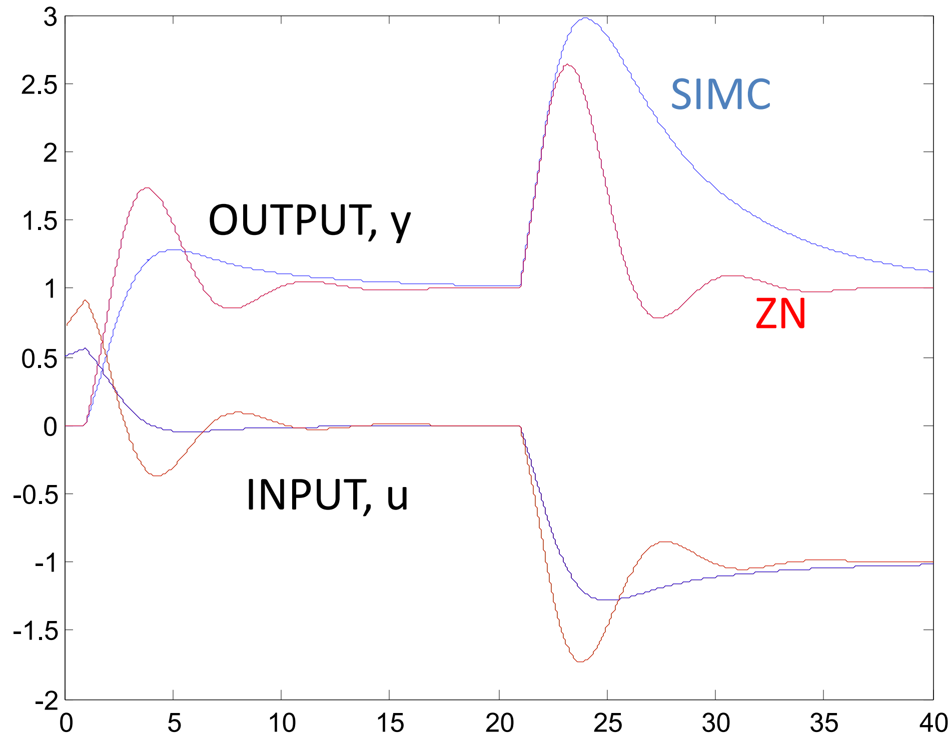
SIMC: $\Delta\theta$ = 46.9*(3.14/180)/0.515 = 1.882s



ZN
SIMC
Kc 0.707/k'θ 0.5/k'θ
τI 3.33θ 8θ
ω180 ∠L(jω180)=-180° 1.35 rad/s 1.49
ωc |L(jωc)|=1 0.76 0.515
GM 1/|L(jω180)| 1.87 2.97
PM ∠L(jωc)+180° 24.9° =0.43 rad 47° =0.82 rad
DM Δθ = PM/ωc 0.57s 1.88s

Delay = 1

Closed-loop response: PI-control of $g(s) = \frac{e^{-s}}{s}$



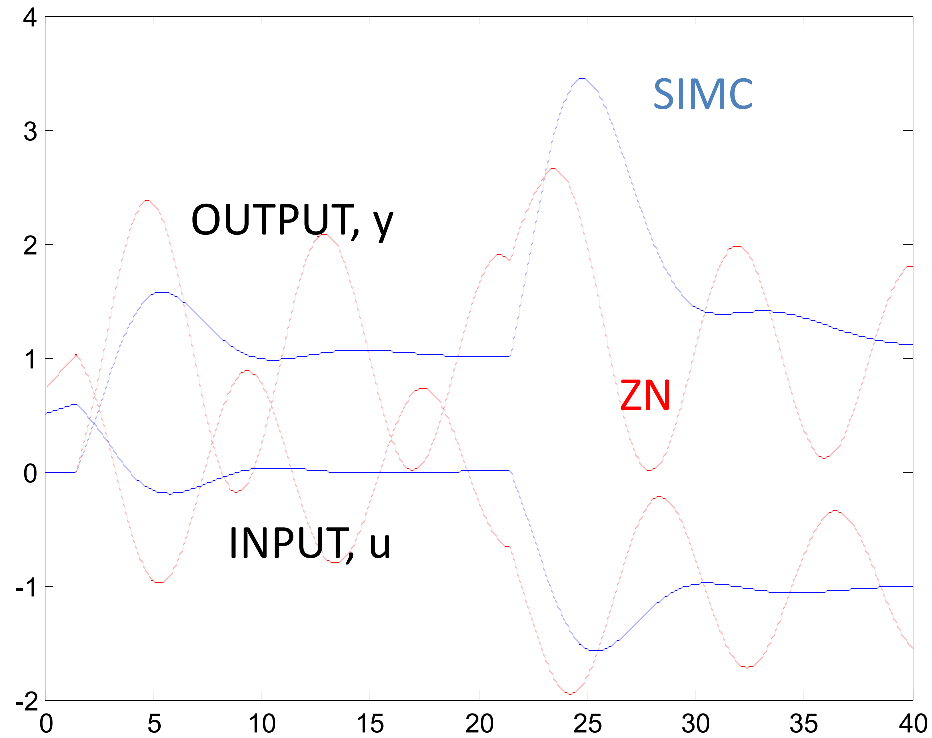
t=0: setpoint change, t=20: input disturbance

```
Simulink file: tunepid4
s=tf('s')
g = exp(-s)/s
Kc=0.707, taui=3.33, taud=0 % ZN
sim('tunepid4')
plot(Tid,y,'red',Tid,u,'red')
Kc=0.5, taui=8, taud=0 % SIMC
sim('tunepid4')
hold
plot(Tid,y,'blue',Tid,u,'blue')
hold off
```

Conclusion: Ziegler-Nichols (ZN) responds faster to the input disturbance, but is much less robust.

- ZN goes unstable if we increase delay from 1s to 1.57s.
- SIMC goes unstable if we increase delay from 1s to 2.88s.

INCREASE DELAY: $g(s) = \frac{e^{-1.5s}}{s}$



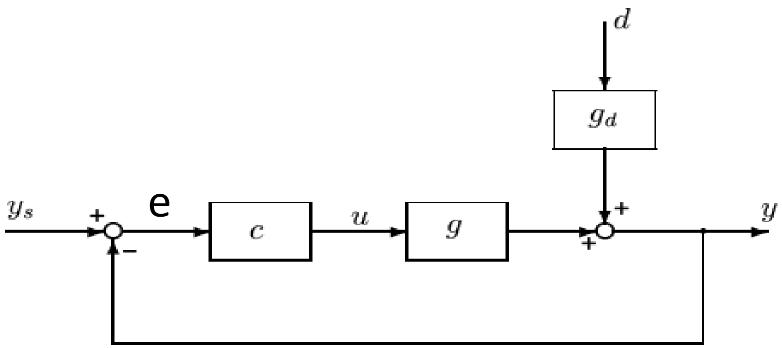
t=0: setpoint change,   t=20: input disturbance

ZN is almost unstable when the delay is increased from 1s to 1.5s.
SIMC does not change very much

Bode stability condition. Why may D-action help in some cases?

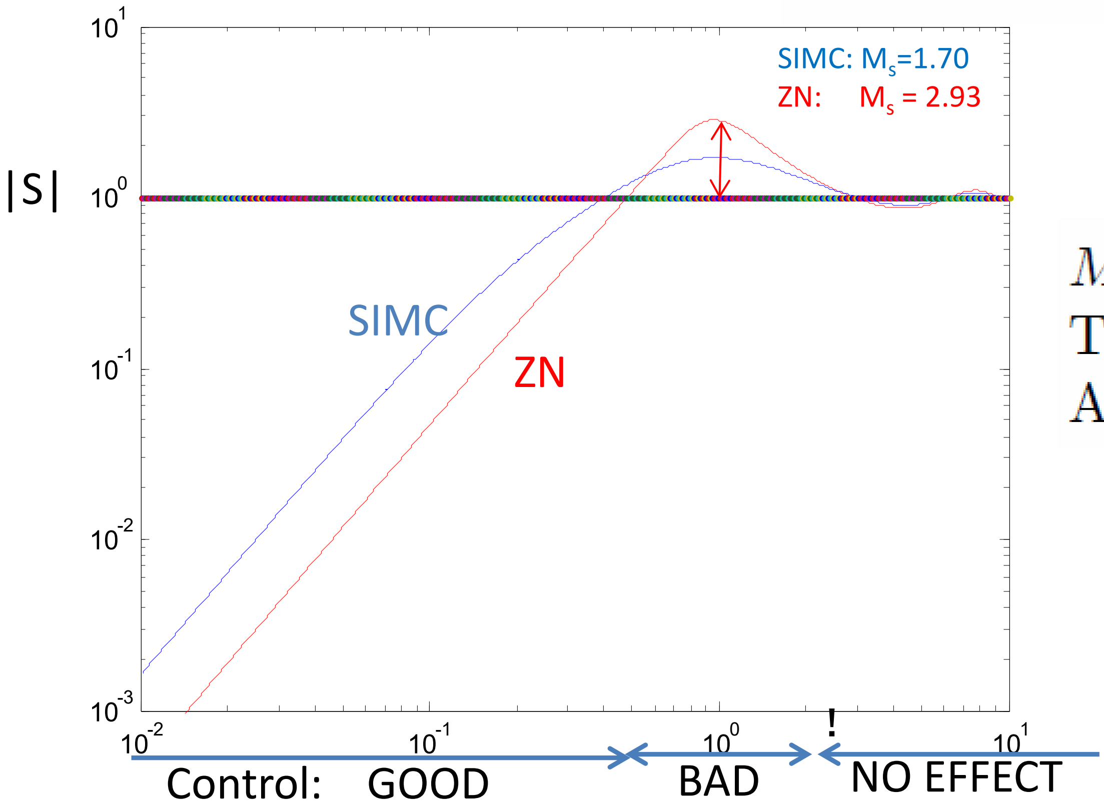
- **Some unstable processes, for example a double integrating process, may need D-action for stabilization.** The reason is to add positive phase and therefore stabilize the system. Why does this help?
  - Recall the Bode stability condition. It says that the loop gain should be less than 1 at the frequency where the phase shift around the loop -180 degrees.
  - Another statement is that phase shift should be less than -180^{o} at the frequency where the loop gain is 1.
  - So for stability and robustness we want as little phase shift as possible (to improve the phase margin). The things that add negative phase shift are time delay (the worst), poles and RHP-zeros.
  - LHP-zeros (D-action, (Td*s +1)) have the opposite (positive) effect on the phase, and this is why they may be added for stabilization in difficult cases, for example, an unstable process. Of course, zeros will also affect the loop gain, but at frequencies up to the break frequency, 1/Td, the positive effect on the phase is most important.
  - **So why don't we always add D-action?** One reason is that it increases the controller gain and therefore the input usage. However, the main reason is probably that it does not help very much in most cases and it makes the controller design more complicated (and easier to do mistakes).

Closed-loop frequency response



No control ($c = 0$): $e_{OL} = y_s - y = y_s - g_d d$
With control: $e = y_s - y = S y_s - S g_d d = S e_{OL}$
$S = \frac{1}{1+L}$ - sensitivity function = effect of control
$L = gc$ - loop transfer function

Low ω where $|S| < 1$: Control reduces error
Intermediate ω where $|S| > 1$: Control increases error
High ω where $S = 1$ ($L \to 0$): Control has no effect



M_s = peak of $|S|$
Typical requirement: $M_s < 2$
At stability limit: $M_s \to \infty$

```
w = logspace(-2,1,1000);
[mag1,phase]=bode(1/(1+L1),w);
[mag2,phase]=bode(1/(1+L2),w);
figure(1), loglog(w,mag1(:),'red',w,mag2(:),'blue',w,1,'-.')
axis([0.01,10,0.001,10])
```