TKP4140 Process Control
Department of Chemical Engineering NTNU
Autumn 2021 - Exam

Solution

Problem 1: Simple process

a) The system response to a step change in the input for a first-order process is given by

$$y(s) = g(s)u(s) = \frac{k}{\tau s + 1}u(s). \tag{1}$$

The time-domain response can be calculated taking the inverse Laplace transform:

$$y(t) = k(1 - e^{-t/\tau})u(t). \tag{2}$$

Using the transfer functions given in the problem for a unit step input, we get:

$$y_1(t) = 6(1 - e^{-t/18}), \tag{3a}$$
$$y_2(t) = -3(1 - e^{-t/2}), \tag{3b}$$
$$y(t) = y_1(t) + y_2(t). \tag{3c}$$

Values of $y_1(t)$, $y_2(t)$, and $y(t)$ at different time points are shown in Table 1. The system responses are also shown in Figure 1.

Table 1: Time-domain system responses to a unit step input.

t	$y_1(t)$	$y_2(t)$	$y(t)$
0	0	0	0
1	0.324	-1.180	-0.856
2	0.631	-1.896	-1.266
5	1.455	-2.754	-1.299
10	2.557	-2.979	-0.422
20	4.025	-3	1.025
40	5.350	-3	2.350
100	5.977	-3	2.977

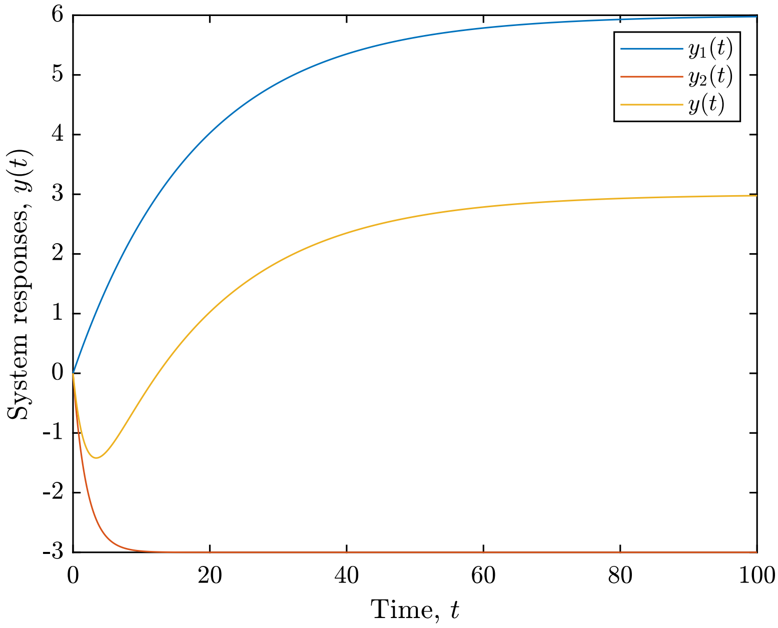

Figure 1: Time-domain system responses to a unit step input.

b) The response of $y(t)$ is called an inverse response and can be predicted from the RHP zero in the transfer function $g(s)$ from u to y, which can be calculated as

$$g(s) = g_1(s) + g_2(s) = \frac{6}{18s+1} + \frac{-3}{2s+1} = \frac{3(-14s+1)}{(18s+1)(2s+1)}. \tag{4}$$

The transfer function has a zero at $s = 1/14$ (RHP zero).

c) The half rule approximation of $g(s)$ to get a first-order model is:

$$k = 3, \tag{5a}$$
$$\tau = 18 + 2/2 = 19, \tag{5b}$$
$$\theta = 14 + 2/2 = 15, \tag{5c}$$
$$g_{app}(s) \approx \frac{3e^{-15s}}{19s+1}. \tag{5d}$$

The PI-controller tuning are found by applying the SIMC tuning method to $g_{app}(s)$. For "tight control", we select $\tau_c = \theta = 15$.

$$K_c = \frac{1}{k}\frac{\tau}{\tau_c+\theta} = \frac{1}{3}\frac{19}{2\cdot 15} = 0.211, \tag{6a}$$
$$\tau_I = \min(\tau, 4(\tau_c+\theta)) = \min(19, 4(2\cdot 15)) = 19. \tag{6b}$$

Problem 2: PI control of first-order with delay process

a) The SIMC PI-rules for a first-order plus delay process are:

$$K_c = \frac{1}{k}\frac{\tau}{\tau_c + \theta}, \tag{7a}$$

$$\tau_I = \min(\tau, 4(\tau_c + \theta)). \tag{7b}$$

For "tight control", we select $\tau_c = \theta$.

b) Given the process

$$g(s) = \frac{6}{(32s+1)(8s+1)}, \tag{8}$$

the half-rule approximation to get a first-order plus delay model can be calculated as

$$k = 6, \tag{9a}$$

$$\tau = 32 + 8/2 = 36, \tag{9b}$$

$$\theta = 8/2 = 4, \tag{9c}$$

$$g_{app}(s) \approx \frac{6e^{-4s}}{36s+1}. \tag{9d}$$

The PI-controller tuning are found by applying the SIMC tuning method to $g_{app}(s)$. For "tight control", we select $\tau_c = \theta = 4$.

$$K_c = \frac{1}{k}\frac{\tau}{\tau_c + \theta} = \frac{1}{6}\frac{36}{2 \cdot 4} = 0.75, \tag{10a}$$

$$\tau_I = \min(\tau, 4(\tau_c + \theta)) = \min(36, 4(2 \cdot 4)) = 32. \tag{10b}$$

c) PID control is recommended because this is a second-order process where $\tau_2 > \theta$. Given a second-order plus delay process

$$g(s) = \frac{k}{(\tau_1 s + 1)(\tau_2 s + 1)} e^{-\theta s}, \tag{11}$$

the PID settings can be calculated using the SIMC tuning method as follows:

$$K_c = \frac{1}{k}\frac{\tau_1}{\tau_c + \theta} = \frac{1}{6}\frac{32}{\tau_c}, \tag{12a}$$

$$\tau_I = \min(\tau_1, 4(\tau_c + \theta)) = \min(32, 4\tau_c), \tag{12b}$$

$$\tau_D = \tau_2. \tag{12c}$$

The PID settings above are a function of the closed-loop time constant τ_c, which is a tuning parameter. For this process, we cannot set $\tau_c = 0$ because $\theta = 0$, and this would result in infinitely fast control with $K_c = \infty$. We do not have enough information to select a good value for τ_c. However, we generally use a smaller τ_c for PID than for the corresponding PI, so we should choose τ_c smaller than 4. Given this information, we could tune it online.

d) The block diagram of the control system is shown in Fig. 2. The open-loop response (no control) to a step disturbance at the input is:

$$y(s) = g(s)d(s). \tag{13}$$

On the other hand, the closed-loop response (with control) is:

$$y(s) = T_d(s)d(s), \tag{14}$$

where

$$T_d(s) = \frac{g(s)}{1 + g(s)c(s)}. \tag{15}$$

Note that $T_d(s)$ has a zero at the origin because of I-action. So $T_d(0) = 0$. Both responses are depicted in Fig. 3. The open-loop response is s-shaped because $g(s)$ is a second-order process. The closed-loop response follows the open-loop response initially, then reaches a peak somewhere between $\tau_c = 4$ and $\tau_c = 36$, and finally it goes back to zero because of I-action.

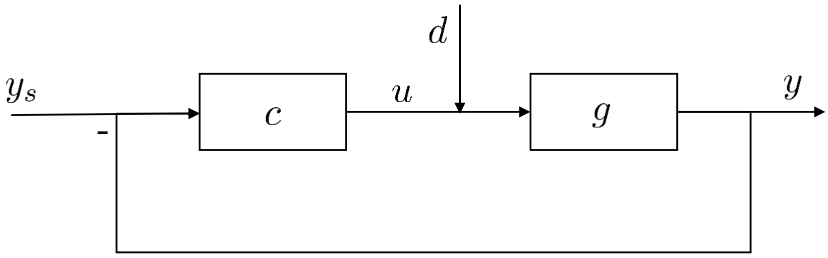

Figure 2: Block diagram of the control system.

e) Given the process

$$g(s) = \frac{6}{(32s+1)(8s+1)}, \tag{16}$$

and the controller

$$c(s) = K_c\left(\frac{\tau_I s + 1}{\tau_I s}\right) = 0.75\left(\frac{32s+1}{32s}\right) = 0.023\left(\frac{32s+1}{s}\right), \tag{17}$$

we can calculate the loop transfer function as follows:

$$\begin{aligned} L(s) &= c(s)g(s) && \text{(18a)} \\ &= 0.75\left(\frac{32s+1}{32s}\right)\frac{6}{(32s+1)(8s+1)} && \text{(18b)} \\ &= \frac{0.1406}{s(8s+1)}. && \text{(18c)} \end{aligned}$$

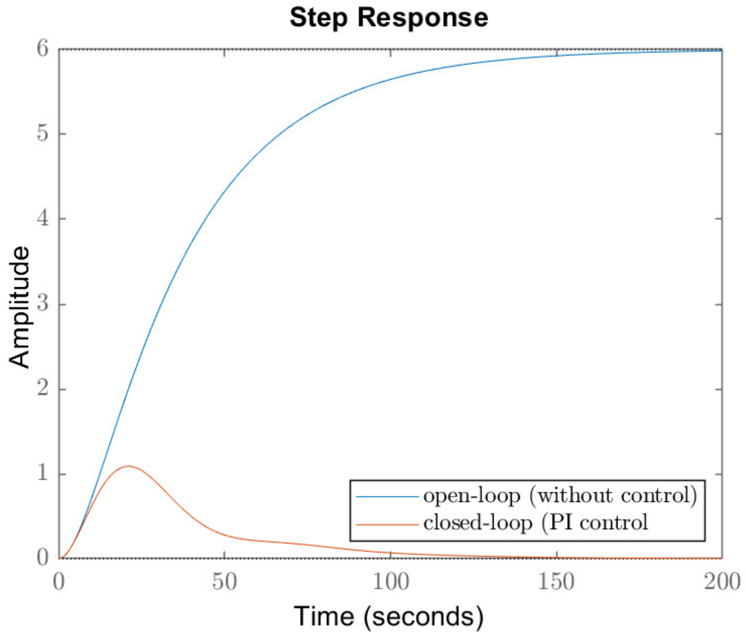

Figure 3: Open-loop and closed-loop responses to a step disturbance at the input.

The gain and phase as a function of the frequency can be calculated as

$$|L| = \frac{0.1406}{\omega} \cdot \frac{1}{\sqrt{8^2\omega^2+1}}, \tag{19}$$

$$\angle L = -90^\circ - \tan^{-1}(8\omega). \tag{20}$$

Values for the gain and phase at different frequencies are shown in Table 2. The Bode plot of the loop transfer function $L = gc$ is shown in Fig. 4. The critical frequency is $\omega_c \approx 0.106$. The phase phase margin (PM), gain margin (GM) and maximum time delay error ($\theta_{\max}$) are calculated as follows:

$$\text{PM} = 180^\circ + \angle L(j\omega_c) = 180^\circ - 130.30^\circ = 49.7^\circ, \tag{21}$$

$$\text{GM} = \frac{1}{|L(j\omega_{180})|} = \frac{1}{0} = \infty,^{1} \tag{22}$$

$$\text{DM} = \theta_{\max} = \frac{\text{PM [rad]}}{\omega_c} = \frac{0.867}{0.106} = 8.18s. \tag{23}$$

[1] The phase of L reaches -180° at infinite frequency ($\omega_{180} = \infty$), where $|L(j\omega_{180})| = 0$.

Table 2: Gain and phase shifts at different frequencies.

ω	$\|L\|$	$\angle L$
0.001	140.6	-90.458
0.01	14.015	-94.574
0.1	1.0979	-128.66
0.106	1.0116	-130.30
1	0.0174	-172.87

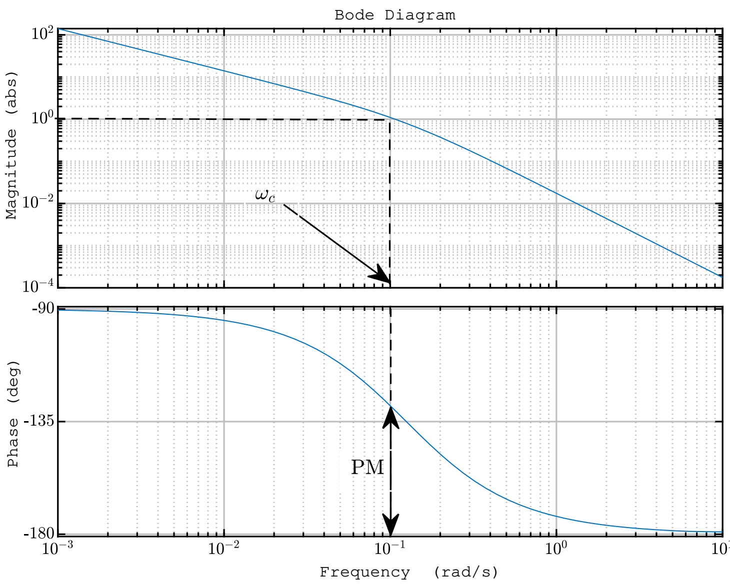

Figure 4: Bode plot of $L = gc$.

Problem 3. Mixing process

a) Assumptions:

- perfect level control, i.e. the volume in the tank is constant, $V = 100$ l
- constant density
- $c_v \approx c_p \approx$ constant
- the reference temperature is $T_{ref} = 0\,K$

The steady-state mass balance for the tank is given by

$$q_1 + q_2 + q_3 = q, \tag{24}$$

the steady-state component balance is given by

$$q_1 c_1 + q_2 c_2 + q_3 c_3 = qc, \tag{25}$$

and the steady-state energy balance is given by

$$q_1 T_1 + q_2 T_2 + q_3 T_3 = qT. \tag{26}$$

Using the nominal data, we get

$$q_1 + q_2 + 0.45 = 1.25, \tag{27a}$$
$$0.5 q_2 + 3.8 \cdot 0.45 = 1.25c, \tag{27b}$$
$$20 q_1 + 120 q_2 + 40 \cdot 0.45 = 54 \cdot 1.25. \tag{27c}$$

This system of equations can be written in matrix form as

$$\begin{bmatrix} 1 & 1 & 1 \\ 20 & 120 & 0 \\ 0 & 0.5 & -1.25 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ c \end{bmatrix} = \begin{bmatrix} 0.8 \\ 49.5 \\ -1.71 \end{bmatrix}. \tag{28}$$

We can calculate the nominal flows by solving this system of equations:

$$\begin{bmatrix} q_1 \\ q_2 \\ c \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 20 & 120 & 0 \\ 0 & 0.5 & -1.25 \end{bmatrix}^{-1} \begin{bmatrix} 0.8 \\ 49.5 \\ -1.71 \end{bmatrix}, \tag{29a}$$

$$\begin{bmatrix} q_1 \\ q_2 \\ c \end{bmatrix} = \begin{bmatrix} 1.2 & -0.01 & 0 \\ -0.2 & 0.01 & 0 \\ -0.08 & 0.004 & -0.8 \end{bmatrix} \begin{bmatrix} 0.8 \\ 49.5 \\ -1.71 \end{bmatrix}, \tag{29b}$$

$$\begin{bmatrix} q_1 \\ q_2 \\ c \end{bmatrix} = \begin{bmatrix} 0.465 \\ 0.335 \\ 1.502 \end{bmatrix}. \tag{29c}$$

The nominal steady-state flows are $q_1 = 0.465$ l/s and $q_2 = 0.335$ l/s. The nominal steady-state concentration is $c = 1.502$ g/l.

b) Under the same assumptions as before, the dynamic mass balance for the tank is given by

$$\frac{dV}{dt} = 0 = q_1 + q_2 + q_3 - q, \tag{30}$$

and the dynamic energy balance is given by

$$V\frac{dT}{dt} = q_1T_1 + q_2T_2 + q_3T_3 - qT. \tag{31}$$

c) i) *Dynamic mass balance.*

$$0 = q_1 + q_2 + q_3 - q = f_1. \tag{32}$$

Linearizing around the nominal point, we get

$$0 = \left(\frac{\partial f_1}{\partial q_1}\right)_* \Delta q_1 + \left(\frac{\partial f_1}{\partial q_2}\right)_* \Delta q_2 + \left(\frac{\partial f_1}{\partial q}\right)_* \Delta q, \tag{33a}$$

$$0 = \Delta q_1 + \Delta q_2 - \Delta q. \tag{33b}$$

Rearranging:

$$\Delta q = \Delta q_1 + \Delta q_2. \tag{34}$$

Taking the Laplace transform:

$$q(s) = q_1(s) + q_2(s). \tag{35}$$

ii) *Dynamic energy balance.* Using $q = q_1 + q_2 + q_3$, we get

$$V\frac{dT}{dt} = q_1(T_1 - T) + q_2(T_2 - T) + q_3(T_3 - T) = f_2. \tag{36}$$

Linearizing around the nominal point, we get

$$V\frac{\Delta T}{dt} = \left(\frac{\partial f_2}{\partial q_1}\right)_* \Delta q_1 + \left(\frac{\partial f_2}{\partial q_2}\right)_* \Delta q_2 + \left(\frac{\partial f_2}{\partial T}\right)_* \Delta T, \tag{37a}$$

$$V\frac{\Delta T}{dt} = (T_1^* - T^*)\Delta q_1 + (T_2^* - T^*)\Delta q_2 - (q_1^* + q_2^* + q_3^*)\Delta T. \tag{37b}$$

Using nominal data and rearranging, we get

$$100\frac{\Delta T}{dt} = -34\Delta q_1 + 66\Delta q_2 - 1.25\Delta T. \tag{38}$$

Taking the Laplace transform:

$$100sT(s) = -34q_1(s) + 66q_2(s) - 1.25T(s) \tag{39a}$$
$$(80s+1)T(s) = -27.2q_1(s) + 52.8q_2(s) \tag{39b}$$
$$T(s) = -\frac{27.2}{80s+1}q_1(s) + \frac{52.8}{80s+1}q_2(s) \tag{39c}$$

Writing this system of equations in matrix form, we get

$$y(s) = G(s)u(s), \tag{40a}$$
$$\begin{bmatrix} q(s) \\ T(s) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -\frac{27.2}{80s+1} & \frac{52.8}{80s+1} \end{bmatrix} \begin{bmatrix} q_1(s) \\ q_2(s) \end{bmatrix}. \tag{40b}$$

d) The steady-state gain matrix (setting $s = 0$) is

$$G(0) = \begin{bmatrix} 1 & 1 \\ -27.2 & 52.8 \end{bmatrix}. \tag{41}$$

The steady-state relative gain array (RGA) can be calculated as

$$\text{RGA} = G(0) \times (G(0)^{-1})^\top, \tag{42a}$$
$$\text{RGA} = \begin{bmatrix} 0.66 & 0.34 \\ 0.34 & 0.66 \end{bmatrix}. \tag{42b}$$

e) Based on the RGA analysis, we pair inputs and outputs following the main diagonal of the matrix (whose elements are closer to one). The suggested pairing is then:

$$y_1 \Longleftrightarrow u_1, \tag{43a}$$
$$y_2 \Longleftrightarrow u_2, \tag{43b}$$

or equivalently,

$$q \Longleftrightarrow q_1, \tag{44a}$$
$$T \Longleftrightarrow q_2. \tag{44b}$$

Physically, we should select the pairing with largest effect (gain) from input to output:

(a) Pair largest stream (q_1) with total flow (q)

(b) Pair stream with temperature most different from desired (q_2) with temperature (T).

In conclusion, the physical arguments agree with the RGA analysis. The proposed flow sheet is shown in Fig. 5.

f) The control structure shown in Fig. 6 gives up controlling the total flow when the constraint on the concentration is reached. We choose a max selector because the constraint for c is satisfied by a large value of q_1 (increasing q_1 will lower c and keep it below $c_{\max} = 2$ g/l). It is important that both controllers (FC and CC) that enter the MAX-selector have anti windup because both controllers are always running.

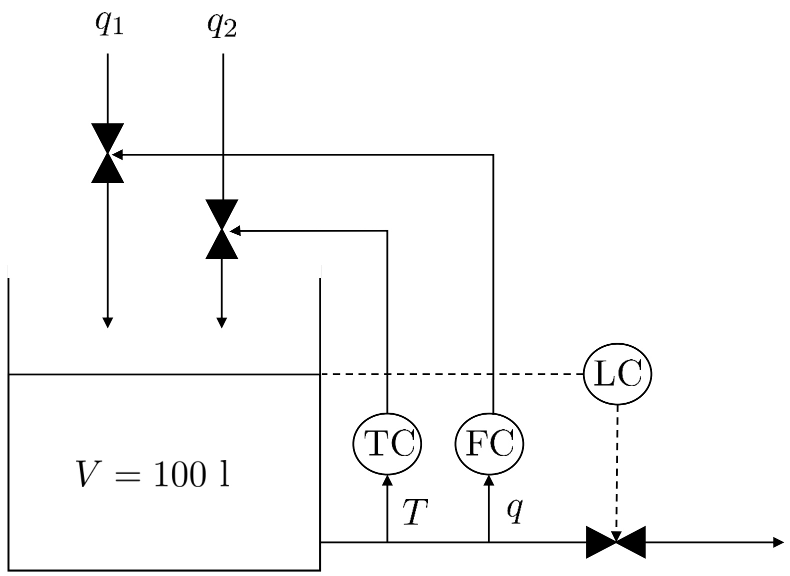

Figure 5: Flow sheet with proposed control system.

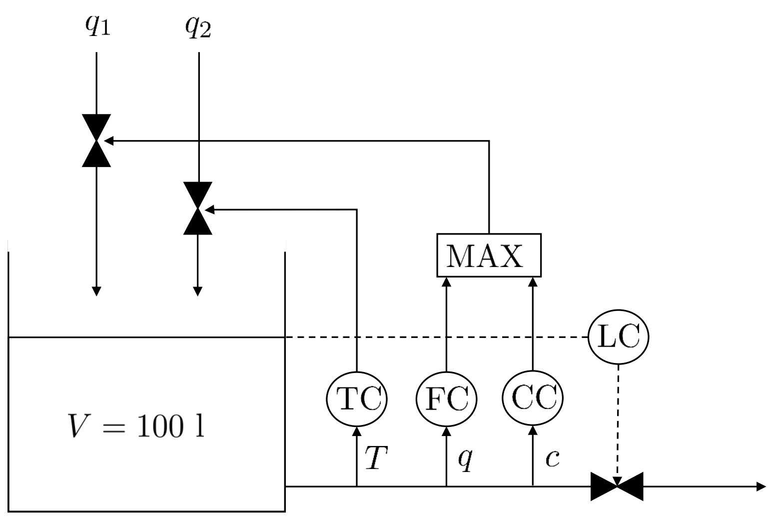

Figure 6: Proposed control structure with a selector.