Appendix K

Contour Mapping and the Principle of the Argument

The concept of contour mapping is illustrated in Fig. K.1. A closed contour C_s in the complex s-plane is mapped by a transfer function $H(s)$ into another closed contour C_H in the complex H-plane. For each point on C_s, there is a corresponding point on C_H. For example, three arbitrarily chosen points s_1, s_2, and s_3 on the C_s contour map into points $H(s_1)$, $H(s_2)$, and $H(s_3)$ on the C_H contour. Suppose that C_s is traversed in the clockwise direction, starting at s_1 and continuing along C_s to s_2 and s_3, before eventually returning to s_1. Then C_H will also be traversed, starting at $H(s_1)$ and continuing to $H(s_2)$ and to $H(s_3)$ before eventually returning to the starting point. In Fig. K.1, a clockwise traverse of C_s results in a clockwise traverse of C_H. However, this is not always the case; a counterclockwise traverse of C_H could result, depending on the particular $H(s)$ that is considered.

The concept of contour encirclement plays a key role in Nyquist stability theory. A contour is said to make a *clockwise encirclement* of a point if the point is always to the right of the contour as the contour is traversed in the clockwise direction. Thus, a single traverse of either C_H or C_s in Fig. K.1 results in a clockwise encirclement of the origin. The number of encirclements by C_H is related to the poles and zeroes of $H(s)$ that are located inside of C_s, by a well-known result from complex variable theory (Brown and Churchill, 2004; Franklin et al., 2005).

Principle of the Argument. *Consider a transfer function $H(s)$ and a closed contour C_s in the complex s-plane that is traversed in the clockwise (positive) direction. Assume that C_s does not pass through any poles or zeroes of $H(s)$. Let N be the number of clockwise (positive) encirclements of the origin by contour C_H in the complex H-plane. Define P and Z to be the numbers of poles and zeroes of $H(s)$, respectively, that are encircled by C_s in the clockwise direction. Then $N = Z - P$.*

Note that N is negative when $P > Z$. For this situation, the C_H contour encircles the origin in the *counterclockwise* (or *negative*) direction. Next, we show that the Nyquist Stability Criterion is based on a direct application of the Principle of the Argument.

K.1 DEVELOPMENT OF THE NYQUIST STABILITY CRITERION

According to the General Stability Criterion of Chapter 10, a feedback control system is stable if and only if all roots of the characteristic equation lie to the left of the imaginary axis. This condition motivates the following choices for function $H(s)$ and contour C_s:

1. Let $H(s) = 1 + G_{OL}(s)$, where $G_{OL}(s)$ is the open-loop transfer function, $G_{OL}(s) = G_c(s)G_v(s)G_p(s)G_m(s)$. Assume that $G_{OL}(s)$ is strictly proper (more poles than zeros) and does not contain any unstable pole-zero cancellations.
2. Contour C_s is chosen to be the boundary of the open right-half-plane (RHP). We assume that it is traversed in the clockwise (positive) direction.

This choice of C_s creates a dilemma—how do we evaluate $H(s)$ on the boundary of an infinite region? This problem is solved by choosing C_s to be the *Nyquist contour* shown in Fig. K.2. The Nyquist contour consists of the imaginary axis and a semicircle with radius, $R \to \infty$. Because $G_{OL}(s)$ is strictly proper (that is, it has more poles than zeros), $G_{OL}(s) \to 0$ as $R \to \infty$ and the semicircular arc of the Nyquist contour maps into the origin of the H-plane. Thus, the imaginary axis is the only

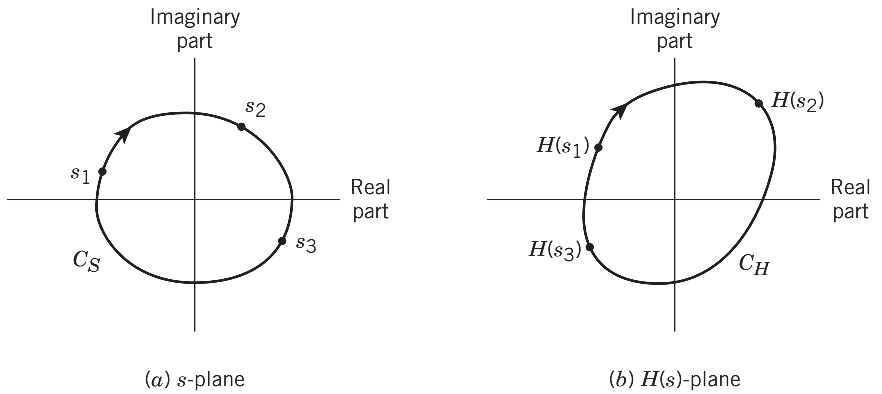

Figure K.1 Contour mapping for a transfer function, $H(s)$.

portion of the Nyquist contour that needs to be considered. In other words, we only have to evaluate $G_{OL}(s)$ for $s = j\omega$ and $-\infty < \omega < \infty$.

In order to apply the Principle of the Argument, we need to determine P, the number of poles of $H(s) = 1 + G_{OL}(s)$ that lie in the RHP. Fortunately, this is easy, because P is equal to the number of poles of $G_{OL}(s)$ that lie in the RHP. To prove this, suppose that $G_{OL}(s)$ has the general form,

$$G_{OL}(s) = \frac{A(s)e^{-\theta s}}{B(s)} \tag{K-1}$$

where $A(s)$ and $B(s)$ are polynomials in s and $G_{OL}(s)$ does not contain any unstable pole-zero cancellations. Then,

$$H(s) = 1 + G_{OL}(s) = 1 + \frac{A(s)e^{-\theta s}}{B(s)} = \frac{B(s) + A(s)e^{-\theta s}}{\mathrm{B}(s)} \tag{K-2}$$

Because $H(s)$ and $G_{OL}(s)$ have the same denominator, they have the same number of RHP poles.

Recall that $H(s)$ was defined as $H(s) = 1 + G_{OL}(s)$. Thus, the C_H and $C_{G_{OL}}$ contours have the same shape, but the C_H contour is shifted to the left by -1, relative to the $C_{G_{OL}}$ contour. Consequently, encirclements of the origin by C_H are identical to encirclements of the -1 point by $C_{G_{OL}}$. As a result, it is more convenient to express the Nyquist Stability Criterion in terms of $G_{OL}(s)$ rather than $H(s)$.

One more issue needs to be addressed, namely, the condition that C_s contour cannot pass through any pole or zero of $G_{OL}(s)$. Open-loop transfer functions often have a pole at the origin owing to an integrating element or integral control action. This difficulty is avoided by using the modified Nyquist contour in Fig. K.2, where $\varepsilon << 1$. A similar modification is available for the unusual situation where $G_{OL}(s)$ has a pair of complex poles on the imaginary axis. These modifications are described elsewhere (Kuo, 2003; Franklin et al., 2005). Although conceptually important, we do not have to be overly concerned with these modifications, because they are typically incorporated in software for control applications.

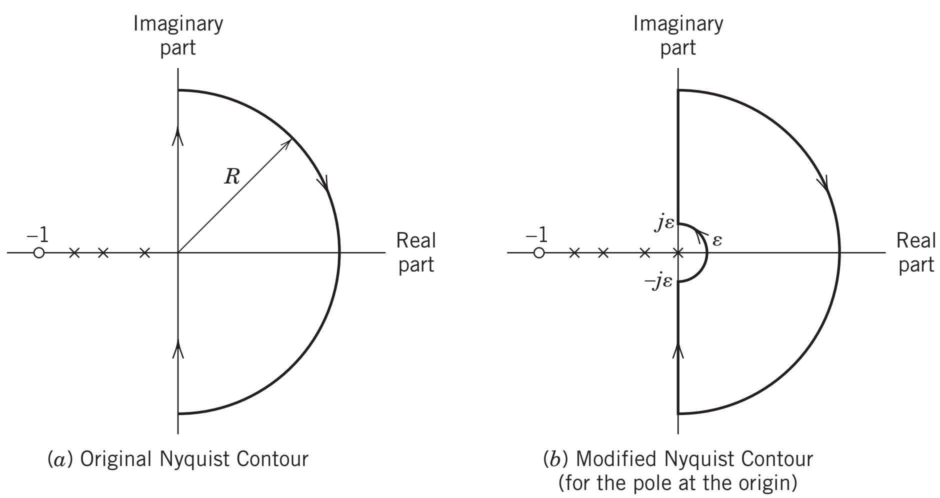

Figure K.2 Original and modified Nyquist contours.

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Franklin, G. F., J. D. Powell, and A. Emami-Naeini, *Feedback Control of Dynamic Systems*, 5th ed., Prentice Hall, Upper Saddle River, NJ, 2005.

Kuo, B. C., *Automatic Control Systems*, 8th ed., Prentice Hall, Englewood Cliffs, NJ, 2003, Chapter 9.