Studies on Controllability Analysis and Control Structure Design

by

Kjetil Havre

Chemical Engineering, Norwegian University of Science and Technology,
N-7034 Trondheim, Norway

Thesis overview

Ch. 1: Introduction.

Ch. 2: Directions of zeros and poles in multivariable systems.

Ch. 3–5: Limitations imposed by unstable zeros and poles:

- Ch. 3: Effect of RHP zeros and poles on the sensitivity functions.
- Ch. 4: Performance limitations for unstable SISO plants.
- Ch. 5: Achievable $\mathcal{H}_\infty$ performance of multivariable systems.

Ch. 6–9: Control structure design:

- Ch. 6: Selection of variables for regulatory control using pole vectors.
- Ch. 7: I/O selection and partial control.
- Ch. 8: I/O selection for indirect and cascade control.
- Ch. 9: I/O selection for indirect two-point control of product compositions in binary distillation.

Ch. 10: Effect of input/output uncertainty on performance in multivariable systems.

Ch. 11: Summary.

Appendix. A: Factorizations of zeros and poles in multivariable systems.

Notation

- Linear-time-invariant dynamical systems:

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

  where A, B, C and D are real matrices.

- Rational transfer function matrix:

$$G(s) = C(sI - A)^{-1}B + D$$

- $\mathcal{H}_2$-norm of a strictly proper stable rational transfer function matrix $M(s)$

$$||M(s)||_2 \triangleq \sqrt{\frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{tr}\left\{M^H(j\omega)M(j\omega)\right\} d\omega}$$

- $\mathcal{H}_\infty$-norm of a proper linear stable system $M(s)$

$$||M(s)||_\infty \triangleq \sup_{\omega} \bar{\sigma}(M(jw))$$

Directionality of zeros and poles

Zeros: z is a zero if and only if:

1) SISO: $z \quad \Leftrightarrow \quad G(z) = 0$.
2) MIMO: rank of $G(z)$ is less than the normal rank of $G(s)$.

Poles: The poles p are the singular points at which $G(s)$ fails to be analytic ("infinite"), Bode (1945).

Zero and pole directions:

	Input direction:	Output direction:
Zeros:	$G(z)u_z = 0;$	$y_z^H G(z) = 0$
Poles:	$G(p)u_p = \infty;$	$y_p^H G(p) = \infty$

Directionality of zeros and poles

Computation:

	Input:	Output:
Zero directions:	$\begin{bmatrix} A - zI & B \\ C & D \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_{zi} \\ u_z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$	$[\, \boldsymbol{x}_{zo}^H \quad y_z^H \,] \begin{bmatrix} A - zI & B \\ C & D \end{bmatrix} = [\, 0 \quad 0 \,]$
Pole vectors:	$\boldsymbol{u}_p = B^H x_{pi}, \quad x_{pi}^H A = p x_{pi}^H;$	$\boldsymbol{y}_p = C x_{po}, \quad A x_{po} = p x_{po}$
Pole directions:	$u_p = \begin{cases} \boldsymbol{u}_p / \lVert \boldsymbol{u}_p \rVert_2 & \text{if } \lVert \boldsymbol{u}_p \rVert_2 \neq 0 \\ 0 & \text{if } \lVert \boldsymbol{u}_p \rVert_2 = 0 \end{cases}$	$y_p = \begin{cases} \boldsymbol{y}_p / \lVert \boldsymbol{y}_p \rVert_2 & \text{if } \lVert \boldsymbol{y}_p \rVert_2 \neq 0 \\ 0 & \text{if } \lVert \boldsymbol{y}_p \rVert_2 = 0 \end{cases}$

State controllability and observability

THEOREM 1 (PBH EIGENVECTOR TESTS IN TERMS OF POLE DIRECTIONS).

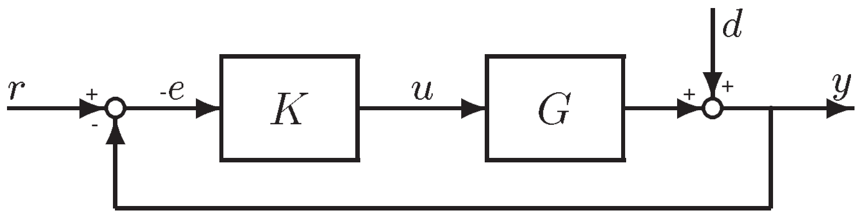
1) The pair $(A,\ B)$ *is controllable if and only if for all poles* p

$$\boldsymbol{u}_p \neq 0 \tag{1}$$

2) The pair $(C,\ A)$ *is observable if and only if for all poles* p

$$\boldsymbol{y}_p \neq 0 \tag{2}$$

Closing the control loop



- Closed-loop:

$$y = T(s)r + S(s)d \quad \text{and} \quad e = -S(s)r + S(s)d$$

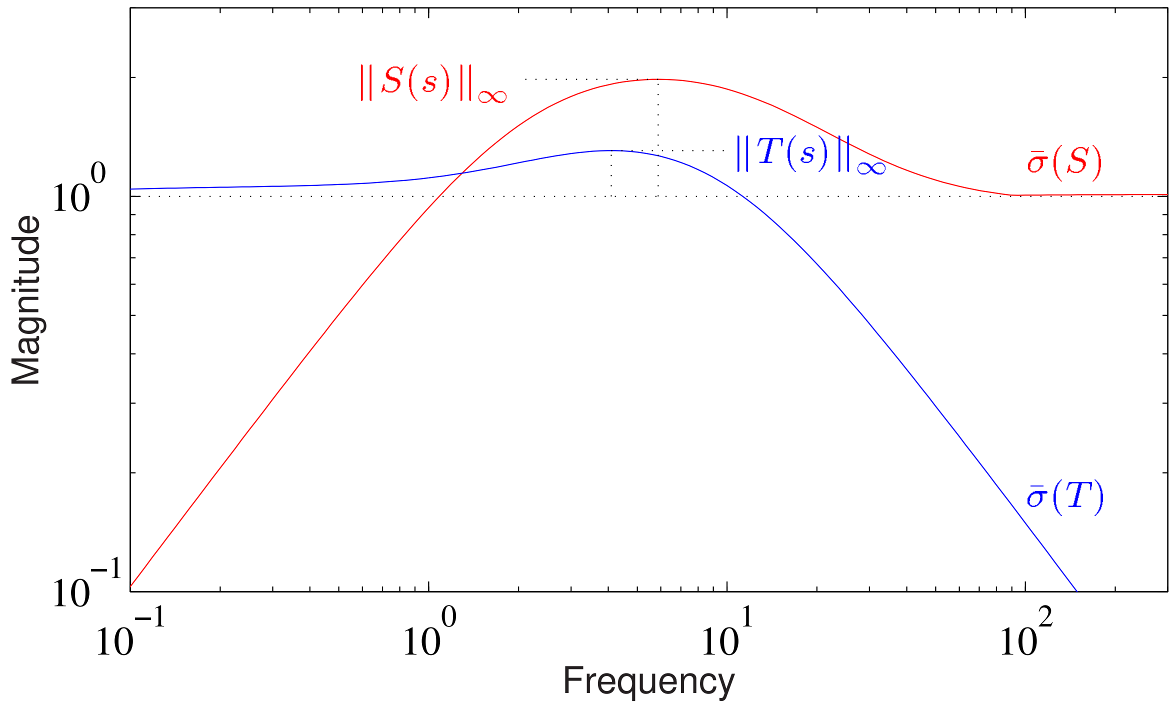
  ⋆ S is the sensitivity

$$S \triangleq (I + GK)^{-1}$$

  ⋆ T is the complementary sensitivity

$$T \triangleq GK(I + GK)^{-1}$$

Typical plot of $\bar{\sigma}(S(j\omega))$ and $\bar{\sigma}(T(jw))$



- $$\|S(s)\|_\infty = \sup_\omega \bar{\sigma}(S(j\omega))$$
- $$\|T(s)\|_\infty = \sup_\omega \bar{\sigma}(T(j\omega))$$

Why use the $\mathcal{H}_\infty$-norm?

- The $\mathcal{H}_\infty$-norm gives information about worst-case performance.
- May use weights:
  - Weighted sensitivity $w_P S$.
- RHP-zeros and poles imply that peaks are unavoidable.
- “Easier” to derive results.

Interpolation constraints

MIMO-interpolation constraints (required for internal stability):

RHP-zero z: $\begin{cases} \text{Input:} & S_I(z)u_z = u_z; \quad T_I(z)u_z = 0 \\ \text{Output:} & y_z^H S(z) = y_z^H; \quad y_z^H T(z) = 0 \end{cases}$

RHP-pole p: $\begin{cases} \text{Input:} & u_p^H S_I(p) = 0; \quad u_p^H T_I(p) = u_p^H \\ \text{Output:} & S(p)y_p = 0; \quad T(p)y_p = y_p \end{cases}$

SISO-interpolation constraints (required for internal stability):

RHP-zero z: $S(z) = 1; \quad T(z) = 0$

RHP-zero p: $S(p) = 0; \quad T(p) = 1$

Previous work

Given a plant G with RHP-zero z, then for closed-loop stability (Zames, 1981):

$$\boxed{\|w_P S(s)\|_\infty \;\geq\; |w_P(z)|} \tag{3}$$

Proof.

1) Select the scalar function $f(s) = y_z^H w_P S(s) y_z$, then:

$$\|w_P S(s)\|_\infty \geq \|f(s)\|_\infty$$

2) Apply maximum modulus theorem:

$$\|f(s)\|_\infty \geq |f(z)| = |y_z^H w_P(z) S(z) y_z|$$

3) Switch the order of y_z^H and $w_P(z)$ and use interpolation constraint:

$$y_z^H S(z) = y_z^H$$

□

“Symmetric” result

Given a plant G with RHP-pole p, then for closed-loop stability (Chapter 3):

$$\boxed{\|w_T T(s)\|_\infty \;\geq\; |w_T(p)|} \qquad (4)$$

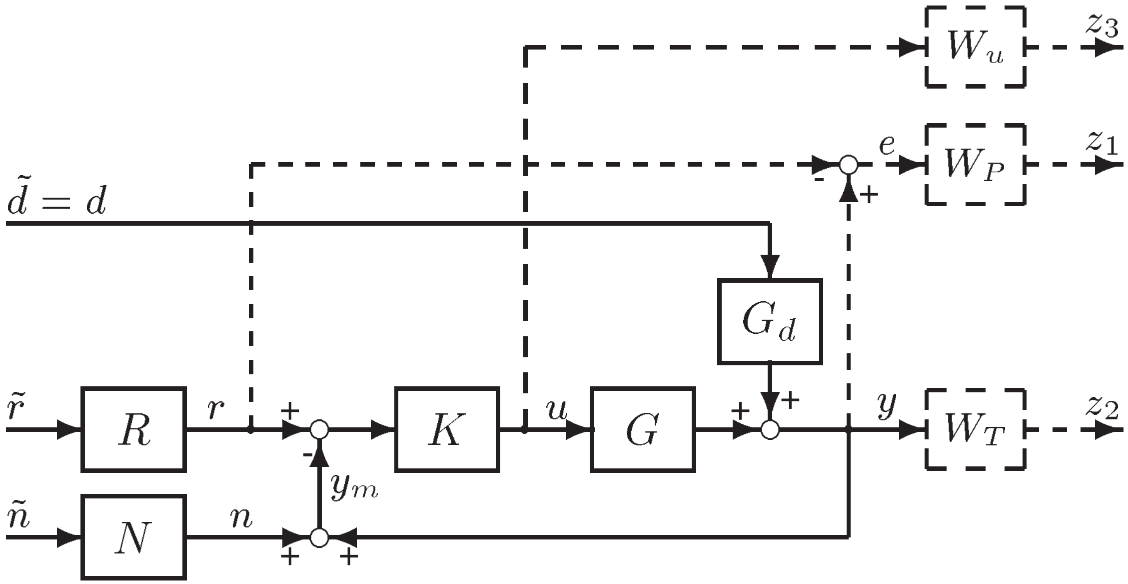
“Symmetric” results $(S$ and $T)$

Given a plant G with RHP-zeros z_j and RHP-poles p_i, then for closed-loop stability the following must apply (Chapter 3):

$$\boxed{\|w_P S(s)\|_\infty \geq \max_{z_j} |w_P(z_j)| \cdot \|y_{z_j}^H \mathcal{B}_{po}^{-1}(G)|_{s=z_j}\|_2} \tag{5}$$

$$\boxed{\|w_T T(s)\|_\infty \geq \max_{p_i} |w_T(p_i)| \cdot \|\mathcal{B}_{zo}^{-1}(G)|_{s=p_i} y_{p_i}\|_2} \tag{6}$$

Control configuration



Closed-loop transfer functions:

$$
\begin{aligned}
y &= T(s)r + SG_d(s)d - T(s)n \\
e &= -S(s)r + SG_d(s)d - T(s)n \\
u &= KS(s)r - KSG_d(s)d - KS(s)n
\end{aligned}
\qquad
\begin{aligned}
L(s) &\triangleq GK(s) \\
S(s) &\triangleq (I + L(s))^{-1} \\
T(s) &\triangleq L(s)(I + L(s))^{-1}
\end{aligned}
$$

Extensions

1) Other closed-loop transfer functions than S and T

- Disturbance rejection: $SG_d(s)$.
- Input usage: $KS(s)$ and $KSG_d(s)$.
- etc.

2) Matrix-valued weights, i.e. $W_P S$, $W_P SG_d$, etc.

3) Non-minimum phase and unstable "weights".

- Write $KS(s) = T_I G^{-1}(s)$ and use "weight"=G^{-1},
- or $KSG_d(s) = T_I G^{-1} G_d(s)$ and use "weight"=$G^{-1}G_d$.

Main result on lower bounds

THEOREM 2 (LOWER BOUNDS ON $\|WXV(s)\|_\infty$). *Plant G with:*

- RHP-*zeros z_j: input directions u_{z_j} and output directions y_{z_j}.*
- RHP-*poles p_i: input directions u_{p_i} and output directions y_{p_i}.*

Then for closed-loop stability the following bounds must apply:

$$\|WSV(s)\|_\infty \geq \max_{z_j} \|W_{mo}(z_j)y_{z_j}\|_2 \cdot \|y_{z_j}^H V \mathcal{B}_{zi}^{-1}(\mathcal{B}_{po}(G)\,V)|_{s=z_j}\|_2 \tag{7}$$

$$\|WS_IV(s)\|_\infty \geq \max_{z_j} \|\mathcal{B}_{zo}^{-1}(W\mathcal{B}_{pi}(G))\,W|_{s=z_j}u_{z_j}\|_2 \cdot \|u_{z_j}^H V_{mi}(z_j)\|_2 \tag{8}$$

$$\|WTV(s)\|_\infty \geq \max_{p_i} \|\mathcal{B}_{zo}^{-1}(W\mathcal{B}_{zo}(G))\,W|_{s=p_i}y_{p_i}\|_2 \cdot \|y_{p_i}^H V_{mi}(p_i)\|_2 \tag{9}$$

$$\|WT_IV(s)\|_\infty \geq \max_{p_i} \|W_{mo}(p_i)u_{p_i}\|_2 \cdot \|u_{p_i}^H V \mathcal{B}_{zi}^{-1}(\mathcal{B}_{zi}(G)\,V)|_{s=p_i}\|_2 \tag{10}$$

Applications of lower bounds

1) Output performance: Includes matrix-valued weights.

   a) Weighted sensitivity: $W_P S(s)$.
   b) Weighted complementary sensitivity: $W_T T(s)$.
   c) Disturbance rejection: $W_P S G_d(s)$.
   d) Reference tracking 1-DOF: $W_P S R(s)$.
   e) Measurement noise rejection: $W_P T N(s)$.

   1c) and 1d) can be combined using $V = [\, G_d \quad R \,]$.

2) Input usage: Basis for new bounds $KS(s) = T_I G^{-1}(s)$.

   a) Effect of disturbances on inputs: $W_u T_I G^{-1} G_d(s)$.
   b) Effect of reference changes on inputs: $W_u T_I G^{-1} R(s)$.
   c) Effect of measurement noise on inputs: $W_u T_I G^{-1} N(s)$.

   2a) to 2c) can be combined using $V = [\, G_d \quad R \quad N \,]$.

Tightness of lower bounds

- The lower bounds on $\|WSV(s)\|_\infty$ and $\|WS_IV(s)\|_\infty$ are tight when the plant G has one RHP-zero and any number of RHP-poles.
- The lower bounds on $\|WTV(s)\|_\infty$ and $\|WT_IV(s)\|_\infty$ are tight when the plant G has one RHP-pole and any number of RHP-zeros.

Proof.

- Find controllers which achieve these lower bounds.
- The thesis contains analytical expressions for the $\mathcal{H}_\infty$-optimal controllers.

$\square$

Lower bounds for SISO-systems

THEOREM 3 (LOWER BOUNDS ON $\|XV(s)\|_\infty$). *Let G be* SISO *plant with* RHP-*zeros z_j and* RHP-*poles p_i. Then for closed-loop stability the following bounds must apply:*

$$\|SV(s)\|_\infty \geq \max_{z_j} |\mathcal{B}_p^{-1}(z_j)| \cdot |V_{ms}(z_j)| \tag{11}$$

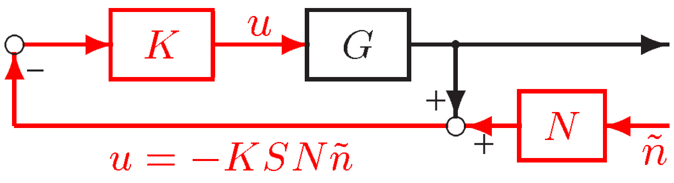
$$\|TV(s)\|_\infty \geq \max_{p_i} |\mathcal{B}_z^{-1}(p_i)| \cdot |V_{ms}(p_i)| \tag{12}$$

Implications of results (SISO)

RHP-pole p and input usage due to measurement noise:

Requirement: $\|KSN(s)\|_\infty \leq 1,$ BUT:

$$\|KSN(s)\|_\infty \geq |\mathcal{B}_z^{-1}(p)| \cdot |(G^{-1}N)_{ms}(p)|$$



EXAMPLE: G has RHP-pole only.

$$G(s) = \frac{1}{s-10}$$

$$p = 10, \quad \mathcal{B}_z(G) = 1 \quad \text{and} \quad |(G^{-1}N)_{ms}(p)| = |p+10| \cdot |N(p)| = 20 \cdot |N(p)|$$

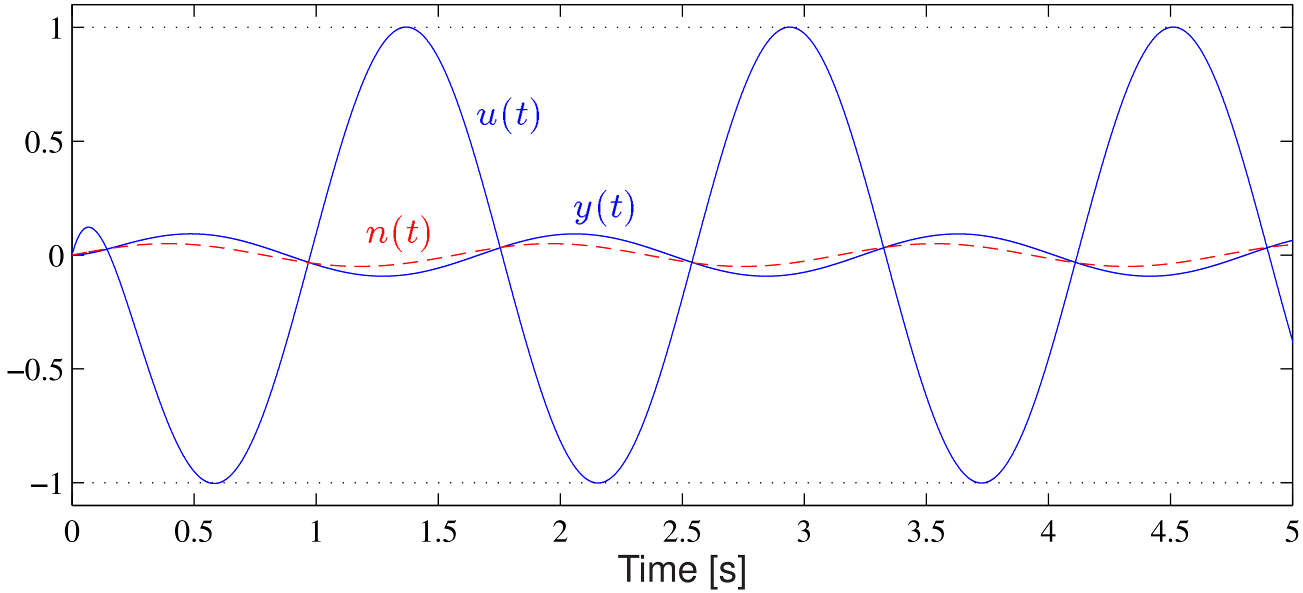
$$\|KSN(s)\|_\infty \geq |(G^{-1}N)_{ms}(p)| = 20 \cdot |N(p)| \quad \Rightarrow \quad |N(p)| \leq 0.05 \quad (5\%)$$

EXAMPLE CONTINUED: Set $N(p) = N = 1$, then $K_\infty = 2p = 20$.

Measurement noise:

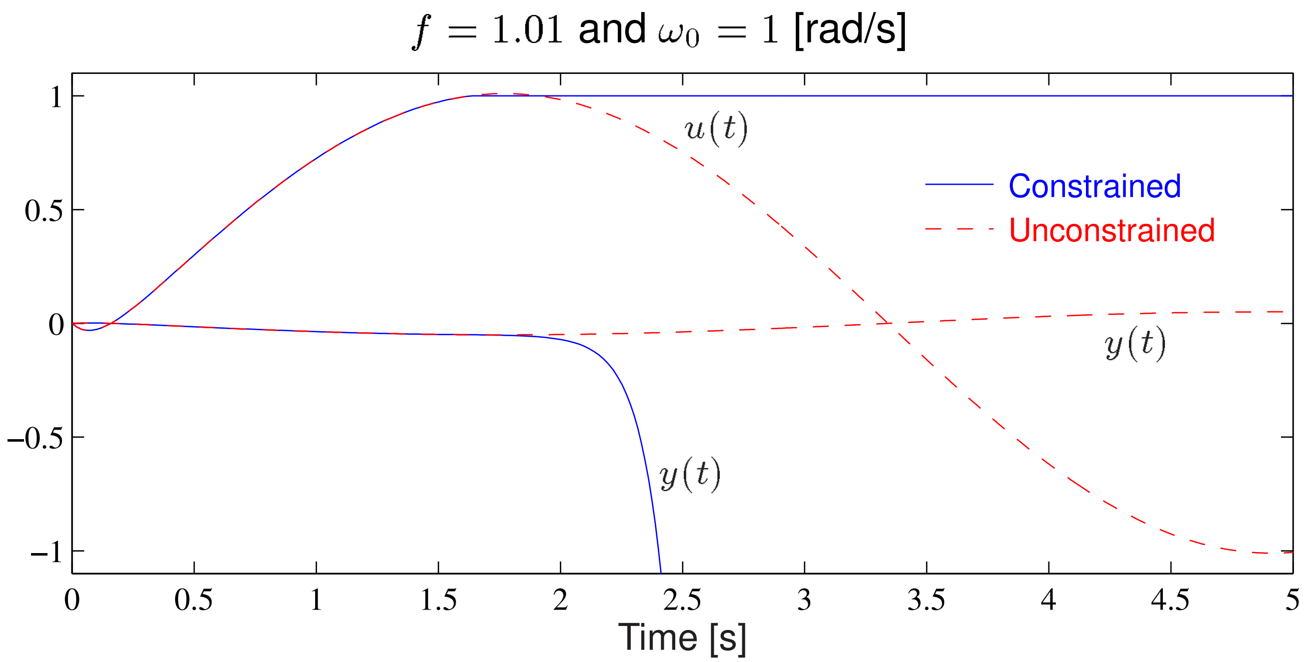
$$n(t) = f \cdot 0.05 \sin(\omega_0 t) \quad \Leftrightarrow \quad u(t) = f \sin(\omega_0 t + \varphi)$$

$$f = 1 \text{ and } \omega_0 = 4 \text{ [rad/s]}$$



Example continued:

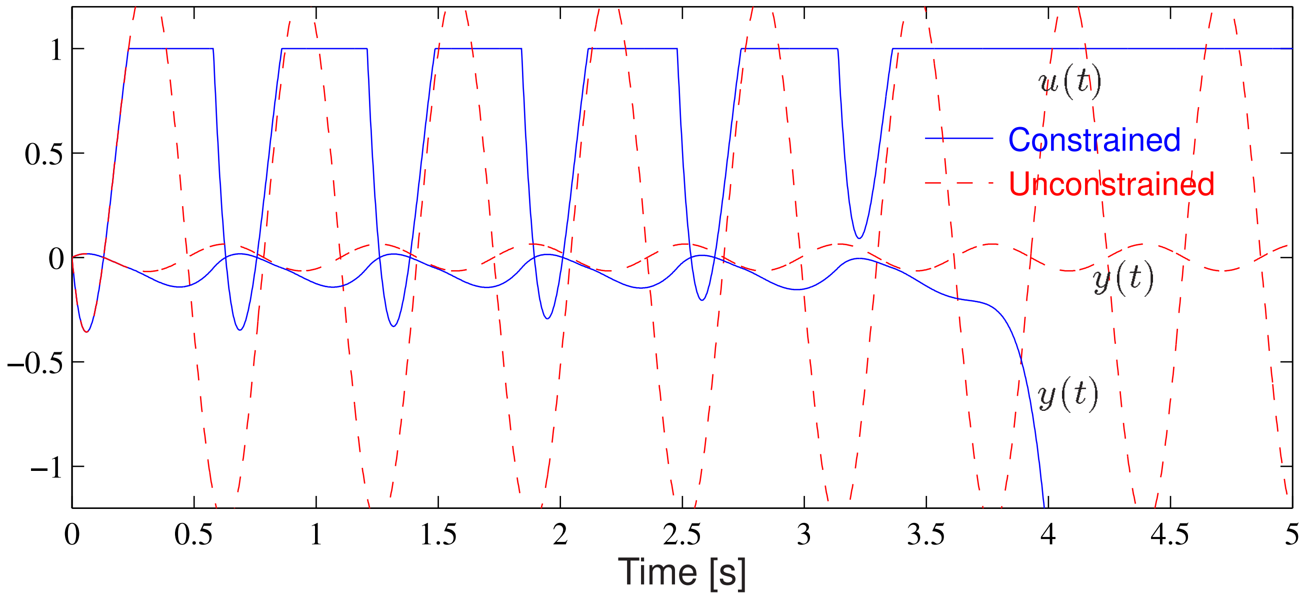
$$n(t) = f \cdot 0.05 \sin(\omega_0 t) \quad \Leftrightarrow \quad u(t) = f \sin(\omega_0 t + \varphi)$$



EXAMPLE CONTINUED:

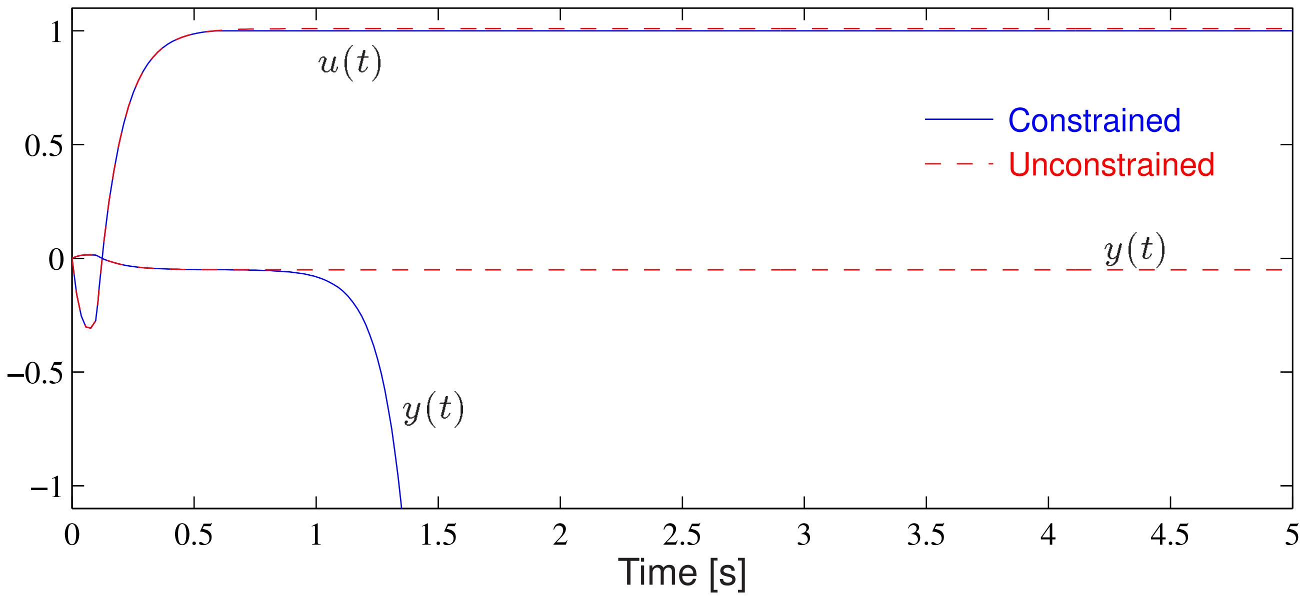
$$n(t) = f \cdot 0.05 \sin(\omega_0 t) \quad \Leftrightarrow \quad u(t) = f \sin(\omega_0 t + \varphi)$$

$f = 1.29$ and $\omega_0 = 10$ [rad/s]



EXAMPLE CONTINUED:

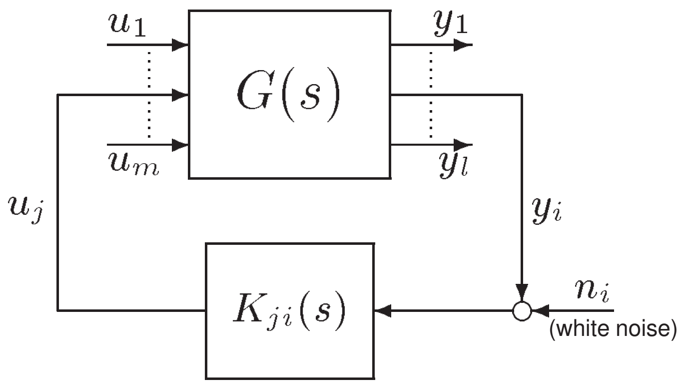
$$n(t) = 1.01 \cdot \frac{0.05}{s}$$



Summary on the effect of RHP-zeros and poles

- New tight and controller independent lower bounds on the $\mathcal{H}_\infty$-norm of "simple" closed-loop transfer functions are derived.
- Applications of lower bounds:
  - Quantify "worst case" performance both at input and the output of the plant.
  - Quantify the required input usage needed to stabilize an unstable plant in the presence of disturbances and measurement noise.
- Analytical $\mathcal{H}_\infty$-optimal controllers which achieve the lower bounds are derived.
- A special tight lower bound applicable to reference tracking for the 2-DOF control configuration are also derived.

Stabilizing control minimizing $\|K_{ji}S_{ii}(s)\|_2$



Consider a plant G, with one unstable mode p.

- Problem: Find pairing $u_j \leftrightarrow y_i$ such that the plant is stabilized with minimum input energy:

$$J(i,j) = E\left\{ \lim_{T\to\infty} \frac{1}{T}\int_0^T u_j^2(t)dt \right\}$$

- Solution:

$$\boxed{J = \frac{8p^3}{\boldsymbol{u}_{p,j}^2\,\boldsymbol{y}_{p,i}^2}(x_{pi}^H x_{po})^2}$$

- Select u_j corresponding to $\max_j \boldsymbol{u}_{p,j}$.
- Select y_i corresponding to $\max_i \boldsymbol{y}_{p,i}$.

Multivariable control minimizing $\|KS(s)\|_2$

- Problem: For a plant with one unstable mode p, quantify the minimum value of:

$$J = E\left\{\lim_{T\to\infty}\frac{1}{T}\int_0^T u^T(t)u(t)dt\right\}$$

- Solution:

$$\boxed{J = \frac{8p^3}{\|\boldsymbol{u}_p\|_2^2\,\|\boldsymbol{y}_p\|_2^2}(x_{pi}^T x_{po})^2}$$

- Performance degradation due to SISO control:

$$\varrho_2(i,j) = \frac{\|K_{ji}S_{ii}(s)\|_2}{\|K_{\mathrm{LQG}}S_{\mathrm{LQG}}(s)\|_2} = \frac{\sqrt{J(i,j)}}{\sqrt{J}} = \frac{\|\boldsymbol{u}_p\|_2\,\|\boldsymbol{y}_p\|_2}{|\boldsymbol{u}_{p,j}|\cdot|\boldsymbol{y}_{p,i}|} \geq 1$$

Stabilizing control minimizing $\|K_{ji}S_{ii}(s)\|_\infty$ and $\|KS(s)\|_\infty$

For a plant with one unstable mode p:

- SISO-control:

$$\boxed{\min_{K_{ji}(s)} \|K_{ji}S_{ii}(s)\|_\infty = |(G_{ij})_s^{-1}(p)| = \frac{2p}{|\boldsymbol{u}_{p,j}| \cdot |\boldsymbol{y}_{p,i}|}|x_{pi}^H x_{po}|}$$

- MIMO-control:

$$\min_{K(s)} \|KS(s)\|_\infty = \|u_p^H G_{so}^{-1}(p)\|_2 = \|G_{si}^{-1}(p)y_p\|_2$$

- Performance degradation due to SISO control:

$$\varrho_\infty(i,j) = \frac{\min\limits_{K_{ji}(s)} \|K_{ji}S_{ii}(s)\|_\infty}{\min\limits_{K(s)} \|KS(s)\|_\infty} = \frac{|(G_{ij})_s^{-1}(p)|}{\|u_p^H G_{so}^{-1}(p)\|_2} \geq 1$$

SISO stabilizing control minimizing the input usage

$\mathcal{H}_2$-norm

$$\min_{K_{ji}(s)} \|K_{ji}S_{ii}\|_2 = \frac{2p\sqrt{2p}}{|\boldsymbol{u}_{p,j}|\cdot|\boldsymbol{y}_{p,i}|}|x_{pi}^H x_{po}|$$

$\mathcal{H}_\infty$-norm

$$\min_{K_{ji}(s)} \|K_{ji}S_{ii}\|_\infty = \frac{2p}{|\boldsymbol{u}_{p,j}|\cdot|\boldsymbol{y}_{p,i}|}|x_{pi}^H x_{po}|$$

- For this special problem the $\mathcal{H}_2$ and the $\mathcal{H}_\infty$ norms are closely related:

$$\min_{K_{ji}(s)} \|K_{ji}S_{ii}(s)\|_\infty = \frac{1}{\sqrt{2p}} \min_{K_{ji}(s)} \|K_{ji}S_{ii}(s)\|_2$$

- The choices for the “best” input j and output i are independent of the norm.

Procedure for I/O-selection

1) Scale the inputs so that a change in each input is of equal importance on the overall objective.
2) Scale outputs relative to measurement noise (tolerable control error).
3) Compute pole vectors.
4) Select input j corresponding to a large element in $\boldsymbol{u}_p$.
5) Select output i corresponding to a large element in $\boldsymbol{y}_p$.
6) Synthesize a controller.

Repeat steps (3)–(6) until the plant is "stabilized". Steps (1), (2), (4) and (5) are implications from the results.

Tennessee Eastman

Outputs:

Variable name	No.	PID No.	Scaling
Reactor pressure	y_1	y_7	54.1 [kPa]
Reactor level	y_2	y_8	1.5 %
Reactor temperature	y_3	y_9	1.2 [$^\circ$C]
Separator temperature	y_4	y_{11}	1.0 [$^\circ$C]
Separator level	y_5	y_{12}	1.0 %
Separator pressure	y_6	y_{13}	52.6 [kPa]
Stripper level	y_7	y_{15}	1.0 %
Stripper pressure	y_8	y_{16}	62.0 [kPa]
Stripper temperature	y_9	y_{18}	1.0 [$^\circ$C]
Reactor cooling water outlet temperature	y_{10}	y_{21}	0.2 [$^\circ$C]
Separator cooling water outlet temperature	y_{11}	y_{22}	0.2 [$^\circ$C]

Inputs:

Variable name	No.	Stream no.	Scaling
D feed flow	u_1	2	10%
E feed flow	u_2	3	10%
A feed flow	u_3	3	10%
A and C feed flow	u_4	4	10%
Compressor recycle valve	u_5		10%
Purge valve	u_6	9	10%
Separator pot liquid flow	u_7	10	10%
Stripper liquid product flow	u_8	11	10%
Stripper steam valve	u_9	Stm	10%
Reactor cooling water flow	u_{10}	CWS	10%
Condenser cooling water flow	u_{11}	CWS	10%
Agitator speed	u_{12}		10%

Tennessee Eastman

Unstable poles: $P_u = [\,0 \quad 0.001 \quad 0.023 \pm 0.156i \quad 3.066 \pm 5.079i\,]$

Output pole vectors:

$$|Y_p| = \begin{array}{cl} \left[\begin{array}{cccc} 0.000 & 0.001 & 0.041 & 0.112 \\ 0.000 & 0.004 & 0.169 & 0.065 \\ 0.000 & 0.000 & 0.013 & 0.366 \\ 0.000 & 0.001 & 0.051 & 0.410 \\ 0.009 & 0.580 & 0.488 & 0.315 \\ 0.000 & 0.001 & 0.041 & 0.115 \\ \boxed{1.605} & 1.192 & 0.754 & 0.131 \\ 0.000 & 0.001 & 0.039 & 0.107 \\ 0.000 & 0.001 & 0.038 & 0.217 \\ 0.000 & 0.001 & 0.055 & \boxed{1.485} \\ 0.000 & 0.002 & 0.132 & 0.272 \end{array}\right] & \begin{array}{l} \\ \\ \\ \\ \\ \\ \leftarrow y_{15} \\ \\ \\ \leftarrow y_{21} \\ \\ \end{array} \end{array}$$

Input pole vectors:

$$|U_p| = \begin{array}{cl} \left[\begin{array}{cccc} 6.815 & 6.909 & 2.573 & 0.964 \\ 6.906 & 7.197 & 2.636 & 0.246 \\ 0.148 & 1.485 & 0.768 & 0.044 \\ 3.973 & 11.550 & 5.096 & 0.470 \\ 0.012 & 0.369 & 0.519 & 0.356 \\ 0.597 & 0.077 & 0.066 & 0.033 \\ 0.132 & 1.850 & 1.682 & 0.110 \\ \boxed{22.006} & 0.049 & 0.000 & 0.000 \\ 0.007 & 0.054 & 0.009 & 0.013 \\ 0.247 & 0.708 & 1.501 & \boxed{2.020} \\ 0.109 & 0.976 & 1.446 & 0.753 \\ 0.033 & 0.094 & 0.201 & 0.302 \end{array}\right] & \begin{array}{l} \\ \\ \\ \\ \\ \\ \\ \leftarrow u_8 \\ \\ \leftarrow u_{10} \\ \\ \\ \end{array} \end{array}$$

Tennessee Eastman

Suggested pairings:

stripper level y_{15}

$\updownarrow$

stripper liquid product flow u_8

reactor cooling water outlet temperature y_{21}

$\updownarrow$

reactor cooling water flow u_{10}

PI-controllers (from McAvoy and Ye, 1994):

- Reactor/cooling water, $y_{21} \leftrightarrow u_{10}$:

$$k_{p,1} = -0.1\,[1/^\circ\mathrm{C}], \quad T_{i,1} = 1\,[\mathrm{min}]$$

- Stripper level/stripper outflow, $y_{15} \leftrightarrow u_8$:

$$k_{p,2} = -0.05\,[\mathrm{m}^3/\mathrm{h}], \quad T_{i,2} = 300\,[\mathrm{min}]$$

Tennessee Eastman

Remaining unstable pole: $P_u = 0.001$

Output and input pole vectors:

$$Y_p = \begin{bmatrix} -0.001 \\ -0.005 \\ 0.000 \\ 0.001 \\ \boxed{-0.867} \\ -0.001 \\ 0.000 \\ -0.001 \\ 0.001 \\ 0.000 \\ -0.002 \end{bmatrix} \begin{array}{l} \\ \\ \\ \\ \leftarrow y_{12} \\ \\ \\ \\ \\ \\ \\ \end{array} \qquad U_p = \begin{bmatrix} -7.363 \\ -7.536 \\ 1.410 \\ 11.515 \\ -0.346 \\ -0.065 \\ \boxed{2.465} \\ 0.000 \\ -0.062 \\ 0.008 \\ 0.901 \\ -0.078 \end{bmatrix} \begin{array}{l} \\ \\ \\ \\ \\ \\ \leftarrow u_7 \\ \\ \\ \\ \\ \\ \end{array}$$

Suggested pairing:

separator level y_{12}

$\updownarrow$

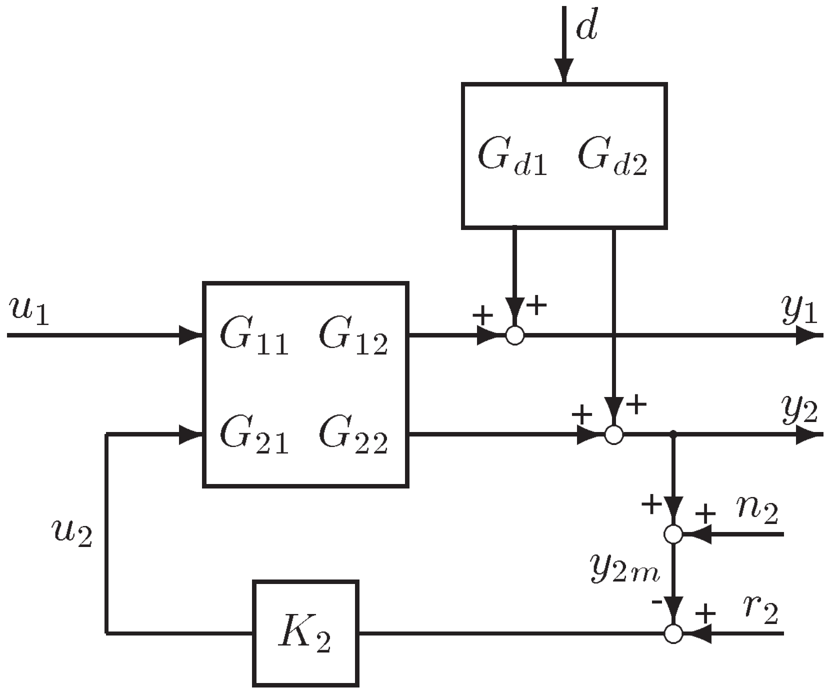
separator liquid flow u_7

PI-controller:

$$k_{p,3} = -0.025\,[\mathrm{m}^3/\mathrm{h}]$$

$$T_{i,3} = 200\,[\mathrm{min}]$$

Partial control



y_1 uncontrolled outputs.

y_2 controlled outputs.

u_1 inputs not used for control.

u_2 inputs used to control y_2.

Conclusions

- Directionality of zeros and poles are studied.
- New tight and controller independent lower bounds on the $\mathcal{H}_\infty$-norm of "simple" closed-loop transfer functions are derived.
- Applications of lower bounds:
  - Quantify "worst case" performance both at input and the output of the plant.
  - Quantify the required input usage needed to stabilize an unstable plant in the presence of disturbances and measurement noise.

Conclusions

- Find the "best" input–output pairing to stabilize a plant with one unstable mode:
  1) Minimizing the $\mathcal{H}_2$-norm of input usage (input energy).
  2) Minimizing the $\mathcal{H}_\infty$-norm of the input usage.
- Quantify the effect of disturbances, measurement noise and reference changes in partially controlled systems.
- Use the controllability tools derived in partial control to address the input/output selection.
- Quantify the effect of input and output uncertainty on the performance at the output of the plant.
- Several realistic case studies are given.

Future work

- Generalize the definition of zero and pole directions to rational transfer functions described by polynomial matrices.
- Extensions to the lower bounds and the controllers proving tightness:
  1) Improved bounds on $\|WSV(s)\|_\infty$ and $\|WS_IV(s)\|_\infty$ which takes into account more than one RHP-zero a time.
  2) Improved bounds on $\|WTV(s)\|_\infty$ and $\|WT_IV(s)\|_\infty$ which takes into account more than one RHP-pole a time.
  3) Find analytical expressions for the controllers which achieve these improved lower bounds.
  4) Find analytical controllers which minimize the $\mathcal{H}_\infty$-norm of stacked closed-loop transfer functions.
- Extension of the work in Chapter 10: consider output uncertainty and measure the performance at the input of the plant.