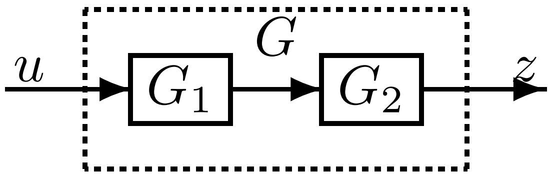
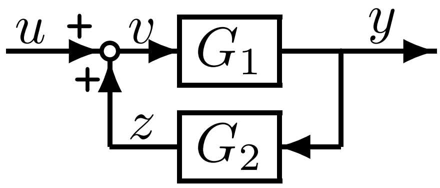
6 INTRODUCTION TO MULTIVARIABLE CONTROL [3]

6.1 Transfer functions for MIMO systems [3.2]



(a) Cascade system



(b) Positive feedback system

Figure 52: Block diagrams for the cascade rule and the feedback rule

1. **Cascade rule.** (Figure 52(a)) $G = G_2 G_1$
2. **Feedback rule.** (Figure 52(b)) $v = (I - L)^{-1} u$ where $L = G_2 G_1$
3. **Push-through rule.**

$$G_1 (I - G_2 G_1)^{-1} \quad = \quad (I - G_1 G_2)^{-1} G_1$$

MIMO Rule: *Start from the output, move backwards. If you exit from a feedback loop then include a term $(I - L)^{-1}$ where L is the transfer function around that loop (evaluated against the signal flow starting at the point of exit from the loop).*

Example

$$z = (P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21})w \qquad (6.1)$$

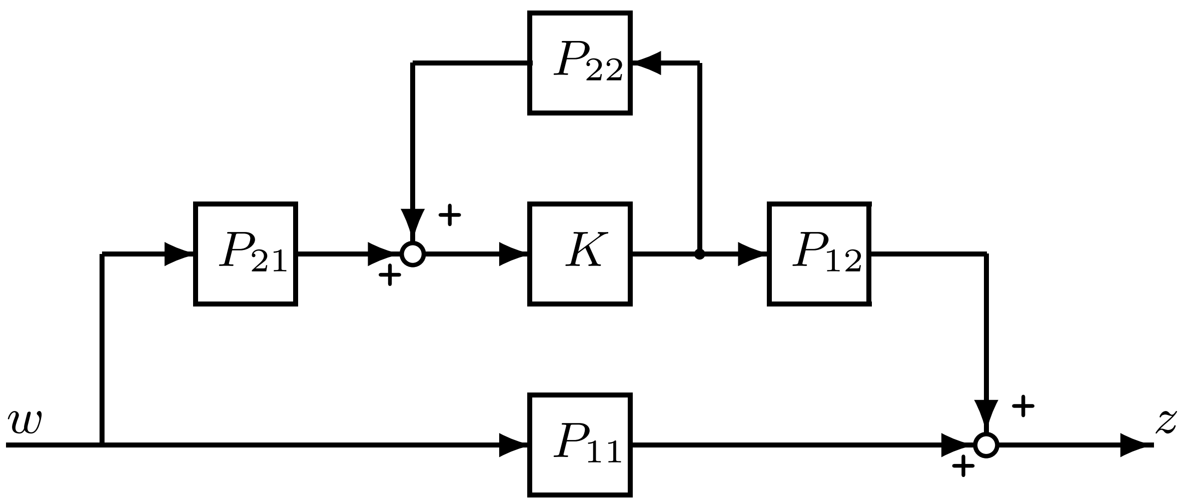


Figure 53: Block diagram corresponding to (6.1)

Negative feedback control systems

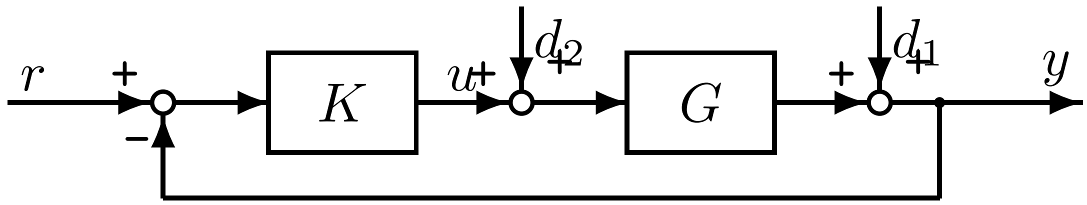


Figure 54: Conventional negative feedback control system

- L is the loop transfer function when breaking the loop at the *output* of the plant.

$$L = GK \tag{6.2}$$

Accordingly

$$S \triangleq (I + L)^{-1}$$

$$\textit{output sensitivity} \tag{6.3}$$

$$T \triangleq I - S = (I + L)^{-1}L = L(I + L)^{-1}$$

$$\textit{output complementary sensitivity} \tag{6.4}$$

$L_O \equiv L$, $S_O \equiv S$ and $T_O \equiv T$.

- L_I is the loop transfer function at the *input* to the plant

$$L_I = KG \tag{6.5}$$

Input sensitivity:

$$S_I \triangleq (I + L_I)^{-1}$$

Input complementary sensitivity:

$$T_I \triangleq I - S_I = L_I(I + L_I)^{-1}$$

- Some relationships:

$$(I + L)^{-1} + (I + L)^{-1}L = S + T = I \tag{6.6}$$

$$G(I + KG)^{-1} = (I + GK)^{-1}G \tag{6.7}$$

$$GK(I+GK)^{-1} = G(I+KG)^{-1}K = (I+GK)^{-1}GK \tag{6.8}$$

$$T = L(I + L)^{-1} = (I + L^{-1})^{-1} = (I + L)^{-1}L \tag{6.9}$$

Rule to remember: "G comes first and then G and K alternate in sequence".

6.2 Multivariable frequency response analysis [3.3]

$$\begin{aligned} G(s) &= \text{transfer (function) matrix} \\ G(j\omega) &= \text{complex matrix representing response} \\ &\quad \text{to sinusoidal signal of frequency } \omega \end{aligned}$$

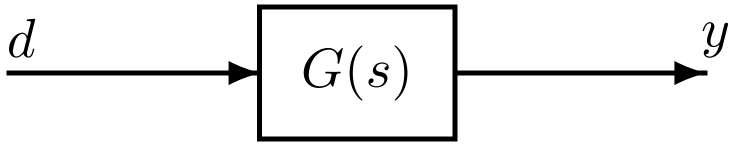


Figure 55: System $G(s)$ with input d and output y

$$y(s) = G(s)d(s) \tag{6.10}$$

Sinusoidal input to channel j

$$d_j(t) = d_{j0} \sin(\omega t + \alpha_j) \tag{6.11}$$

starting at $t = -\infty$. Output in channel i is a sinusoid with the same frequency

$$y_i(t) = y_{i0} \sin(\omega t + \beta_i) \tag{6.12}$$

Amplification (gain):

$$\frac{y_{io}}{d_{jo}} = |g_{ij}(j\omega)| \tag{6.13}$$

Phase shift:

$$\beta_i - \alpha_j = \angle g_{ij}(j\omega) \tag{6.14}$$

$g_{ij}(j\omega)$ represents the sinusoidal response from input j to output i.

Example 2×2 multivariable system, sinusoidal signals of the same frequency ω to the two input channels:

$$d(t) = \begin{bmatrix} d_1(t) \\ d_2(t) \end{bmatrix} = \begin{bmatrix} d_{10}\sin(\omega t + \alpha_1) \\ d_{20}\sin(\omega t + \alpha_2) \end{bmatrix} \qquad (6.15)$$

The output signal

$$y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} y_{10}\sin(\omega t + \beta_1) \\ y_{20}\sin(\omega t + \beta_2) \end{bmatrix} \qquad (6.16)$$

can be computed by multiplying the complex matrix $G(j\omega)$ by the complex vector $d(\omega)$:

$$\begin{aligned} y(\omega) &= G(j\omega)d(\omega) \\ y(\omega) &= \begin{bmatrix} y_{10}e^{j\beta_1} \\ y_{20}e^{j\beta_2} \end{bmatrix}, \; d(\omega) = \begin{bmatrix} d_{10}e^{j\alpha_1} \\ d_{20}e^{j\alpha_2} \end{bmatrix} \end{aligned} \qquad (6.17)$$

6.2.1 Directions in multivariable systems [3.3.2]

SISO system ($y = Gd$): gain

$$\frac{|y(\omega)|}{|d(\omega)|} = \frac{|G(j\omega)d(\omega)|}{|d(\omega)|} = |G(j\omega)|$$

The gain depends on ω, but is independent of $|d(\omega)|$.

MIMO system: input and output are vectors.

$\Rightarrow$ need to "sum up" magnitudes of elements in each vector by use of some norm

$$\|d(\omega)\|_2 = \sqrt{\sum_j |d_j(\omega)|^2} = \sqrt{d_{10}^2 + d_{20}^2 + \cdots} \quad (6.18)$$

$$\|y(\omega)\|_2 = \sqrt{\sum_i |y_i(\omega)|^2} = \sqrt{y_{10}^2 + y_{20}^2 + \cdots} \quad (6.19)$$

The *gain* of the system $G(s)$ is

$$\frac{\|y(\omega)\|_2}{\|d(\omega)\|_2} = \frac{\|G(j\omega)d(\omega)\|_2}{\|d(\omega)\|_2} = \frac{\sqrt{y_{10}^2 + y_{20}^2 + \cdots}}{\sqrt{d_{10}^2 + d_{20}^2 + \cdots}} \quad (6.20)$$

The gain depends on ω, and is independent of $\|d(\omega)\|_2$. However, for a MIMO system the gain depends on the *direction* of the input d.

Example Consider the five inputs (all $\|d\|_2 = 1$)

$$d_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix},\ d_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\ d_3 = \begin{bmatrix} 0.707 \\ 0.707 \end{bmatrix},$$
$$d_4 = \begin{bmatrix} 0.707 \\ -0.707 \end{bmatrix},\ d_5 = \begin{bmatrix} 0.6 \\ -0.8 \end{bmatrix}$$

For the 2×2 system

$$G_1 = \begin{bmatrix} 5 & 4 \\ 3 & 2 \end{bmatrix} \tag{6.21}$$

The five inputs d_j lead to the following output vectors

$$y_1 = \begin{bmatrix} 5 \\ 3 \end{bmatrix},\ y_2 = \begin{bmatrix} 4 \\ 2 \end{bmatrix},\ y_3 = \begin{bmatrix} 6.36 \\ 3.54 \end{bmatrix},\ y_4 = \begin{bmatrix} 0.707 \\ 0.707 \end{bmatrix}, y_5 = \begin{bmatrix} -0.2 \\ 0.2 \end{bmatrix}$$

with the 2-norms (i.e. the gains for the five inputs)

$$\|y_1\|_2 = 5.83,\ \|y_2\|_2 = 4.47,\ \|y_3\|_2 = 7.30,$$
$$\|y_4\|_2 = 1.00,\ \|y_5\|_2 = 0.28$$

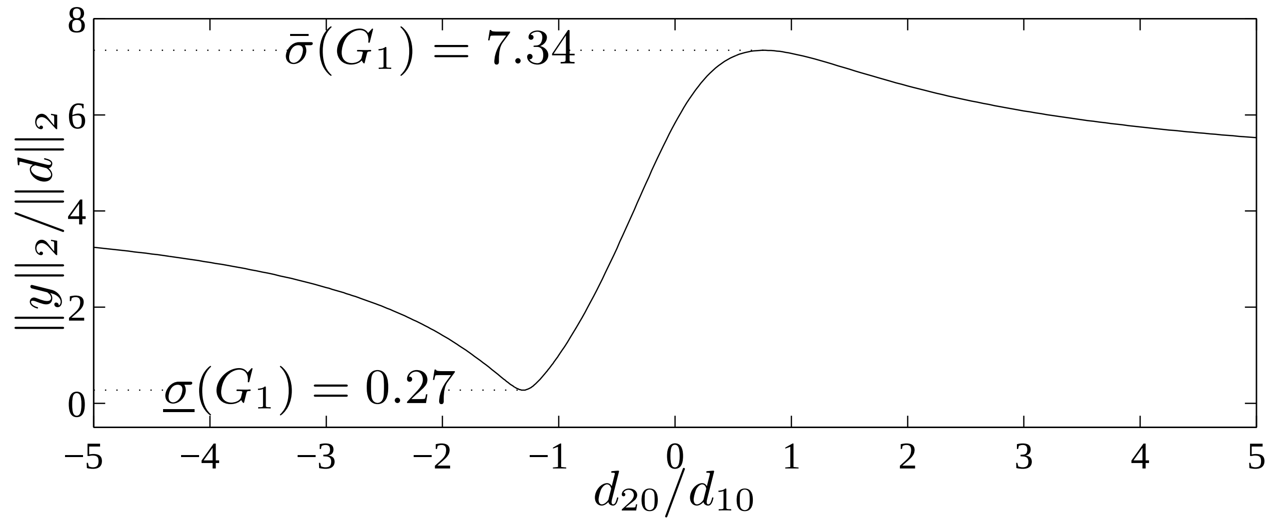


Figure 56: Gain $\|G_1 d\|_2/\|d\|_2$ as a function of d_{20}/d_{10} for G_1 in (6.21)

The maximum value of the gain in (6.20) as the direction of the input is varied, is the maximum singular value of G,

$$\max_{d \neq 0} \frac{\|Gd\|_2}{\|d\|_2} = \max_{\|d\|_2=1} \|Gd\|_2 = \bar{\sigma}(G) \qquad (6.22)$$

whereas the minimum gain is the minimum singular value of G,

$$\min_{d \neq 0} \frac{\|Gd\|_2}{\|d\|_2} = \min_{\|d\|_2=1} \|Gd\|_2 = \underline{\sigma}(G) \qquad (6.23)$$

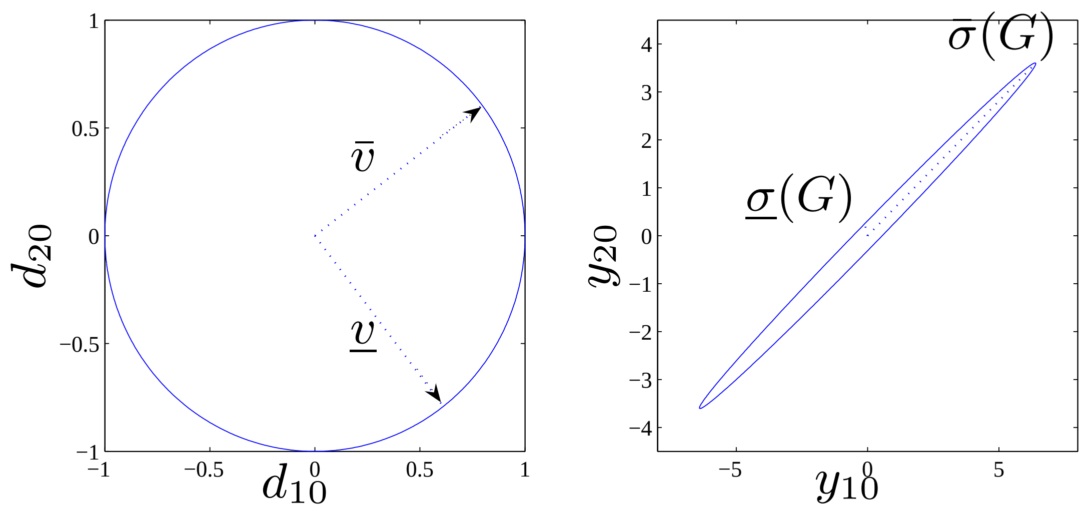


Figure 1: Outputs (right plot) resulting from use of $\|d\|_2 = 1$ (unit circle in left plot) for system G. The maximum ($\bar{\sigma}(G)$) and minimum ($\underline{\sigma}(G)$) gains are obtained for $d = (\bar{v})$ and $d = (\underline{v})$ respectively.

6.2.2 Eigenvalues are a poor measure of gain [3.3.3]

Example

$$G = \begin{bmatrix} 0 & 100 \\ 0 & 0 \end{bmatrix}; \quad G \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 100 \\ 0 \end{bmatrix} \tag{6.24}$$

Both eigenvalues are equal to zero, but gain is equal to 100.

Problem: eigenvalues measure the gain for the special case when the inputs and the outputs are in the same direction (in the direction of the eigenvectors).

For generalizations of $|G|$ when G is a matrix, we need the concept of a *matrix norm*, denoted $\|G\|$. Two important properties: *triangle inequality*

$$\|G_1 + G_2\| \leq \|G_1\| + \|G_2\| \tag{6.25}$$

and the multiplicative property

$$\|G_1 G_2\| \leq \|G_1\| \cdot \|G_2\| \tag{6.26}$$

$\rho(G) \triangleq |\lambda_{max}(G)|$ (the spectral radius), does *not* satisfy the properties of a matrix norm

6.2.3 Singular value decomposition [3.3.4]

Any matrix G may be decomposed into its singular value decomposition,

$$G = U\Sigma V^H \tag{6.27}$$

where

Σ is an $l \times m$ matrix with $k = \min\{l, m\}$ non-negative singular values, σ_i, arranged in descending order along its main diagonal;

U is an $l \times l$ unitary matrix of output singular vectors, u_i,

V is an $m \times m$ unitary matrix of input singular vectors, v_i,

Example SVD of a real 2×2 matrix can always be written as

$$G = \underbrace{\begin{bmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix}}_{\Sigma} \underbrace{\begin{bmatrix} \cos\theta_2 & \pm\sin\theta_2 \\ -\sin\theta_2 & \pm\cos\theta_2 \end{bmatrix}^T}_{V^T} \tag{6.28}$$

U and V involve rotations and their columns are orthonormal.

Input and output directions.

The column vectors of U, denoted u_i, represent the *output directions* of the plant. They are orthogonal and of unit length (orthonormal), that is

$$\|u_i\|_2 = \sqrt{|u_{i1}|^2 + |u_{i2}|^2 + \ldots + |u_{il}|^2} = 1 \qquad (6.29)$$

$$u_i^H u_i = 1, \quad u_i^H u_j = 0, \quad i \neq j \qquad (6.30)$$

The column vectors of V, denoted v_i, are orthogonal and of unit length, and represent the *input directions*.

$$G v_i = \sigma_i u_i \qquad (6.31)$$

If we consider an *input* in the direction v_i, then the *output* is in the direction u_i. Since $\|v_i\|_2 = 1$ and $\|u_i\|_2 = 1$ σ_i gives the gain of the matrix G in this direction.

$$\sigma_i(G) = \|G v_i\|_2 = \frac{\|G v_i\|_2}{\|v_i\|_2} \qquad (6.32)$$

Maximum and minimum singular values.

The largest gain for *any* input direction is

$$\bar{\sigma}(G) \equiv \sigma_1(G) = \max_{d \neq 0} \frac{\|Gd\|_2}{\|d\|_2} = \frac{\|Gv_1\|_2}{\|v_1\|_2} \qquad (6.33)$$

The smallest gain for any input direction is

$$\underline{\sigma}(G) \equiv \sigma_k(G) = \min_{d \neq 0} \frac{\|Gd\|_2}{\|d\|_2} = \frac{\|Gv_k\|_2}{\|v_k\|_2} \qquad (6.34)$$

where $k = \min\{l, m\}$. For any vector d we have

$$\underline{\sigma}(G) \leq \frac{\|Gd\|_2}{\|d\|_2} \leq \bar{\sigma}(G) \qquad (6.35)$$

Define $u_1 = \bar{u}, v_1 = \bar{v}, u_k = \underline{u}$ and $v_k = \underline{v}$. Then

$$G\bar{v} = \bar{\sigma}\bar{u}, \qquad G\underline{v} = \underline{\sigma}\ \underline{u} \qquad (6.36)$$

$\bar{v}$ corresponds to the input direction with largest amplification, and $\bar{u}$ is the corresponding output direction in which the inputs are most effective. The directions involving $\bar{v}$ and $\bar{u}$ are sometimes referred to as the “strongest”, “high-gain” or “most important” directions.

Example

$$G_1 = \begin{bmatrix} 5 & 4 \\ 3 & 2 \end{bmatrix} \tag{6.37}$$

The singular value decomposition of G_1 is

$$G_1 = \underbrace{\begin{bmatrix} 0.872 & 0.490 \\ 0.490 & -0.872 \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} 7.343 & 0 \\ 0 & 0.272 \end{bmatrix}}_{\Sigma} \underbrace{\begin{bmatrix} 0.794 & -0.608 \\ 0.608 & 0.794 \end{bmatrix}^H}_{V^H}$$

The largest gain of 7.343 is for an input in the direction $\bar{v} = \begin{bmatrix} 0.794 \\ 0.608 \end{bmatrix}$, the smallest gain of 0.272 is for an input in the direction $\underline{v} = \begin{bmatrix} -0.608 \\ 0.794 \end{bmatrix}$. Since in (6.37) both inputs affect both outputs, we say that the system is *interactive*. The system is *ill-conditioned*, that is, some combinations of the inputs have a strong effect on the outputs, whereas other combinations have a weak effect on the outputs. Quantified by the *condition number*; $\bar{\sigma}/\underline{\sigma} = 7.343/0.272 = 27.0$.

Example

Shopping cart. Consider a shopping cart (supermarket trolley) with fixed wheels which we may want to move in three directions; forwards, sideways and upwards. For the shopping cart the gain depends strongly on the input direction, i.e. the plant is ill-conditioned.

Example: Distillation process.

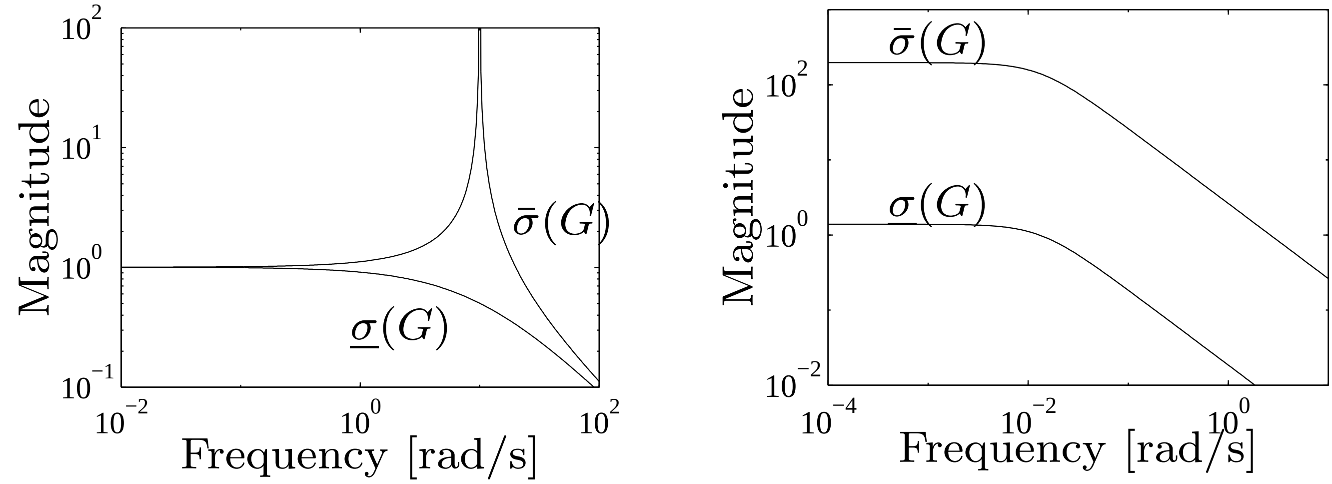
Steady-state model of a distillation column

$$G = \begin{bmatrix} 87.8 & -86.4 \\ 108.2 & -109.6 \end{bmatrix} \tag{6.38}$$

Since the elements are much larger than 1 in magnitude there should be no problems with input constraints. However, the gain in the low-gain direction is only just above 1.

$$G = \underbrace{\begin{bmatrix} 0.625 & -0.781 \\ 0.781 & 0.625 \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} 197.2 & 0 \\ 0 & 1.39 \end{bmatrix}}_{\Sigma} \underbrace{\begin{bmatrix} 0.707 & -0.708 \\ -0.708 & -0.707 \end{bmatrix}^H}_{V^H} \tag{6.39}$$

The distillation process is *ill-conditioned*, and the condition number is $197.2/1.39 = 141.7$. For dynamic systems the singular values and their associated directions vary with frequency (Figure 57).



(a) Spinning satellite in (6.44)

(b) Distillation process in (6.49)

Figure 57: Typical plots of singular values

6.2.4 Singular values for performance [3.3.5]

Maximum singular value is very useful in terms of frequency-domain performance and robustness.

Performance measure for SISO systems:

$$|e(\omega)|/|r(\omega)| = |S(j\omega)|$$

.

Generalization for MIMO systems $\|e(\omega)\|_2/\|r(\omega)\|_2$

$$\underline{\sigma}(S(j\omega)) \leq \frac{\|e(\omega)\|_2}{\|r(\omega)\|_2} \leq \bar{\sigma}(S(j\omega)) \tag{6.40}$$

For *performance* we want the gain $\|e(\omega)\|_2/\|r(\omega)\|_2$ small for any direction of $r(\omega)$

$$\begin{aligned} \bar{\sigma}(S(j\omega)) < 1/|w_P(j\omega)|, \ \forall\omega \quad &\Leftrightarrow \quad \bar{\sigma}(w_P S) < 1, \ \forall\omega \\ &\Leftrightarrow \quad \|w_P S\|_\infty < 1 \end{aligned} \tag{6.41}$$

where the $\mathcal{H}_\infty$ norm is defined as the peak of the maximum singular value of the frequency response

$$\|M(s)\|_\infty \triangleq \max_\omega \bar{\sigma}(M(j\omega)) \tag{6.42}$$

Typical singular values of $S(j\omega)$ in Figure 58.

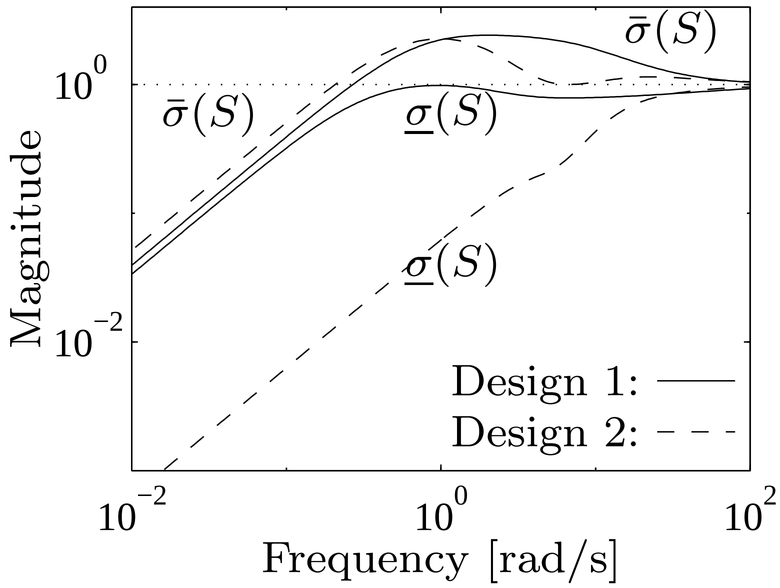


Figure 58: Singular values of S for a 2×2 plant with RHP-zero

- *Bandwidth*, ω_B: frequency where $\bar{\sigma}(S)$ crosses $\frac{1}{\sqrt{2}} = 0.7$ from below.

Since $S = (I + L)^{-1}$, the singular values inequality $\underline{\sigma}(A) - 1 \leq \frac{1}{\bar{\sigma}(I+A)^{-1}} \leq \underline{\sigma}(A) + 1$ yields

$$\underline{\sigma}(L) - 1 \leq \frac{1}{\bar{\sigma}(S)} \leq \underline{\sigma}(L) + 1 \qquad (6.43)$$

- low ω : $\underline{\sigma}(L) \gg 1 \Rightarrow \bar{\sigma}(S) \approx \frac{1}{\underline{\sigma}(L)}$
- high ω: $\bar{\sigma}(L) \ll 1 \Rightarrow \bar{\sigma}(S) \approx 1$

5.4 Poles [4.4]

Definition

Poles. The poles p_i of a system with state-space description (5.1)–(5.2) are the eigenvalues $\lambda_i(A), i = 1, \ldots, n$ of the matrix A. The pole or characteristic polynomial $\phi(s)$ is defined as $\phi(s) \stackrel{\Delta}{=} \det(sI - A) = \prod_{i=1}^{n}(s - p_i)$. Thus the poles are the roots of the characteristic equation

$$\phi(s) \stackrel{\Delta}{=} \det(sI - A) = 0 \tag{5.36}$$

5.4.1 Poles and stability

Theorem 6 *A linear dynamic system $\dot{x} = Ax + Bu$ is stable if and only if all the poles are in the open left-half plane (LHP), that is,* $\mathrm{Re}\{\lambda_i(A)\} < 0, \forall i$. *A matrix A with such a property is said to be "stable" or Hurwitz.*

5.4.2 Poles from transfer functions

Theorem 7 *The pole polynomial $\phi(s)$ corresponding to a minimal realization of a system with transfer function $G(s)$, is the least common denominator of all non-identically-zero minors of all orders of $G(s)$.*

Example:

$$G(s) = \frac{1}{1.25(s+1)(s+2)} \begin{bmatrix} s-1 & s \\ -6 & s-2 \end{bmatrix} \qquad (5.37)$$

The minors of order 1 are the four elements all have $(s+1)(s+2)$ in the denominator.

Minor of order 2

$$\det G(s) = \frac{(s-1)(s-2)+6s}{1.25^2(s+1)^2(s+2)^2} = \frac{1}{1.25^2(s+1)(s+2)} \qquad (5.38)$$

Least common denominator of all the minors:

$$\phi(s) = (s+1)(s+2) \qquad (5.39)$$

Minimal realization has two poles: $s = -1$; $s = -2$.

Example: Consider the 2×3 system, with 3 inputs and 2 outputs,

$$G(s) = \frac{1}{(s+1)(s+2)(s-1)} *$$

$$* \begin{bmatrix} (s-1)(s+2) & 0 & (s-1)^2 \\ -(s+1)(s+2) & (s-1)(s+1) & (s-1)(s+1) \end{bmatrix} \qquad (5.40)$$

Minors of order 1:

$$\frac{1}{s+1},\ \frac{s-1}{(s+1)(s+2)},\ \frac{-1}{s-1},\ \frac{1}{s+2},\ \frac{1}{s+2} \qquad (5.41)$$

Minor of order 2 corresponding to the deletion of column 2:

$$M_2 = \frac{(s-1)(s+2)(s-1)(s+1) + (s+1)(s+2)(s-1)^2}{((s+1)(s+2)(s-1))^2} =$$

$$= \frac{2}{(s+1)(s+2)} \tag{5.42}$$

The other two minors of order two are

$$M_1 = \frac{-(s-1)}{(s+1)(s+2)^2}, \quad M_3 = \frac{1}{(s+1)(s+2)} \tag{5.43}$$

Least common denominator:

$$\phi(s) = (s+1)(s+2)^2(s-1) \tag{5.44}$$

The system therefore has four poles: $s = -1$, $s = 1$ and two at $s = -2$.

Note MIMO-poles are essentially the poles of the elements. A procedure is needed to determine multiplicity.

5.5 Zeros [4.5]

- SISO system: zeros z_i are the solutions to $G(z_i) = 0$.

In general, zeros are values of s at which $G(s)$ loses rank.

Example

$$\left[Y = \frac{s+2}{s^2+7s+12} U \right]$$

Compute the response when

$$\begin{aligned}
u(t) &= e^{-2t},\ y(0) = 0,\ \dot{y}(0) = -1 \\
\mathcal{L}\{u(t)\} &= \frac{1}{s+2} \\
s^2 Y &- sy(0) - \dot{y}(0) + 7sY - 7y(0) + 12Y = 1 \\
s^2 Y &+ 7sY + 12Y + 1 = 1 \\
&\Rightarrow Y(s) = 0
\end{aligned}$$

Assumption: $g(s)$ has a zero z, $g(z) = 0$. Then for input $u(t) = u_0 e^{zt}$ the output is $y(t) \equiv 0$, $t > 0$. (with appropriate initial conditions)

5.5.2 Zeros from transfer functions [4.5.2]

Definition Zeros. z_i is a zero of $G(s)$ if the rank of $G(z_i)$ is less than the normal rank of $G(s)$. The zero polynomial is defined as $z(s) = \prod_{i=1}^{n_z}(s - z_i)$ where n_z is the number of finite zeros of $G(s)$.

Theorem The zero polynomial $z(s)$, corresponding to a minimal realization of the system, is the greatest common divisor of all the numerators of all order-r minors of $G(s)$, where r is the normal rank of $G(s)$, provided that these minors have been adjusted in such a way as to have the pole polynomial $\phi(s)$ as their denominators.

Example

$$G(s) = \frac{1}{s+2}\begin{bmatrix} s-1 & 4 \\ 4.5 & 2(s-1) \end{bmatrix} \tag{5.45}$$

The normal rank of $G(s)$ is 2.

Minor of order 2: $\det G(s) = \frac{2(s-1)^2-18}{(s+2)^2} = 2\frac{s-4}{s+2}$.

Pole polynomial: $\phi(s) = s + 2$.

Zero polynomial: $z(s) = s - 4$.

Note Multivariable zeros have no relationship with the zeros of the transfer function elements.

Example

$$G(s) = \frac{1}{1.25(s+1)(s+2)} \begin{bmatrix} s-1 & s \\ -6 & s-2 \end{bmatrix} \tag{5.46}$$

Minor of order 2 is the determinant

$$\det G(s) = \frac{(s-1)(s-2)+6s}{1.25^2(s+1)^2(s+2)^2} = \frac{1}{1.25^2(s+1)(s+2)} \tag{5.47}$$

$$\phi(s) = 1.25^2(s+1)(s+2)$$

Zero polynomial = numerator of (5.47)

$\Rightarrow$ no multivariable zeros.

Example

$$G(s) = \left[\frac{s-1}{s+1} \quad \frac{s-2}{s+2} \right] \tag{5.48}$$

- The normal rank of $G(s)$ is 1
- no value of s for which $G(s) = 0$

  $\Rightarrow G(s)$ has no zeros.

5.6 More on poles and zeros[4.6]

5.6.1 *Directions of poles and zeros

Let $G(s) = C(sI - A)^{-1}B + D$.

Zero directions. Let $G(s)$ have a zero at $s = z$. Then $G(s)$ loses rank at $s = z$, and there exist non-zero vectors u_z and y_z such that

$$G(z)u_z = 0, \quad y_z^H G(z) = 0 \tag{5.49}$$

$u_z =$ input zero direction

$y_z =$ output zero direction

y_z gives information about which output (or combination of outputs) may be difficult to control.

SVD:

$$G(z) = U\Sigma V^H$$

$u_z =$ last column in V

$y_z =$ last column of U

(corresponding to the zero singular value of $G(z)$)

Pole directions. Let $G(s)$ have a pole at $s = p$. Then $G(p)$ is infinite, and we may write

$$G(p)u_p = \infty, \quad y_p^H G(p) = \infty \tag{5.50}$$

$u_p =$ input pole direction

$y_p =$ output pole direction.

Example

Plant in (5.45) has a RHP-zero at $z = 4$ and a LHP-pole at $p = -2$.

$$\begin{aligned} G(z) &= G(4) = \frac{1}{6}\begin{bmatrix} 3 & 4 \\ 4.5 & 6 \end{bmatrix} \\ &= \frac{1}{6}\begin{bmatrix} 0.55 & -0.83 \\ 0.83 & 0.55 \end{bmatrix}\begin{bmatrix} 9.01 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0.6 & -0.8 \\ 0.8 & 0.6 \end{bmatrix}^H \end{aligned}$$

$$u_z = \begin{bmatrix} -0.80 \\ 0.60 \end{bmatrix} \quad y_z = \begin{bmatrix} -0.83 \\ 0.55 \end{bmatrix} \tag{5.51}$$

For pole directions consider

$$G(p+\epsilon) = G(-2+\epsilon) = \frac{1}{\epsilon^2}\begin{bmatrix} -3+\epsilon & 4 \\ 4.5 & 2(-3+\epsilon) \end{bmatrix} \tag{5.52}$$

The SVD as $\epsilon \to 0$ yields

$$G(-2+\epsilon) = \frac{1}{\epsilon^2}\begin{bmatrix} -0.55 & -0.83 \\ 0.83 & -0.55 \end{bmatrix}\begin{bmatrix} 9.01 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0.6 & -0.8 \\ -0.8 & -0.6 \end{bmatrix}^H$$

$$u_p = \begin{bmatrix} 0.60 \\ -0.80 \end{bmatrix} \quad y_p = \begin{bmatrix} -0.55 \\ 0.83 \end{bmatrix} \tag{5.53}$$

Note Locations of poles and zeros are independent of input and output scalings, their directions are *not*.

5.6.2 Remarks on poles and zeros [4.6.2]

1. For square systems the poles and zeros of $G(s)$ are "essentially" the poles and zeros of $\det G(s)$.

   This fails when zero and pole in different parts of the system cancel when forming $\det G(s)$.

   $$G(s) = \begin{bmatrix} (s+2)/(s+1) & 0 \\ 0 & (s+1)/(s+2) \end{bmatrix} \tag{5.54}$$

   $\det G(s) = 1$, although the system obviously has poles at -1 and -2 and (multivariable) zeros at -1 and -2.

2. System (5.54) has poles and zeros at the same locations (at -1 and -2). Their directions are different. They do not cancel or otherwise interact.

3. There are no zeros if the outputs contain direct information about all the states; that is, if from y we can directly obtain x (e.g. $C = I$ and $D = 0$);

4. Zeros usually appear when there are fewer inputs or outputs than states

5. **Moving poles.** (a) feedback control $(G(I+KG)^{-1})$ moves the poles, (b) series compensation (GK, feedforward control) can cancel poles in G by placing zeros in K (but not move them), and (c) parallel compensation $(G+K)$ cannot affect the poles in G.

6. **Moving zeros.** (a) With feedback, the zeros of $G(I+KG)^{-1}$ are the zeros of G plus the poles of K. , i.e. the zeros are unaffected by feedback. (b) Series compensation can counter the effect of zeros in G by placing poles in K to cancel them, but cancellations are not possible for RHP-zeros due to internal stability (see Section 5.7). (c) The only way to move zeros is by parallel compensation, $y=(G+K)u$, which, if y is a physical output, can only be accomplished by adding an extra input (actuator).

Example

Effect of feedback on poles and zeros.

SISO plant $G(s) = z(s)/\phi(s)$ and $K(s) = k$.

$$T(s) = \frac{L(s)}{1+L(s)} = \frac{kG(s)}{1+kG(s)} = \frac{kz(s)}{\phi(s)+kz(s)} = k\frac{z_{cl}(s)}{\phi_{cl}(s)} \tag{5.55}$$

Note the following:

1. Zero polynomial: $z_{cl}(s) = z(s)$
   $\Rightarrow$ zero locations are unchanged.
2. Pole locations are changed by feedback.
   For example,

$$k \to 0 \quad \Rightarrow \quad \phi_{cl}(s) \to \phi(s) \tag{5.56}$$

$$k \to \infty \quad \Rightarrow \quad \phi_{cl}(s) \to z(s).\widetilde{z}(s) \tag{5.57}$$

where roots of $\widetilde{z}(s)$ move with k to infinity (complex pattern)
(cf. root locus)

5.10 System norms [4.10]

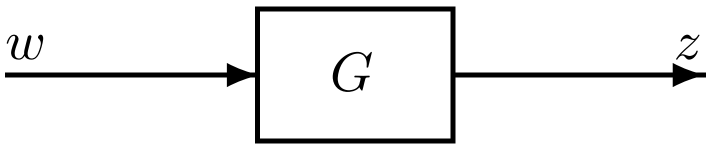


Figure 51: System G

Figure 51: System with stable transfer function matrix $G(s)$ and impulse response matrix $g(t)$.

Question: given information about the allowed input signals $w(t)$, how large can the outputs $z(t)$ become?

We use the 2-norm,

$$\|z(t)\|_2 = \sqrt{\sum_i \int_{-\infty}^{\infty} |z_i(\tau)|^2 d\tau} \tag{5.88}$$

and consider three inputs:

1. $w(t)$ is a series of unit impulses.
2. $w(t)$ is any signal satisfying $\|w(t)\|_2 = 1$.
3. $w(t)$ is any signal satisfying $\|w(t)\|_2 = 1$, but $w(t) = 0$ for $t \geq 0$, and we only measure $z(t)$ for $t \geq 0$.

The relevant system norms in the three cases are the $\mathcal{H}_2$, $\mathcal{H}_\infty$, and Hankel norms, respectively.

5.10.1 $\mathcal{H}_2$ norm [4.10.1]

$G(s)$ strictly proper.

For the $\mathcal{H}_2$ norm we use the Frobenius norm spatially (for the matrix) and integrate over frequency, i.e.

$$\|G(s)\|_2 \triangleq \sqrt{\frac{1}{2\pi}\int_{-\infty}^{\infty} \underbrace{\operatorname{tr}(G(j\omega)^H G(j\omega))}_{\|G(j\omega)\|_F^2 = \sum_{ij} |G_{ij}(j\omega)|^2} \, d\omega} \tag{5.89}$$

$G(s)$ must be strictly proper, otherwise the $\mathcal{H}_2$ norm is infinite. By Parseval's theorem, (5.89) is equal to the $\mathcal{H}_2$ norm of the impulse response

$$\|G(s)\|_2 = \|g(t)\|_2 \triangleq \sqrt{\int_0^{\infty} \underbrace{\operatorname{tr}(g^T(\tau) g(\tau))}_{\|g(\tau)\|_F^2 = \sum_{ij} |g_{ij}(\tau)|^2} \, d\tau} \tag{5.90}$$

- Note that $G(s)$ and $g(t)$ are dynamic *systems* while $G(j\omega)$ and $g(\tau)$ are constant *matrices* (for a given value of ω or τ).

- We can change the order of integration and summation in (5.90) to get

$$\|G(s)\|_2 = \|g(t)\|_2 = \sqrt{\sum_{ij} \int_0^{\infty} |g_{ij}(\tau)|^2 d\tau} \tag{5.91}$$

where $g_{ij}(t)$ is the ij'th element of the impulse response matrix, $g(t)$. Thus $\mathcal{H}_2$ norm can be interpreted as the 2-norm output resulting from applying unit impulses $\delta_j(t)$ to each input, one after another (allowing the output to settle to zero before applying an impulse to the next input). Thus $\|G(s)\|^2 = \sqrt{\sum_{i=1}^{m} \|z_i(t)\|_2^2}$ where $z_i(t)$ is the output vector resulting from applying a unit impulse $\delta_i(t)$ to the i'th input.

5.10.2 $\mathcal{H}_\infty$ norm [4.10.2]

$G(s)$ proper.

For the $\mathcal{H}_\infty$ norm we use the singular value (induced 2-norm) spatially (for the matrix) and pick out the peak value as a function of frequency

$$\|G(s)\|_\infty \triangleq \max_\omega \bar{\sigma}(G(j\omega)) \qquad (5.93)$$

The $\mathcal{H}_\infty$ norm is the peak of the transfer function "magnitude".

Time domain performance interpretations of the $\mathcal{H}_\infty$ norm.

- Worst-case steady-state gain for sinusoidal inputs at any frequency.
- Induced (worst-case) 2-norm in the time domain:

$$\|G(s)\|_\infty = \max_{w(t)\neq 0} \frac{\|z(t)\|_2}{\|w(t)\|_2} = \max_{\|w(t)\|_2=1} \|z(t)\|_2 \qquad (5.94)$$

(In essence, (5.94) arises because the worst input signal $w(t)$ is a sinusoid with frequency ω^* and a direction which gives $\overline{\sigma}(G(j\omega^*))$ as the maximum gain.)

Numerical computation of the $\mathcal{H}_\infty$ norm.
Consider

$$G(s) = C(sI - A)^{-1}B + D$$

$\mathcal{H}_\infty$ norm is the smallest value of γ such that the Hamiltonian matrix H has no eigenvalues on the imaginary axis, where

$$H = \begin{bmatrix} A + BR^{-1}D^TC & BR^{-1}B^T \\ -C^T(I + DR^{-1}D^T)C & -(A + BR^{-1}D^TC)^T \end{bmatrix} \tag{5.95}$$

and $R = \gamma^2 I - D^T D$

5.10.3 Difference between the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms

Frobenius norm in terms of singular values

$$\|G(s)\|_2 = \sqrt{\frac{1}{2\pi}\int_{-\infty}^{\infty} \sum_i \sigma_i^2(G(j\omega))d\omega} \tag{5.96}$$

Thus when optimizing performance in terms of the different norms:

- $\mathcal{H}_\infty$: "push down peak of largest singular value".
- $\mathcal{H}_2$: "push down whole thing" (all singular values over all frequencies).

Example

$$G(s) = \frac{1}{s+a} \tag{5.97}$$

$\mathcal{H}_2$ norm:

$$\begin{aligned} \|G(s)\|_2 &= \left(\frac{1}{2\pi}\int_{-\infty}^{\infty} \underbrace{|G(j\omega)|^2}_{\frac{1}{\omega^2+a^2}} d\omega\right)^{\frac{1}{2}} \\ &= \left(\frac{1}{2\pi a}\left[\tan^{-1}\left(\frac{\omega}{a}\right)\right]_{-\infty}^{\infty}\right)^{\frac{1}{2}} = \sqrt{\frac{1}{2a}} \end{aligned}$$

Alternatively: Consider the impulse response

$$g(t) = \mathcal{L}^{-1}\left(\frac{1}{s+a}\right) = e^{-at}, t \geq 0 \tag{5.98}$$

to get

$$\|g(t)\|_2 = \sqrt{\int_0^{\infty} (e^{-at})^2 dt} = \sqrt{\frac{1}{2a}} \tag{5.99}$$

as expected from Parseval's theorem.

$\mathcal{H}_\infty$ norm:

$$||G(s)||_\infty = \max_\omega |G(j\omega)| = \max_\omega \frac{1}{(\omega^2+a^2)^{\frac{1}{2}}} = \frac{1}{a} \tag{5.100}$$

Example

There is no general relationship between the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms.

$$f_1(s) = \frac{1}{\epsilon s + 1}, \quad f_2(s) = \frac{\epsilon s}{s^2 + \epsilon s + 1} \tag{5.101}$$

$$\begin{aligned} ||f_1||_\infty = 1 \quad ||f_1||_2 = \infty \\ ||f_2||_\infty = 1 \quad ||f_2||_2 = 0 \end{aligned} \tag{5.102}$$

Why is the $\mathcal{H}_\infty$ norm so popular? In robust control convenient for representing unstructured model uncertainty, and because it satisfies the multiplicative property:

$$\|A(s)B(s)\|_\infty \leq \|A(s)\|_\infty \cdot \|B(s)\|_\infty \tag{5.103}$$

What is wrong with the $\mathcal{H}_2$ norm? It is *not* an induced norm and does *not* satisfy the multiplicative property.

Example

Consider again $G(s) = 1/(s+a)$ in (5.97), for which $\|G(s)\|_2 = \sqrt{1/2a}$.

$$\begin{aligned} \|G(s)G(s)\|_2 &= \sqrt{\int_0^{\infty} |\underbrace{\mathcal{L}^{-1}[(\frac{1}{s+a})^2]}_{te^{-at}}|^2} \\ &= \sqrt{\frac{1}{a}\frac{1}{2a}} = \sqrt{\frac{1}{a}}\ \|G(s)\|_2^2 \end{aligned} \quad (5.104)$$

for $a < 1$,

$$\|G(s)G(s)\|_2 > \|G(s)\|_2 \cdot \|G(s)\|_2 \quad (5.105)$$

which does not satisfy the multiplicative property.

$\mathcal{H}_\infty$ norm does satisfy the multiplicative property

$$\|G(s)G(s)\|_\infty = \frac{1}{a^2} = \|G(s)\|_\infty \cdot \|G(s)\|_\infty$$

1 LIMITATIONS ON PERFORMANCE IN MIMO SYSTEMS

In a MIMO system, disturbances, the plant, RHP-zeros, RHP-poles and delays each have directions associated with them. A multivariable plant may have a RHP-zero and a RHP-pole at the same location, but their effects may not interact.

- y_z: output direction of a RHP-zero, $G(z)u_z = 0 \cdot y_z$
- y_p: output direction of a RHP-pole, $G(p)u_p = \infty \cdot y_p$

1.1 Interpolation constraints

RHP-zero. If $G(s)$ has a RHP-zero at z with output direction y_z, then for internal stability

$$y_z^H T(z) = 0; \quad y_z^H S(z) = y_z^H \tag{1.1}$$

RHP-pole. If $G(s)$ has a RHP-pole at p with output direction y_p, then for internal stability the following interpolation constraints apply:

$$S(p) y_p = 0; \quad T(p) y_p = y_p \tag{1.2}$$

Similar constraints apply to L_I, S_I and T_I, but these are in terms of the input zero and pole directions, u_z and u_p.

1.2 Constraints on S and T [6.2]

From the identity $S + T = I$ we get

$$|1 - \bar{\sigma}(S)| \leq \bar{\sigma}(T) \leq 1 + \bar{\sigma}(S) \tag{1.3}$$

$$|1 - \bar{\sigma}(T)| \leq \bar{\sigma}(S) \leq 1 + \bar{\sigma}(T) \tag{1.4}$$

$\Rightarrow$ S and T cannot be small simultaneously; $\bar{\sigma}(S)$ is large if and only if $\bar{\sigma}(T)$ is large. For example, if $\bar{\sigma}(T)$ is 5 at a given frequency, then $\bar{\sigma}(S)$ must be between 4 and 6 at this frequency.

1.3 Sensitivity peaks [6.2.4]

Theorem 1 Weighted sensitivity. *Suppose the plant $G(s)$ has a RHP-zero at $s = z$. Let $w_P(s)$ be any stable scalar weight. Then for closed-loop stability the weighted sensitivity function must satisfy*

$$\|w_P(s)S(s)\|_\infty = \max_\omega \bar{\sigma}(w_P(j\omega)S(j\omega)) \geq |w_P(z)| \tag{1.5}$$

In MIMO systems we generally have the freedom to move the effect of RHP zeros to different outputs by appropriate control.

Theorem 2 Weighted complementary sensitivity. *Suppose the plant $G(s)$ has a RHP-pole at $s = p$. Let $w_T(s)$ be any stable scalar weight. Then for closed-loop stability the weighted complementary sensitivity function must satisfy*

$$\|w_T(s)T(s)\|_\infty = \max_\omega \bar{\sigma}(w_T(j\omega)T(j\omega)) \geq |w_T(p)| \tag{1.6}$$

For a plant with one RHP-zero z and one RHP-pole p,

$$M_{S,\min} = M_{T,\min} = \sqrt{\sin^2\phi + \frac{|z+p|^2}{|z-p|^2}\cos^2\phi} \quad (1.7)$$

where $\phi = \cos^{-1}|y_z^H y_p|$ is the angle between the output directions of the pole and zero.

If the pole and zero are aligned such that $y_z = y_p$ and $\phi = 0$, then (1.7) simplifies to give the equivalent SISO conditions.

Conversely, if the pole and zero are orthogonal to each other, then $\phi = 90^\circ$ and $M_{S,\min} = M_{T,\min} = 1$, and there is no additional penalty for having both a RHP-pole and a RHP-zero.

1.4 Example

Consider the plant

$$G_\alpha(s) = \begin{bmatrix} \frac{1}{s-p} & 0 \\ 0 & \frac{1}{s+3} \end{bmatrix} U_\alpha \begin{bmatrix} \frac{s-z}{0.1s+1} & 0 \\ 0 & \frac{s+2}{0.1s+1} \end{bmatrix}$$

$$U_\alpha = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}, \quad z = 2, p = 3$$

which has for all values of α a RHP-zero at $z = 2$ and a RHP-pole at $p = 3$.

For $\alpha = 0^\circ$, $U_\alpha = I$,

$$G_0(s) = \begin{bmatrix} \frac{s-z}{(0.1s+1)(s-p)} & 0 \\ 0 & \frac{s+2}{(0.1s+1)(s+3)} \end{bmatrix}$$

g_{11} has both RHP-pole and RHP-zero (bad!).

When $\alpha = 90^\circ$

$$G_{90}(s) = \begin{bmatrix} 0 & -\frac{s+2}{(0.1s+1)(s-p)} \\ \frac{s-z}{(0.1s+1)(s+3)} & 0 \end{bmatrix}$$

No interaction between the RHP-pole and RHP-zero (good!).

α	0°	30°	60°	90°
y_z	$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 0.33 \\ -0.94 \end{bmatrix}$	$\begin{bmatrix} 0.11 \\ -0.99 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$
$\phi = \cos^{-1} \lvert y_z^H y_p \rvert$	0°	70.9°	83.4°	90°
$\lVert S \rVert_\infty \geq$	5.0	1.89	1.15	1.0
$\lVert S \rVert_\infty$	7.00	2.60	1.59	1.98
$\lVert T \rVert_\infty$	7.40	2.76	1.60	1.31
$\gamma_{\min}(S/KS)$	9.55	3.53	2.01	1.59

The table also shows the values of $\|S\|_\infty$ and $\|T\|_\infty$ obtained by an $\mathcal{H}_\infty$ optimal S/KS design using the following weights:

$$W_u = I; \quad W_P = \left(\frac{s/M + \omega_B^*}{s} \right) I; \; M = 2, \omega_B^* = 0.5 \tag{1.8}$$

The weight W_P indicates that we require $\|S\|_\infty$ less than 2, and require tight control up to a frequency of about $\omega_B^* = 0.5\, rad/s$. The minimum $\mathcal{H}_\infty$ norm for the overall S/KS problem is given by the value of γ in Table.

7.3 Limitations imposed by uncertainty [6.10]

7.3.1 Input and output uncertainty

In a multiplicative (relative) form, the output and input uncertainties (as in Figure 72) are given by

$$\text{Output uncertainty:} \quad G' = (I + E_O)G \quad \text{or}$$

$$E_O = (G' - G)G^{-1} \tag{7.5}$$

$$\text{Input uncertainty:} \quad G' = G(I + E_I) \quad \text{or}$$

$$E_I = G^{-1}(G' - G) \tag{7.6}$$

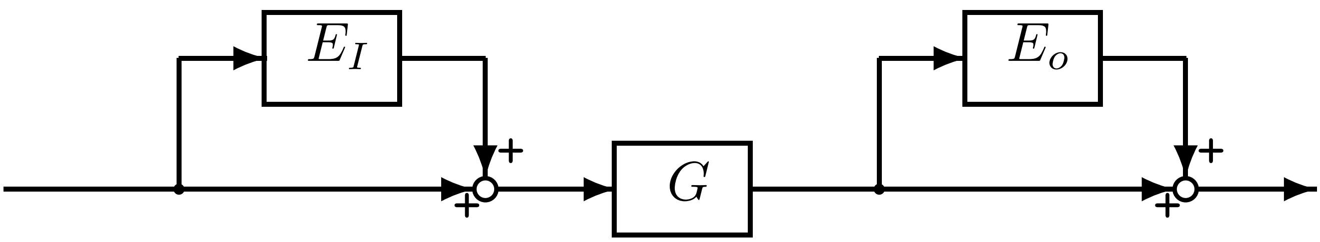


Figure 72: Plant with multiplicative input and output uncertainty

7.3.3 Uncertainty and the benefits of feedback [6.10.3]

Feedback control. With one degree-of-freedom feedback control the nominal transfer function is $y = Tr$ where $T = L(I+L)^{-1}$ is the complementary sensitivity function. Ideally, $T = I$. The change in response with model error is $y' - y = (T' - T)r$ where

$$T' - T = S' E_O T \tag{7.7}$$

Thus, $y' - y = S' E_O T r = S' E_O y$, and we see that

- with feedback control the effect of the uncertainty is reduced by a factor S' relative to that with feedforward control.

7.3.4 Uncertainty and the sensitivity peak

We will derive *upper bounds* on $\bar{\sigma}(S')$ which involve the plant and controller condition numbers

$$\gamma(G) \;=\; \frac{\bar{\sigma}(G)}{\underline{\sigma}(G)}, \qquad \gamma(K) \;=\; \frac{\bar{\sigma}(K)}{\underline{\sigma}(K)} \tag{7.8}$$

Factorizations of S' in terms of the nominal sensitivity S

Output uncertainty:

$$S' = S(I + E_O T)^{-1} \tag{7.9}$$

Input uncertainty:

$$\begin{aligned} S' &= S(I + G E_I G^{-1} T)^{-1} = \\ &= SG(I + E_I T_I)^{-1} G^{-1} \end{aligned} \tag{7.10}$$

$$\begin{aligned} S' &= (I + T K^{-1} E_I K)^{-1} S = \\ &= K^{-1}(I + T_I E_I)^{-1} K S \end{aligned} \tag{7.11}$$

We assume: G and G' are stable; closed-loop stability, i.e. S and S' are stable; therefore $(I + E_O T)^{-1}$ and $(I + E_I T_I)^{-1}$ are stable; the magnitude of the multiplicative (relative) uncertainty at each frequency can be bounded in terms of its singular value

$$\bar{\sigma}(E_I) \leq |w_I|, \qquad \bar{\sigma}(E_O) \leq |w_O| \tag{7.12}$$

where $w_I(s)$ and $w_O(s)$ are scalar weights. Typically the uncertainty bound, $|w_I|$ or $|w_O|$, is 0.2 at low frequencies and exceeds 1 at higher frequencies.

Upper bound on $\bar{\sigma}(S')$ for output uncertainty

From (7.9) we derive

$$\bar{\sigma}(S') \leq \bar{\sigma}(S)\bar{\sigma}((I + E_O T)^{-1}) \leq \frac{\bar{\sigma}(S)}{1 - |w_O|\bar{\sigma}(T)} \tag{7.13}$$

Upper bounds on $\bar{\sigma}(S')$ for input uncertainty

The sensitivity function can be much more sensitive to input uncertainty than output uncertainty.

From (7.10) and (7.11) we derive:

$$\bar{\sigma}(S') \leq \gamma(G)\bar{\sigma}(S)\bar{\sigma}((I+E_I T_I)^{-1}) \leq \leq \gamma(G)\frac{\bar{\sigma}(S)}{1-|w_I|\bar{\sigma}(T_I)} \qquad (7.14)$$

$$\bar{\sigma}(S') \leq \gamma(K)\bar{\sigma}(S)\bar{\sigma}((I+T_I E_I)^{-1}) \leq \leq \gamma(K)\frac{\bar{\sigma}(S)}{1-|w_I|\bar{\sigma}(T_I)} \qquad (7.15)$$

$\Rightarrow$ If we use a "round" controller ($\gamma(K) \approx 1$) then the sensitivity function is *not* sensitive to input uncertainty.