3 PERFORMANCE LIMITATIONS IN SISO SYSTEMS [5]

3.1 Input-Output Controllability [5.1]

"Control" is not only controller design and stability analysis. Three important questions:

I. How well can the plant be controlled?

II. What control structure should be used?
(What variables should we measure, which variables should we manipulate, and how are these variables best paired together?)

III. How might the process be changed to improve control?

Definition 1 (Input-output) controllability *is the ability to achieve acceptable control performance; that is, to keep the outputs (y) within specified bounds from their references (r), in spite of unknown but bounded variations, such as disturbances (d) and plant changes, using available inputs (u) and available measurements (y_m or d_m).*

Note: controllability is independent of the controller, and is a property of the plant (or process) alone.

It can only be affected by:

- changing the apparatus itself, e.g. type, size, etc.
- relocating sensors and actuators
- adding new equipment to dampen disturbances
- adding extra sensors
- adding extra actuators

3.1.1 Scaling and performance [5.1.2]

We assume that the variables and models have been scaled so that for acceptable performance:

- Output $y(t)$ between $r-1$ and $r+1$ for any disturbance $d(t)$ between -1 and 1 and any reference $r(t)$ between $-R$ and R, using an input $u(t)$ within -1 to 1.

or frequency-by-frequency.

- $|e(\omega)| \leq 1$, for any disturbance $|d(\omega)| \leq 1$ and any reference $|r(\omega)| \leq R(\omega)$, using an input $|u(\omega)| \leq 1$.

Usually for simplicity:

$$\begin{aligned} R(\omega) &= R \qquad && \omega \leq \omega_r \\ R(\omega) &= 0 \qquad && \omega > \omega_r \end{aligned} \tag{3.1}$$

Because:

$$e = y - r = Gu + G_d d - R\widetilde{r} \tag{3.2}$$

we can apply results for disturbances also to references by replacing G_d by $-R$.

3.2 Perfect control & plant inversion [5.2]

$$y = Gu + G_d d \tag{3.3}$$

For "perfect control", i.e. $y = r$ (not realizable) we have feedforward controller:

$$u = G^{-1}r - G^{-1}G_d d \tag{3.4}$$

With feedback control $u = K(r - y)$ we have:

$$u = KSr - KSG_d d$$

or since $T = GKS$,

$$u = G^{-1}Tr - G^{-1}TG_d d \tag{3.5}$$

Where feedback is effective ($T \approx I$) feedback input in (3.5) is the same as perfect control input in (3.4) $\Longrightarrow$ High gain feedback generates an inverse of G even though K may be very simple.

As consequence perfect control *cannot* be achieved if

- G contains RHP-zeros (since then G^{-1} is unstable)
- G contains time delay (since then G^{-1} contains a prediction)
- G has more poles than zeros (since then G^{-1} is unrealizable)

For feedforward control perfect control *cannot* be achieved if

- G is uncertain (since then G^{-1} cannot be obtained exactly)

Because of input constraints perfect control *cannot* be achieved if

- $|G^{-1}G_d|$ is large
- $|G^{-1}R|$ is large

3.3 Constraints on S and T [5.3]

3.3.1 S plus T is one [5.3.1]

$$S + T = 1 \tag{3.6}$$

$\Longrightarrow$ at any frequency $|S(j\omega)| \geq 0.5$ or $|T(j\omega)| \geq 0.5$

3.3.2 The waterbed effects (sensitivity integrals) [5.3.2]

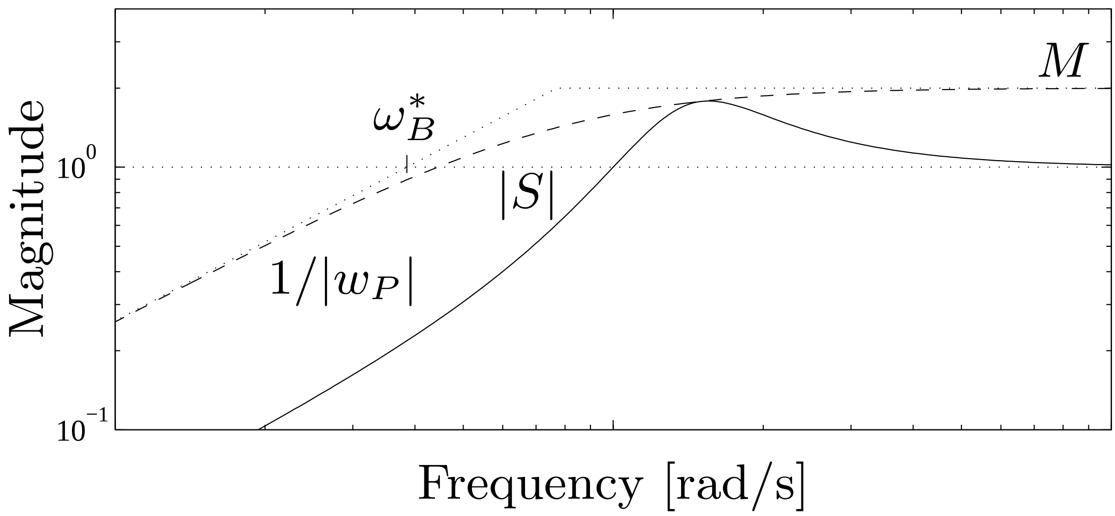


Figure 16: Plot of typical sensitivity, $|S|$, with upper bound $1/|w_P|$

Note: $|S|$ has peak greater than 1; we will show that this is unavoidable in practice.

Pole excess of two: First waterbed formula

Idea:

When $L(s) =$ has a relative degree of two or more, then for some ω the distance between L and -1 is less than one.

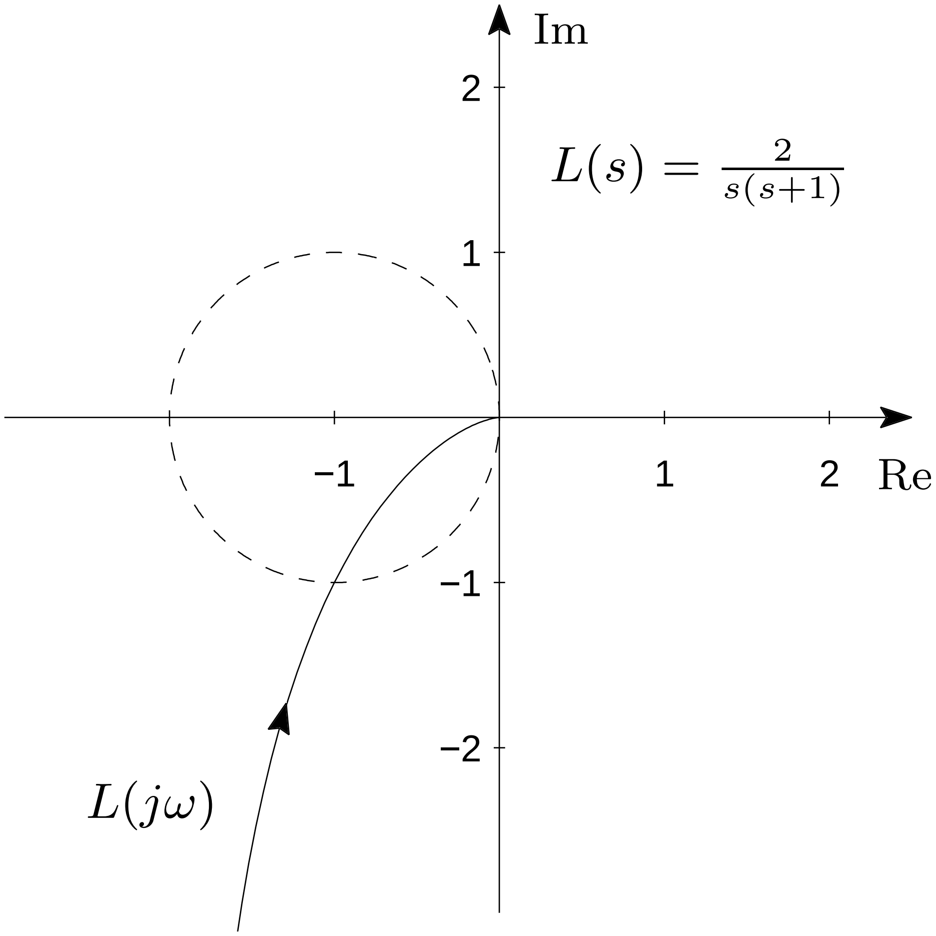


Figure 17: $|S| > 1$ whenever the Nyquist plot of L is inside the circle

Theorem 1 Bode Sensitivity Integral.

Suppose that the open-loop transfer function $L(s)$ is rational and has at least two more poles than zeros (relative degree of two or more).

Suppose also that $L(s)$ has N_p RHP-poles at locations p_i.

Then for closed-loop stability the sensitivity function must satisfy

$$\int_0^\infty \ln|S(j\omega)| d\omega = \pi \cdot \sum_{i=1}^{N_p} Re(p_i) \tag{3.7}$$

where $Re(p_i)$ denotes the real part of p_i.

RHP-zeros: Second waterbed formula

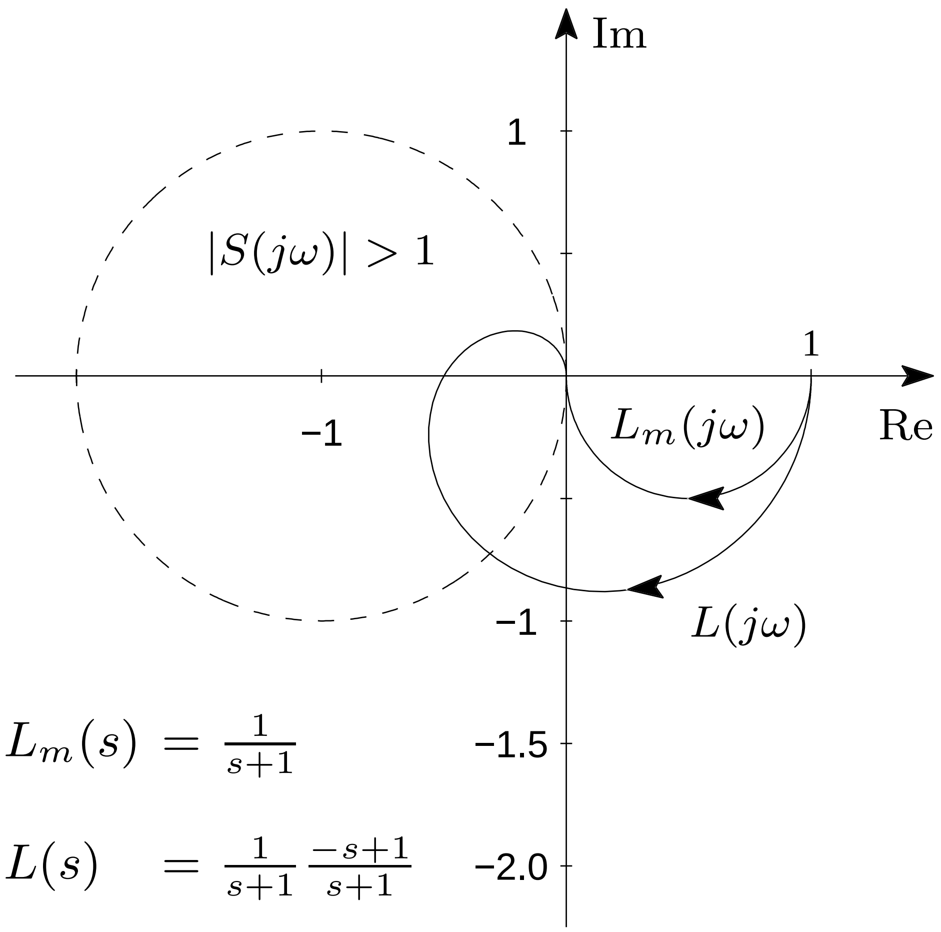


Figure 18: Additional phase lag contributed by RHP-zero causes $|S| > 1$

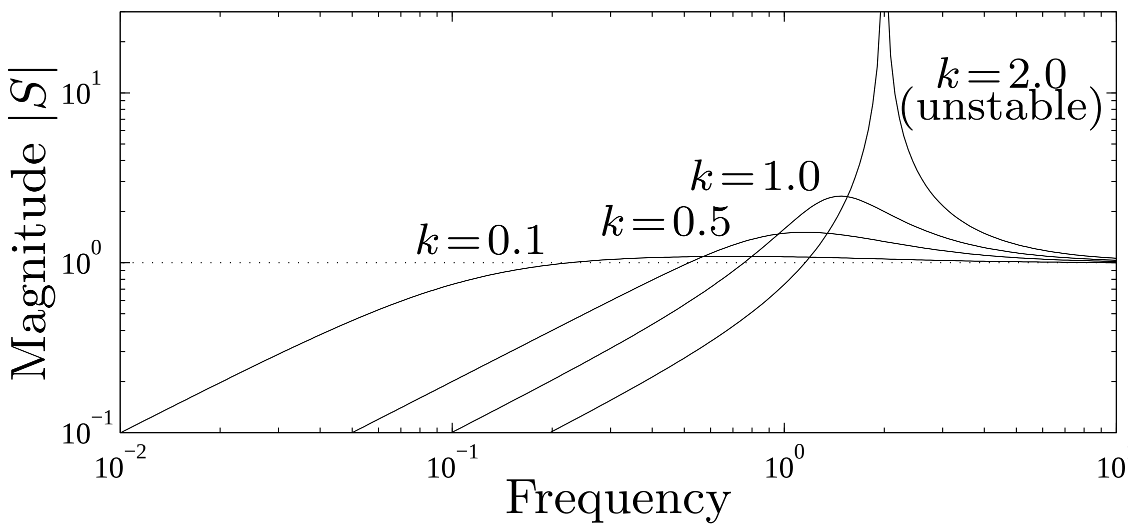


Figure 19: Effect of increased controller gain on $|S|$ for system with RHP-zero at $z = 2$, $L(s) = \frac{k}{s}\frac{2-s}{2+s}$

Theorem 2 Weighted sensitivity integral.
Suppose that $L(s)$ has a single real RHP-zero z and has N_p RHP-poles, p_i. Then for closed-loop stability the sensitivity function must satisfy

$$\int_0^\infty \ln |S(j\omega)| \cdot w(z,\omega) d\omega = \pi \cdot \ln \prod_{i=1}^{N_p} \left| \frac{p_i + z}{p_i - z} \right| \quad (3.8)$$

where:

$$w(z,\omega) = \frac{2z}{z^2 + \omega^2} = \frac{2}{z} \frac{1}{1 + (\omega/z)^2} \quad (3.9)$$

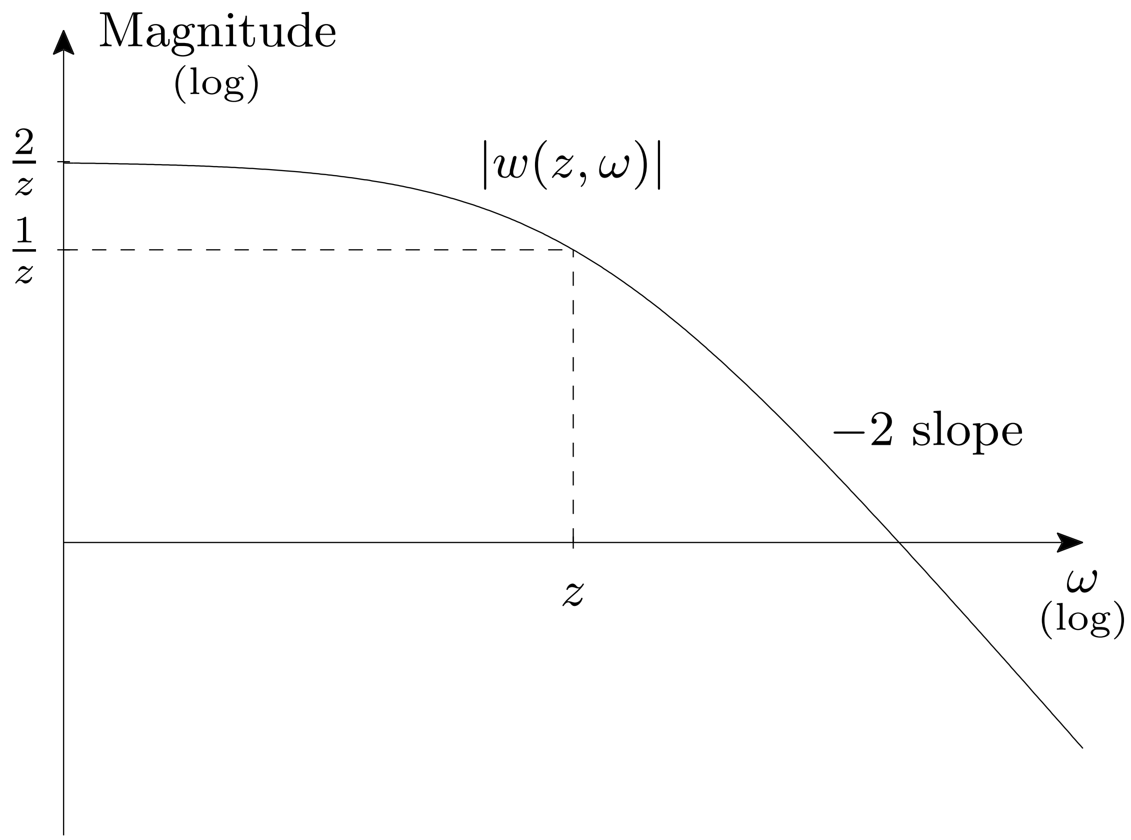


Figure 20: Plot of weight $w(z,\omega)$ for case with real zero at $s = z$

Weight $w(z,\omega)$ "cuts off" contribution of $ln|S|$ at frequencies $\omega > z$. Thus, for a stable plant:

$$\int_0^z \ln|S(j\omega)|d\omega \approx 0 \quad (\text{ for } |S| \approx 1 \text{ at } \omega > z) \quad (3.10)$$

The waterbed is finite, and a large peak for $|S|$ is unavoidable when we reduce $|S|$ at low frequencies (Figure 19).

Note also that when $p_i \to z$ then $\frac{p_i+z}{p_i-z} \to \infty$.

3.3.3 Interpolation constraints from internal stability [5.3.3]

If p is a RHP-pole of $L(s)$ then

$$T(p) = 1, \quad S(p) = 0 \tag{3.11}$$

Similarly, if z is a RHP-zero of $L(s)$ then

$$T(z) = 0, \quad S(z) = 1 \tag{3.12}$$

3.3.4 Sensitivity peaks [5.3.4]

Maximum modulus principle. *Suppose $f(s)$ is stable (i.e. $f(s)$ is analytic in the complex RHP). Then the maximum value of $|f(s)|$ for s in the right-half plane is attained on the region's boundary, i.e. somewhere along the $j\omega$-axis. Hence, we have for a stable $f(s)$*

$$\|f(j\omega)\|_\infty = \max_\omega |f(j\omega)| \geq |f(s_0)| \quad \forall s_0 \in \text{RHP} \tag{3.13}$$

The results below follow from (3.13) with
$f(s) = w_P(s)S(s)$
$f(s) = w_T(s)T(s)$
for weighted sensitivity and weighted complementary sensitivity.

Theorem 3 Weighted sensitivity peak.

Suppose that $G(s)$ has a RHP-zero z and let $w_P(s)$ be any stable weight function.

Then for closed-loop stability the weighted sensitivity function must satisfy

$$\|w_P S\|_\infty \geq |w_P(z)| \tag{3.14}$$

Theorem 4 Weighted complementary sensitivity peak.

Suppose that $G(s)$ has a RHP-pole p and let $w_T(s)$ be any stable weight function.

Then for closed-loop stability the weighted complementary sensitivity function must satisfy

$$\|w_T T\|_\infty \geq |w_T(p)| \tag{3.15}$$

Theorem 5 Combined RHP-poles and RHP-zeros.

Suppose that $G(s)$ has N_z RHP-zeros z_j, and N_p RHP-poles p_i.

Then for closed-loop stability the weighted sensitivity function must satisfy for each *RHP-zero z_j*

$$\|w_P S\|_\infty \geq c_{1j}|w_P(z_j)|, \quad c_{1j} = \prod_{i=1}^{N_p} \frac{|z_j + \bar{p}_i|}{|z_j - p_i|} \geq 1 \tag{3.16}$$

and the weighted complementary sensitivity function must satisfy for each *RHP-pole p_i*

$$\|w_T T\|_\infty \geq c_{2i}|w_T(p_i)|, \quad c_{2i} = \prod_{j=1}^{N_z} \frac{|\bar{z}_j + p_i|}{|z_j - p_i|} \geq 1 \tag{3.17}$$

For $w_P = w_T = 1$:

$$\|S\|_\infty \geq \max_j c_{1j}, \quad \|T\|_\infty \geq \max_i c_{2i} \tag{3.18}$$

$\Longrightarrow$ Large peaks for S and T are unavoidable if a RHP-zero and a RHP-pole are close to each other.

3.3.5 Bandwidth limitation II [5.6.4]

Performance requirement:

$$|S(j\omega)| < 1/|w_P(j\omega)| \quad \forall\omega \quad \Leftrightarrow \quad \|w_P S\|_\infty < 1 \tag{3.19}$$

However, from (3.14) we have that
$\|w_P S\|_\infty \geq |w_P(z)|$,

so the weight must satisfy

$$\boxed{|w_P(z)| < 1} \tag{3.20}$$

For performance weight

$$w_P(s) = \frac{s/M + \omega_B^*}{s + \omega_B^* A} \tag{3.21}$$

and a real zero at z we get:

$$\omega_B^*(1 - A) < z\left(1 - \frac{1}{M}\right) \tag{3.22}$$

e.g. $A = 0, M = 2$:

$$\omega_B^* < \frac{z}{2}$$

3.4 Limitations imposed by RHP-poles [5.8]

Specification:

$$|T(j\omega)| < 1/|w_T(j\omega)| \quad \forall\omega \quad \Leftrightarrow \quad \|w_T T\|_\infty < 1 \tag{3.23}$$

However, from (3.15) we have that:

$$\|w_T T\|_\infty \geq |w_T(p)| \tag{3.24}$$

so the weight must satisfy

$$\boxed{|w_T(p)| < 1} \tag{3.25}$$

For:

$$w_T(s) = \frac{s}{\omega_{BT}^*} + \frac{1}{M_T} \tag{3.26}$$

we get:

$$\boxed{\omega_{BT}^* > p\frac{M_T}{M_T - 1}} \tag{3.27}$$

e.g. $M_T = 2$:

$$\omega_{BT}^* > 2p$$

3.5 Combined RHP-poles and RHP-zeros [5.9]

RHP-zero:

$$\omega_c < z/2$$

RHP-pole:

$$\omega_c > 2p$$

RHP-pole and RHP-zero:

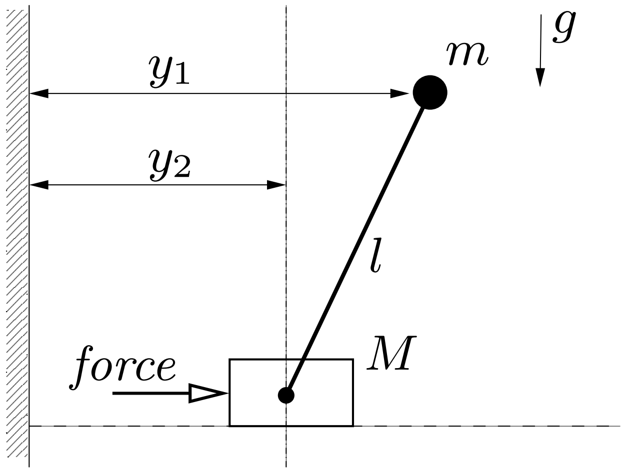
$z > 4p$ for acceptable performance and robustness.

Sensitivity peaks.

From Theorem 5 for a plant with a single real RHP-pole p and a single real RHP-zero z, we always have:

$$\boxed{\|S\|_\infty \geq c, \ \|T\|_\infty \geq c, \quad c = \frac{|z+p|}{|z-p|}} \tag{3.28}$$

Example 1 Balancing a rod. *The objective is to keep the rod upright by movement of the cart, based on observing the rod either at its far end (output y_1) or the cart position (output y_2).*



l [m] = length of rod

m [kg] = mass of rod

M [kg] = mass of hand

$g \approx 10\ m/s^2$ = acceleration due to gravity.

The linearized transfer functions for the two cases are

$$G_1(s) = \frac{-g}{s^2\left(Mls^2 - (M+m)g\right)};$$

$$G_2(s) = \frac{ls^2 - g}{s^2\left(Mls^2 - (M+m)g\right)}$$

Poles: $p = 0, 0, \pm\sqrt{\frac{(M+m)g}{Ml}}$. *For output* $y_1(G_1(s))$ *stabilization requires minimum bandwidth (3.27). For output* $y_2(G_2(s))$ *zero at* $z = \sqrt{\frac{g}{l}}$

- *For light rod* $m << M$, *pole* $\approx$ *zero* $\rightarrow$ *"impossible" to stabilize*
- *For heavy rod (m large) difficult to stabilize because* $p > z$

Example: $m/M = 0.1 \Rightarrow \|S\|_\infty \geq 42$; $\|T\|_\infty \geq 42 \Rightarrow$ *poor control*

3.6 * Ideal Integral Square Error (ISE) optimal control [5.4]

$$\text{ISE} = \int_0^\infty |y(t) - r(t)|^2 dt \tag{3.29}$$

the "ideal" response $y = Tr$ when $r(t)$ is a *unit step* is:

$$T(s) = \prod_i \frac{-s + z_j}{s + \bar{z}_j} e^{-\theta s} \tag{3.30}$$

where $\bar{z}_j$ is the complex conjugate of z_j.

Optimal ISE for three simple stable plants are:

1. with a delay θ:

$\text{ISE} = \theta$

2. with a RHP-zero z:

$\text{ISE} = 2/z$

3. with complex RHP-zeros $z = x \pm jy$:

$\text{ISE} = 4x/(x^2 + y^2)$

3.6.1 * Limitations imposed by time delays [5.5]

Ideal for plant with delay:

$$S = 1 - T = 1 - e^{-\theta s} \tag{3.31}$$

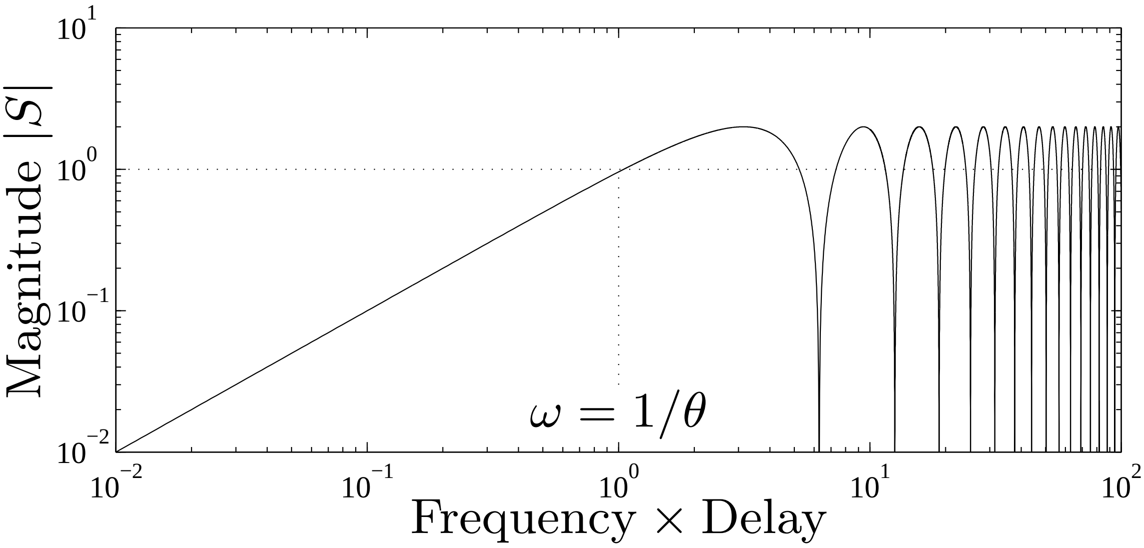


Figure 21: "Ideal" sensitivity function (3.31) for a plant with delay

$|S(j\omega)|$ in Figure 21 crosses 1 at $\frac{\pi}{3}\frac{1}{\theta} = 1.05/\theta$.

Because here $|S| = 1/|L|$, we have:

$$\omega_c < 1/\theta \tag{3.32}$$

3.6.2 * Limitations imposed by RHP-zeros [5.6]

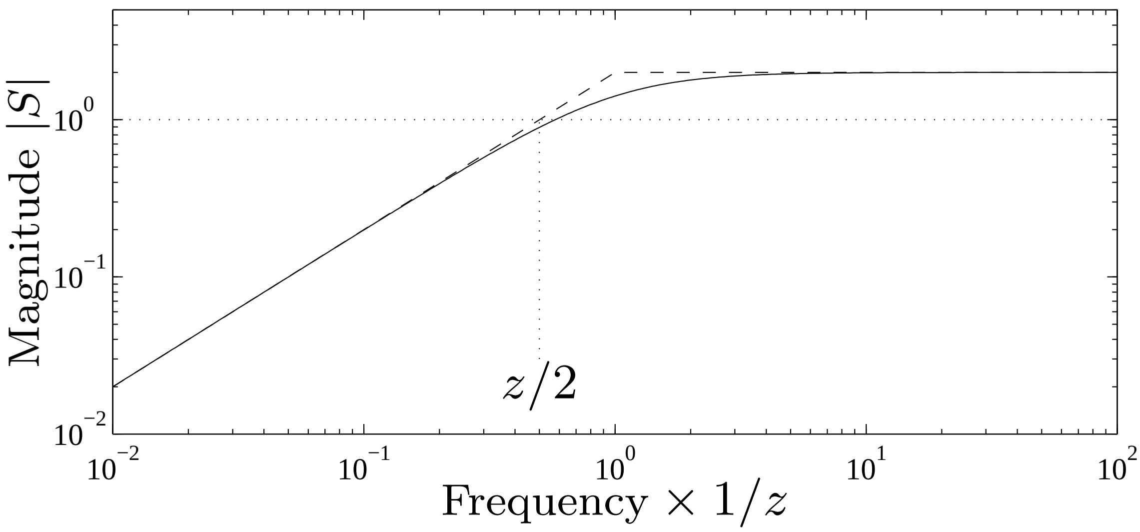
RHP-zeros typically appear when we have competing effects of slow and fast dynamics:

$$G(s) = \frac{1}{s+1} - \frac{2}{s+10} = \frac{-s+8}{(s+1)(s+10)}$$

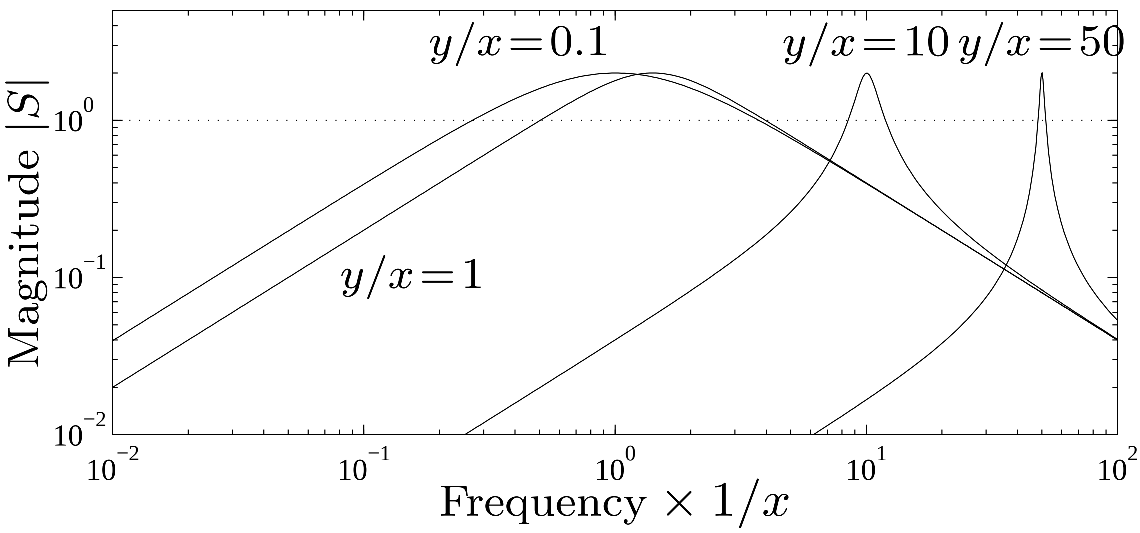
(a) Inverse response [5.6.1]

For a stable plant with n_z RHP-zeros, it may be proven that the output in response to a step change in the input will cross zero (its original value) n_z times, that is, we have *inverse response* behaviour.

(b) Bandwidth limitation I [5.6.3]



(a) Real RHP-zero



(b) Complex pair of RHP-zeros, $z = x \pm jy$

Figure 22: "Ideal" sensitivity functions for plants with RHP-zeros

For a single *real RHP-zero* the “ideal”, i.e. ISE optimal, sensitivity function is

$$S = 1 - T = \frac{2s}{s+z} \tag{3.33}$$

From Figure 22(a):

$$\omega_B \approx \omega_c < \frac{z}{2} \tag{3.34}$$

3.7 * Non-causal controllers [5.7]

Perfect control can be achieved for a plant with a time delay or RHP-zero if we use a non-causal controller, i.e. a controller which uses information about the future. (relevant for servo problems, e.g. in robotics and for batch processing.)

$$G(s) = \frac{-s+z}{s+z}; \quad z > 0 \tag{3.35}$$

$$r(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}$$

Stable *non-causal controller* generates the input

$$u(t) = \begin{cases} 2e^{zt} & t < 0 \\ 1 & t \geq 0 \end{cases}$$

(See (Figure 23))

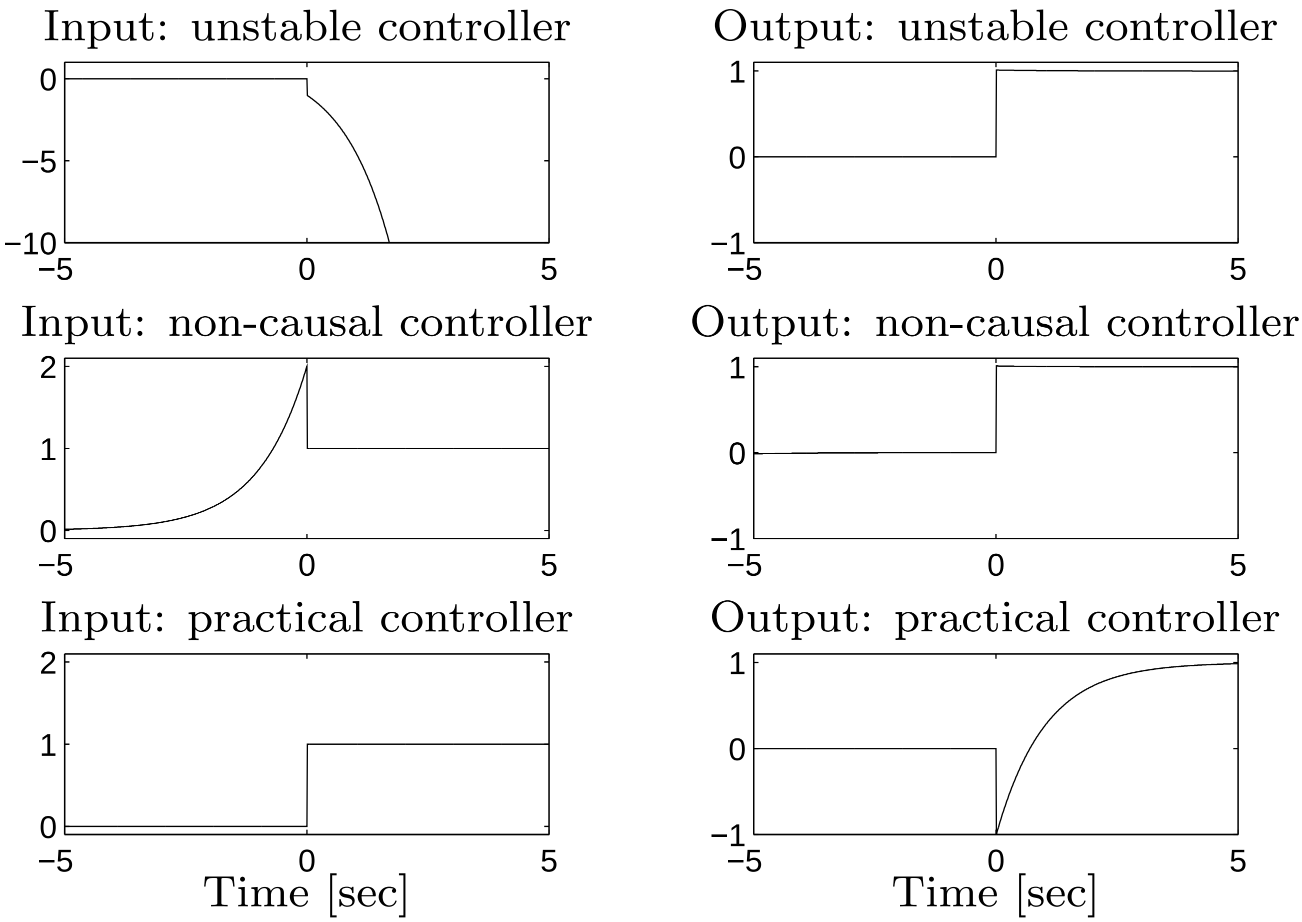


Figure 23: Feedforward control of plant with RHP-zero

3.8 Limitations imposed by input constraints [5.11]

The input required to achieve perfect control ($e = 0$) is

$$u = G^{-1} r - G^{-1} G_d d \tag{3.36}$$

Disturbance rejection. $r = 0$, $|d(\omega)| = 1$; $|u(\omega)| < 1$ implies

$$|G^{-1}(j\omega) G_d(j\omega)| < 1 \quad \forall \omega \tag{3.37}$$

Command tracking. $d = 0$, $|r(\omega)| = R \forall \omega < \omega_r$ $|u(\omega)| < 1$ implies:

$$|G^{-1}(j\omega) R| < 1 \quad \forall \omega \leq \omega_r \tag{3.38}$$

For *acceptable control* (namely $|e(j\omega)| < 1$) requirements change to:

$$\boxed{|G| > |G_d| - 1} \quad \text{at frequencies where} \quad |G_d| > 1 \tag{3.39}$$

$$\boxed{|G| > |R| - 1 < 1} \quad \forall \omega \leq \omega_r \tag{3.40}$$

3.9 Summary: Controllability analysis with feedback control [5.14]

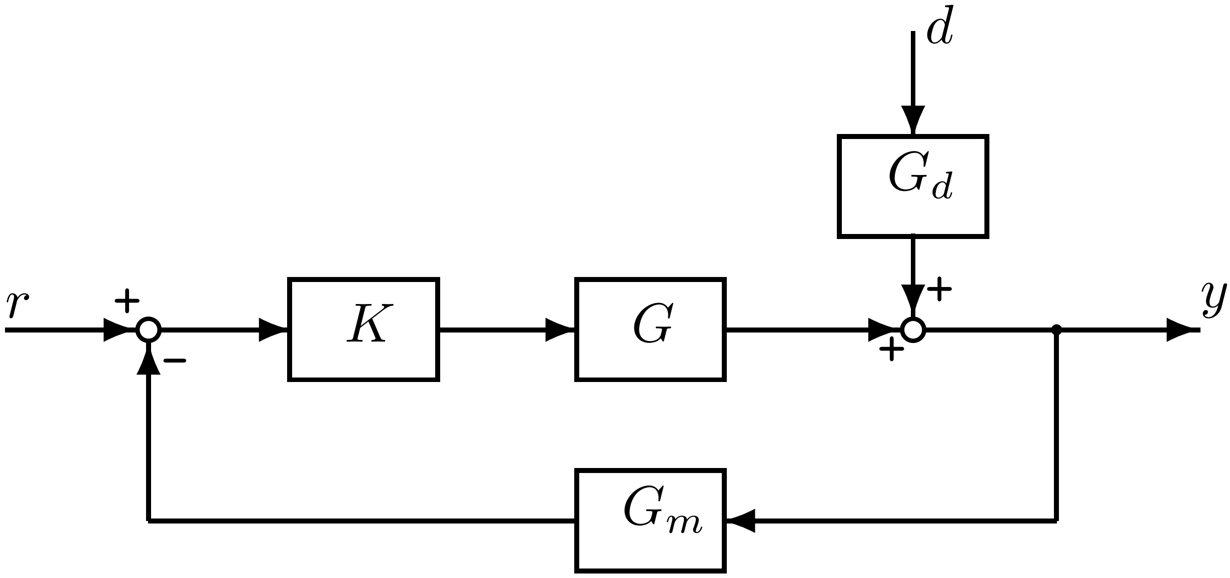


Figure 24: Feedback control system

$$y = G(s)u + G_d(s)d; \quad y_m = G_m(s)y \qquad (3.41)$$

$G_m(0) = 1$ (perfect steady-state measurement);

d, u, y and r are assumed to be scaled;

ω_c = gain crossover frequency (frequency where $|L(j\omega)|$ crosses 1 from above);

ω_d = frequency where $|G_d(j\omega_d)|$ first crosses 1 from above.

The following rules apply:

Rule 1. Speed of response to reject disturbances. *We require* $\omega_c > \omega_d$. *More specifically,* $|S(j\omega)| \leq |1/G_d(j\omega)| \; \forall\omega$.

Rule 2. Speed of response to track reference changes. *We require* $|S(j\omega)| \leq 1/R$ *up to the frequency* ω_r *where tracking is required.*

Rule 3. Input constraints arising from disturbances. *For acceptable control (*$|e| < 1$*) we require* $|G(j\omega)| > |G_d(j\omega)| - 1$ *at frequencies where* $|G_d(j\omega)| > 1$.

Rule 4. Input constraints arising from setpoints. *We require* $|G(j\omega)| > R - 1$ *up to the frequency* ω_r *where tracking is required.* (See (3.40)).

Rule 5. Time delay θ in $G(s)G_m(s)$. *We approximately require $\omega_c < 1/\theta$.* (See (3.32)).

Rule 6. Tight control at low frequencies with a RHP-zero z in $G(s)G_m(s)$. *For a real RHP-zero we require $\omega_c < z/2$.* (See (3.34)).

Rule 7. Phase lag constraint. *We require in most practical cases (e.g. with PID control): $\omega_c < \omega_u$. Here the* ultimate frequency *ω_u is where $\angle GG_m(j\omega_u) = -180°$.*

Rule 8. Real open-loop unstable pole in $G(s)$ at $s = p$. *We need high feedback gains to stabilize the system and require $\omega_c > 2p$.*

In addition, for unstable plants we need $|G| > |G_d|$ up to the frequency p (which may be larger than ω_d where $|G_d| = 1|$). Otherwise, the input may saturate when there are disturbances, and the plant cannot be stabilized.

3.10 Applications of controllability analysis [5.16]

3.10.1 First-order delay process [5.16.1]

Problem statement.

$$G(s) = k\frac{e^{-\theta s}}{1+\tau s}; \quad G_d(s) = k_d\frac{e^{-\theta_d s}}{1+\tau_d s}; \quad |k_d| > 1 \tag{3.42}$$

Also: measurement delays θ_m, θ_{md}

Specification: $|e| < 1$ for $|u| < 1, |d| < 1$.

i) feedback control only

ii) feedforward control only

Give quantitative relationships between the parameters which should be satisfied to achieve controllability.

Solution. For $|u| < 1$ we must from Rule 3 require $|G(j\omega)| > |G_d(j\omega)| \; \forall \omega < \omega_d$. For both feedback and feedforward

$$\boxed{k > k_d; \quad k/\tau > k_d/\tau_d} \tag{3.43}$$

(i) **Feedback control**. From Rule 1 for $|e| < 1$ with disturbances

$$\omega_d \approx k_d/\tau_d < \omega_c \tag{3.44}$$

On the other hand, from Rule 5 we require for stability and performance

$$\omega_c < 1/\theta_{tot} \tag{3.45}$$

where $\theta_{tot} = \theta + \theta_m$ is the total delay around the loop. (3.44) and (3.45) yield the following requirement for controllability

$$\boxed{\text{Feedback:} \qquad \theta + \theta_m < \tau_d/k_d} \tag{3.46}$$

(ii) **Feedforward control.** For $|e| < 1$ we need

$$\boxed{\text{Feedforward:} \qquad \theta + \theta_{md} - \theta_d < \tau_d/k_d} \tag{3.47}$$

3.10.2 Application: Room heating [5.16.2]

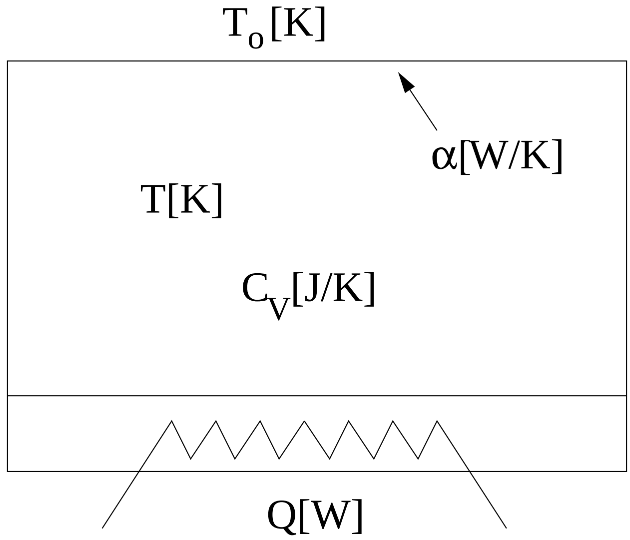


Figure 25: Room heating process

1. **Physical model.** Heat input Q, room temperature T (within $\pm 1K$), outdoor temperature T_o.

   Energy balance:

   $$\frac{d}{dt}(C_V T) = Q + \alpha(T_o - T) \tag{3.48}$$

2. **Operating point.** Heat input Q^* is $2000W$, difference between indoor and outdoor temperatures $T^* - T_o^*$ is 20 K. The steady-state energy balance yields $\alpha^* = 2000/20 = 100W/K$. We assume $C_V = 100kJ/K$.

3. Linear model in deviation variables.

$$\begin{aligned}\delta T(t) &= T(t) - T^*;\\ \delta Q(t) &= Q(t) - Q^*;\\ \delta T_o(t) &= T_o(t) - T_o^*\end{aligned}$$

yields

$$C_V \frac{d}{dt}\delta T(t) = \delta Q(t) + \alpha(\delta T_o(t) - \delta T(t)) \qquad (3.49)$$

On taking Laplace transforms in (3.49), assuming $\delta T(t) = 0$ at $t = 0$ and rearranging we get

$$\delta T(s) = \frac{1}{\tau s + 1}\left(\frac{1}{\alpha}\delta Q(s) + \delta T_o(s)\right); \quad \tau = \frac{C_V}{\alpha} \qquad (3.50)$$

The time constant for this example is $\tau = 100 \cdot 10^3/100 = 1000s \approx 17min$

4. Linear model in scaled variables.

Introduce the following scaled variables

$$y(s) = \frac{\delta T(s)}{\delta T_{max}} \quad (3.51)$$

$$u(s) = \frac{\delta Q(s)}{\delta Q_{max}} \quad (3.52)$$

$$d(s) = \frac{\delta T_o(s)}{\delta T_{o,max}} \quad (3.53)$$

Acceptable variations in room temperature T are $\pm 1K$, i.e. $\delta T_{max} = \delta e_{max} = 1K$. The heat input can vary between $0W$ and $6000W$, since its nominal value is $2000W$ we have $\delta Q_{max} = 2000W$.

Expected variation in temperature are $\pm 10K$, i.e. $\delta T_{o,max} = 10K$.

The model becomes

$$G(s) = \frac{1}{\tau s+1}\frac{\delta Q_{max}}{\delta T_{max}}\frac{1}{\alpha} = \frac{20}{1000s+1} \quad (3.54)$$

$$G_d(s) = \frac{1}{\tau s+1}\frac{\delta T_{o,max}}{\delta T_{max}} = \frac{10}{1000s+1} \quad (3.55)$$

Measurement delay for temperature (y) be $\theta_m = 100s$.

Problem statement.

1. Is the plant controllable with respect to disturbances?

2. Is the plant controllable with respect to setpoint changes of magnitude $R = 3$ (± 3 K) when the desired response time for setpoint changes is $\tau_r = 1000$ s (17 min) ?

Solution.

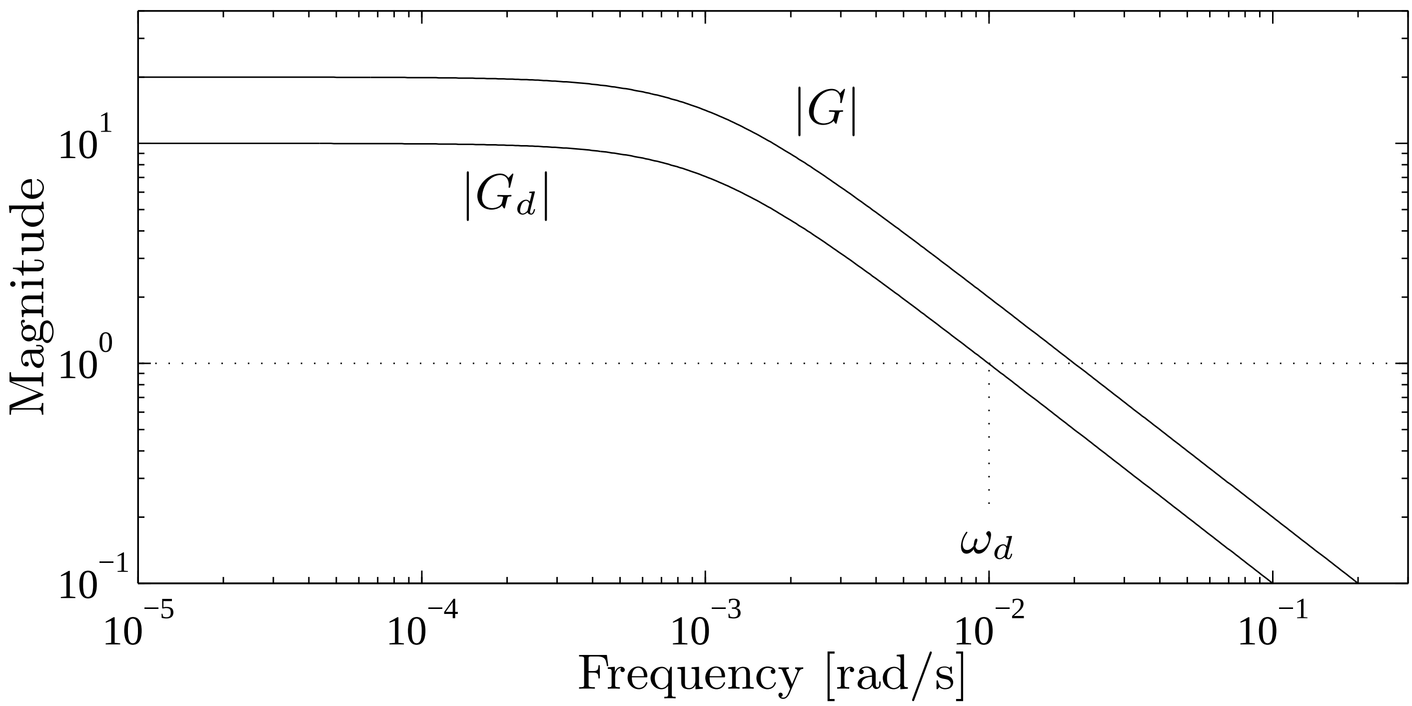
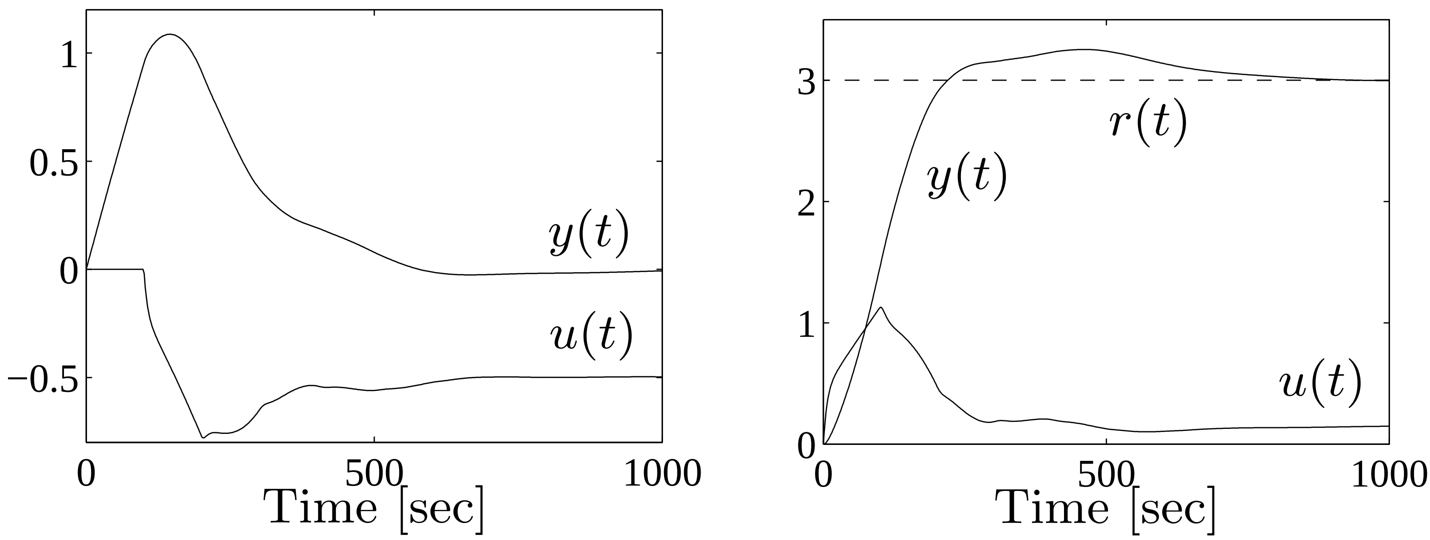


Figure 26: Frequency responses for room heating example

1. Disturbances. From Rule 1 feedback control is necessary up to the frequency $\omega_d = 10/1000 = 0.01$ rad/s, where $|G_d|$ crosses 1 in magnitude ($\omega_c > \omega_d$). This is exactly the same frequency as the upper bound given by the delay, $1/\theta = 0.01$ rad/s ($\omega_c < 1/\theta$). Therefore the system is barely controllable for this disturbance. From Rule 3 no problems with input constraints since $|G| > |G_d|$ at all frequencies. These conclusions are supported by the closed-loop simulation in Figure 27(a) using a PID-controller with $K_c = 0.4$ (scaled variables), $\tau_I = 200$ s and $\tau_D = 60$ s.

2. Setpoints. The plant is controllable with respect to the desired setpoint changes.

1. The delay (100 s) is much smaller than the desired response time of 1000 s
2. $|G(j\omega)| \geq R = 3$ up to about $\omega_1 = 0.007$ [rad/s] which is seven times higher than the required $\omega_r = 1/\tau_r = 0.001$ [rad/s]. This means that input constraints pose no problem. In fact, we achieve response times of about $1/\omega_1 = 150$ s without reaching the input constraints. See Figure 27(b) for a desired setpoint change $3/(150s + 1)$ using the same PID controller as above.



(a) Step disturbance in outdoor temperature

(b) Setpoint change $3/(150s+1)$

Figure 27: PID feedback control of room heating example

3.10.3 * Application: Neutralization process [5.16.3]

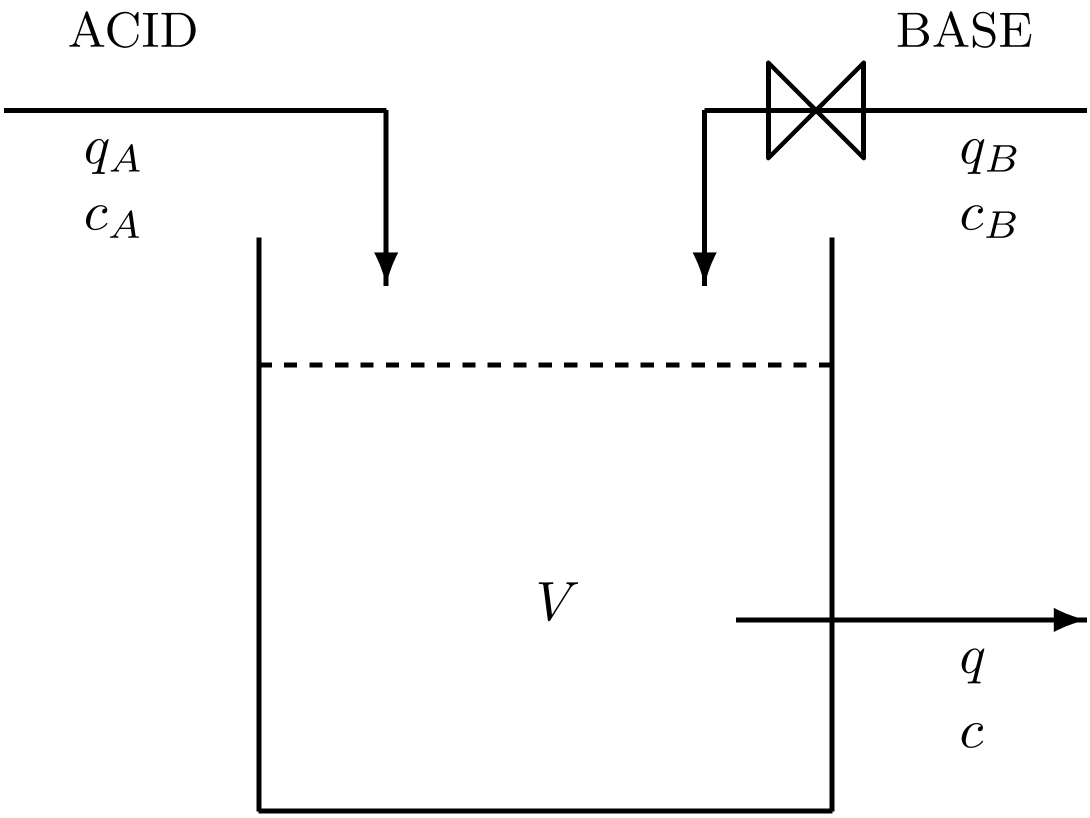


Figure 28: Neutralization process with one mixing tank

Problem statement. Consider process in Figure 28, where a strong acid with pH$= -1$ is neutralized by a strong base (pH=15) in a mixing tank with volume $V = 10\text{m}^3$.

Feedback control to keep the pH in the product stream (output y) in the range 7 ± 1 ("salt water") by manipulating the amount of base, q_B (input u) in spite of variations in the flow of acid, q_A (disturbance d). The delay in the pH-measurement is $\theta_m = 10$ s.

1. Controlled output is the excess of acid, c [mol/l], defined as $c = c_{H^+} - c_{OH^-}$.

2. Objective is to keep $|c| \leq c_{\max} = 10^{-6}$ mol/l, and the plant is

$$\frac{d}{dt}(Vc) = q_A c_A + q_B c_B - qc \qquad (3.56)$$

$q_A^* = q_B^* = 0.005$ [m^3/s] resulting in $q^* = 0.01$ [m^3/s]= 10 [l/s].

3. Scaled variables:

$$y = \frac{c}{10^{-6}}; \quad u = \frac{q_B}{q_B^*}; \quad d = \frac{q_A}{0.5 q_A^*} \qquad (3.57)$$

4. Scaled linear model:

$$G_d(s) = \frac{k_d}{1 + \tau_h s}; \quad G(s) = \frac{-2k_d}{1 + \tau_h s}; \quad k_d = 2.5 \cdot 10^6 \qquad (3.58)$$

where $\tau_h = V/q = 1000$ s is the residence time for the liquid in the tank.

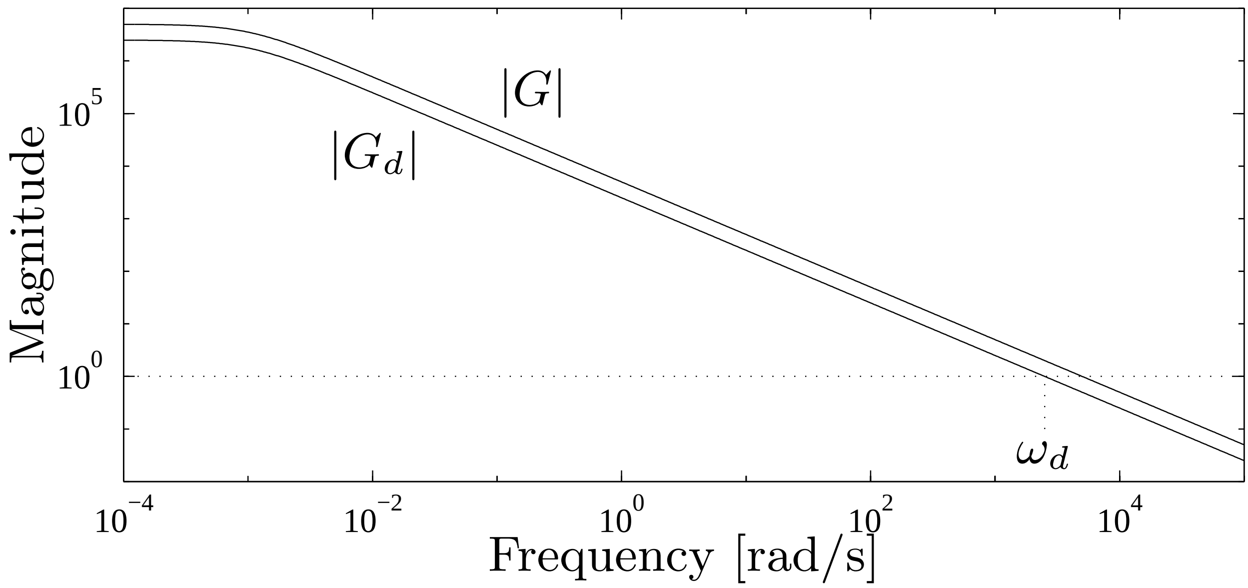


Figure 29: Frequency responses for the neutralization process with one mixing tank

Controllability analysis.

Figure 29: From Rule 2, input constraints do not pose a problem since $|G| = 2|G_d|$ at all frequencies. From Rule 1 we find the frequency up to which feedback is needed

$$\omega_d \approx k_d/\tau = 2500 \text{ rad/s} \tag{3.59}$$

This requires a response time of $1/2500 = 0.4$ milliseconds which is clearly impossible in a process control application (also: delay of 10 s).

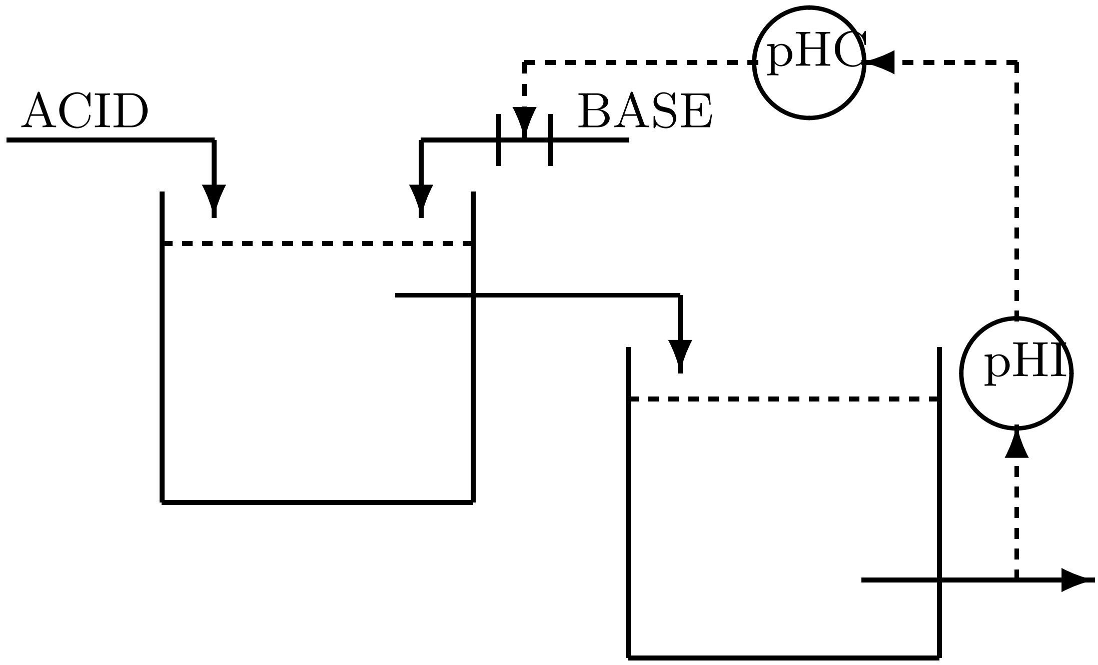


Figure 30: Neutralization process with two tanks and one controller

Design change: Multiple tanks.

To improve controllability modify the process ⇒ Perform the neutralization in several steps as illustrated in Figure 30 for the case of two tanks.

With n equal mixing tanks in series

$$G_d(s) = k_d h_n(s); \quad h_n(s) = \frac{1}{(\frac{\tau_h}{n}s+1)^n} \tag{3.60}$$

$h_n(s)$ is transfer function of the mixing tanks, and τ_h is total residence time, V_{tot}/q.

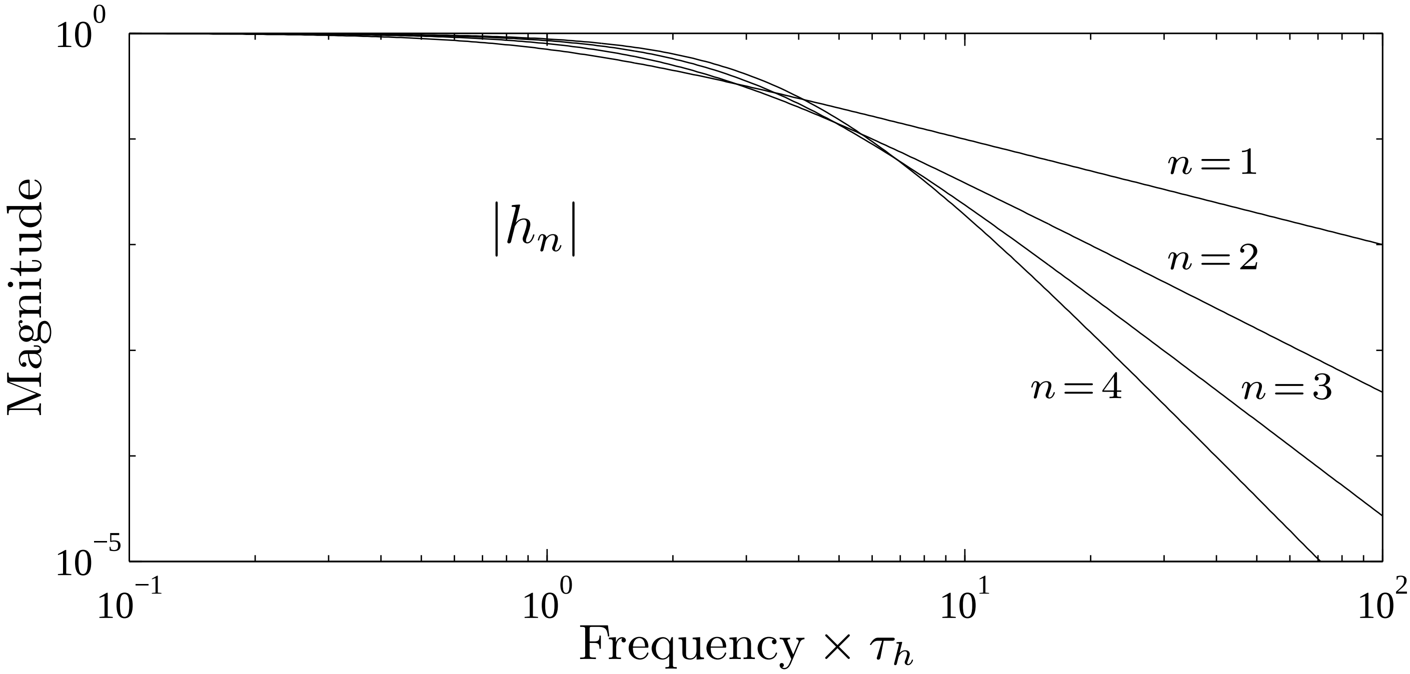


Figure 31: Frequency responses for n tanks in series with the same total residence time τ_h; $h_n(s) = 1/(\frac{\tau_h}{n}s+1)^n,\ n = 1,2,3,4$

From Rules 1 and 5, we must require

$$\boxed{|G_d(j\omega_\theta)| \leq 1} \quad \omega_\theta \overset{\Delta}{=} 1/\theta \tag{3.61}$$

where θ is the delay in the feedback loop. Purpose of mixing tanks $h_n(s)$ is to reduce the effect of the disturbance by a factor $k_d(= 2.5 \cdot 10^6)$ at the frequency $\omega_\theta(= 0.1$ [rad/s]), i.e. $|h_n(j\omega_\theta)| \leq 1/k_d$. Minimum value for the total volume for n equal tanks in series

$$V_{tot} = q\theta n\sqrt{(k_d)^{2/n} - 1} \tag{3.62}$$

where $q = 0.01$ m^3/s.

With $\theta = 10$ s we then find that the following designs have the same controllability

No. of tanks n	Total volume V_{tot} $[m^3]$	Volume each tank $[m^3]$
1	250000	250000
2	316	158
3	40.7	13.6
4	15.9	3.98
5	9.51	1.90
6	6.96	1.16
7	5.70	0.81

$n = 1 \Rightarrow$ Supertanker.

Minimum total volume is 3.662 m^3 with 18 tanks of about 203 liters each

Practical compromise: 3 or 4 tanks.

Control system design. We have $|S| < 1/|G_d|$ at the crossover frequency $\omega_B \approx \omega_c \approx \omega_\theta$. However, from Rule 1 we also require that $|S| < 1/|G_d|$, or approximately $|L| > |G_d|$, at frequencies lower than ω_c, (difficult since $G_d(s) = k_d h(s)$ is of high order). This requires $|L|$ to drop steeply with frequency, which results in a large negative phase for L

Thus, system in Figure 30 with a single feedback controller will *not* work. ⇒ install *local feedback* control system on each tank (Figure 32.).

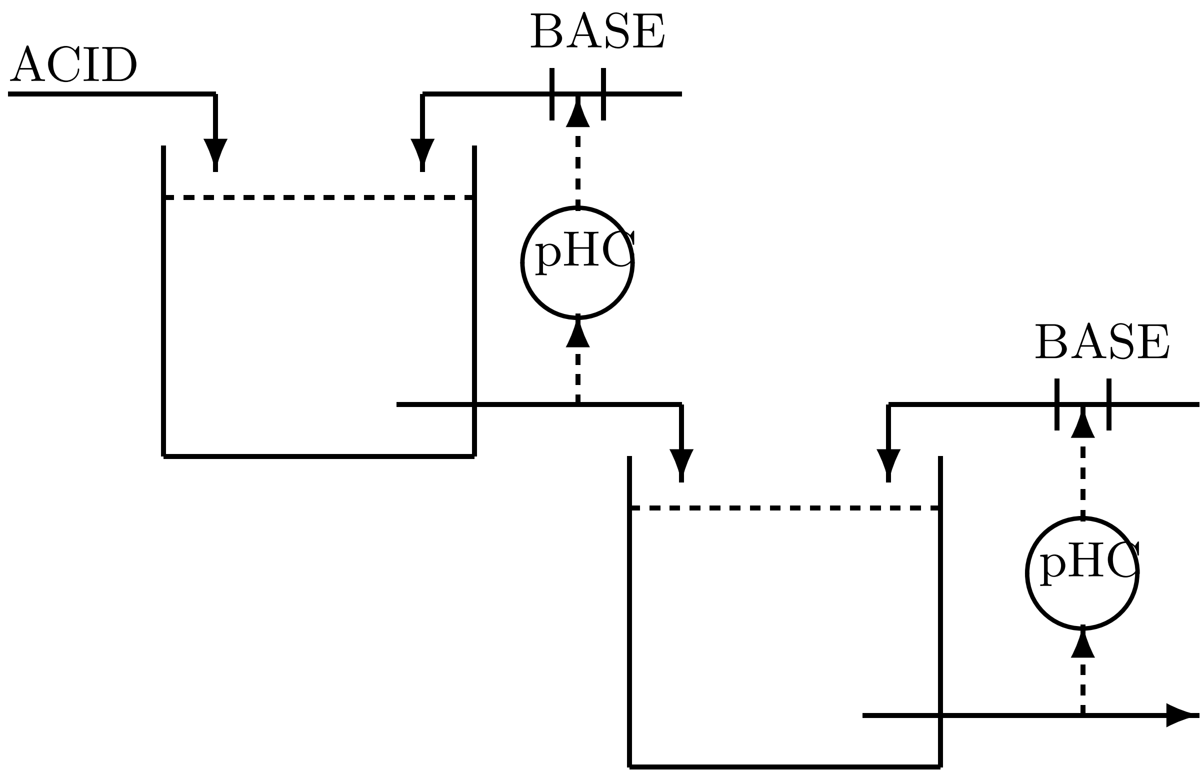


Figure 32: Neutralization process with two tanks and two controllers.

$\Rightarrow$*plant design change*

With n controllers for n tanks the overall closed-loop response from a disturbance into the first tank to the pH in the last tank becomes

$$y = G_d \prod_{i=1}^{n} \left(\frac{1}{1+L_i}\right) d \approx \frac{G_d}{L} d, \quad L \triangleq \prod_{i=1}^{n} L_i \qquad (3.63)$$

where $G_d = \prod_{i=1}^{n} G_i$ and $L_i = G_i K_i$, and the approximation applies at low frequencies where feedback is effective.

Design each loop $L_i(s)$ with a slope of -1 and bandwidth $\omega_c \approx \omega_\theta$, such that the overall loop transfer function L has slope $-n$ and achieves $|L| > |G_d|$ at all frequencies lower than ω_d (the size of the tanks are selected as before such that $\omega_d \approx \omega_\theta$).