Observability & Controllability

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- State Space Model
- Infer State i.c. from Output Measurements
- Important for State Estimation
- Controllability

Rensselaer
Chemical and Biological Engineering

Observability

$$\left.\begin{aligned} x_{k+1} &= \Phi x_k + \Gamma u_k \\ y_k &= C x_k \end{aligned}\right\}$$

Discrete, state space model

Step from initial condition to n-1 for n output measurements

$$y_0 = C x_0$$

$$y_1 = C x_1 = C\left[\Phi x_0 + \Gamma u_0\right] = C\Phi x_0 + C\Gamma u_0$$

$$y_2 = C x_2 = C\left[\Phi x_1 + \Gamma u_1\right] = C\left[\Phi\left\{\Phi x_0 + \Gamma u_0\right\} + \Gamma u_1\right]$$

$$= C\Phi^2 x_0 + C\Phi\Gamma u_0 + C\Gamma u_1$$

$$y_{n-1} = C\Phi^{n-1} x_0 + C\Phi^{n-2}\Gamma u_0 + \cdots + C\Gamma u_{n-1}$$

Can rearrange to solve for the initial condition

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Observability

$$\underbrace{\begin{bmatrix} y_0 \\ y_1 - C\Gamma u_0 \\ y_2 - C\Phi\Gamma u_0 - C\Gamma u_1 \\ \vdots \\ y_{n-1} - C\Phi^{n-2}\Gamma u_0 - \cdots - C\Gamma u_{n-2} \end{bmatrix}}_{(n\times 1)} = \underbrace{\begin{bmatrix} C \\ C\Phi \\ C\Phi^2 \\ \vdots \\ C\Phi^{n-1} \end{bmatrix}}_{\mathrm{O}(n\times n)} \underbrace{x_0}_{(n\times 1)}$$

$$Y = O \quad x_0$$

Solve for the initial condition

$$x_0 = O^{-1} \; Y$$

Solution exists if O is nonsingular (full rank)

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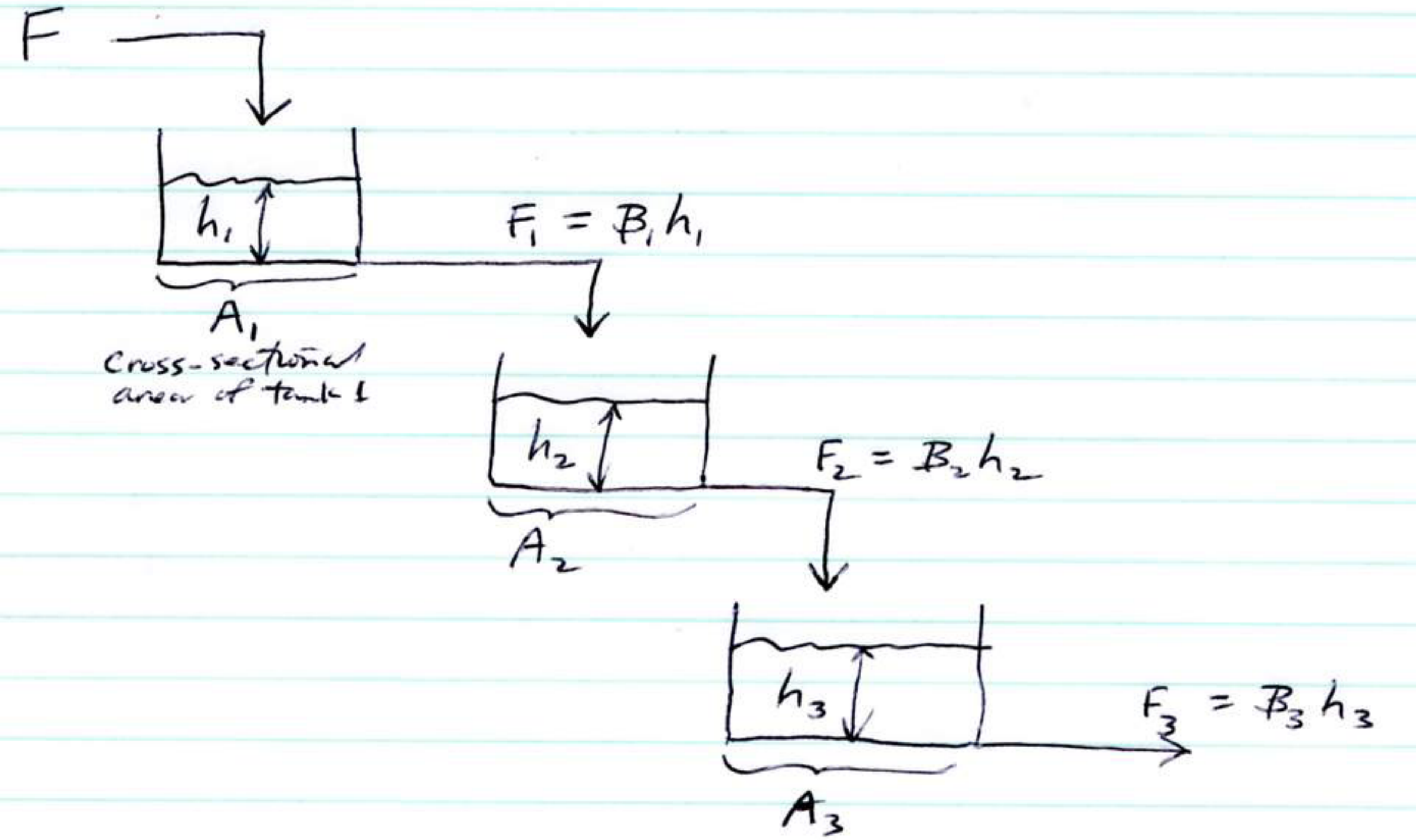
Observability, Continuous-time

$$\dot{x} = Ax + Bu$$

$$y = Cx$$

$$O = \underbrace{\begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}}_{O(n \times n)} = \text{observability matrix}$$

Example 3-tank problem



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Example

$$\underbrace{\begin{bmatrix} \frac{dh_1}{dt} \\ \frac{dh_2}{dt} \\ \frac{dh_3}{dt} \end{bmatrix}}_{\dot{x}} = \underbrace{\begin{bmatrix} -1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} h_1 \\ h_2 \\ h_3 \end{bmatrix}}_{x} + \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{B} \underbrace{F}_{u}$$

$$y = \underbrace{\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}}_{C} \underbrace{\begin{bmatrix} h_1 \\ h_2 \\ h_3 \end{bmatrix}}_{x} \qquad y = \underbrace{\begin{bmatrix} 0 & 1 & 0 \end{bmatrix}}_{C} \underbrace{\begin{bmatrix} h_1 \\ h_2 \\ h_3 \end{bmatrix}}_{x} \qquad y = \underbrace{\begin{bmatrix} 0 & 0 & 1 \end{bmatrix}}_{C} \underbrace{\begin{bmatrix} h_1 \\ h_2 \\ h_3 \end{bmatrix}}_{x}$$

State 1 measured

State 2 measured

State 3 measured

Example, continued

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$$

$C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$ $\quad$ $C = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}$ $\quad$ $C = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$

State 1 measured $\quad$ State 2 measured $\quad$ State 3 measured

$$\underbrace{\begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix}}_{O(3\times 3)} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -1 & 0 \\ -2 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -2 & 1 \end{bmatrix}$$

Rank = 1 $\quad$ Rank = 2 $\quad$ Rank = 3

If state 3 (height of 3rd tank) is measured, states 1 and 2 are observable

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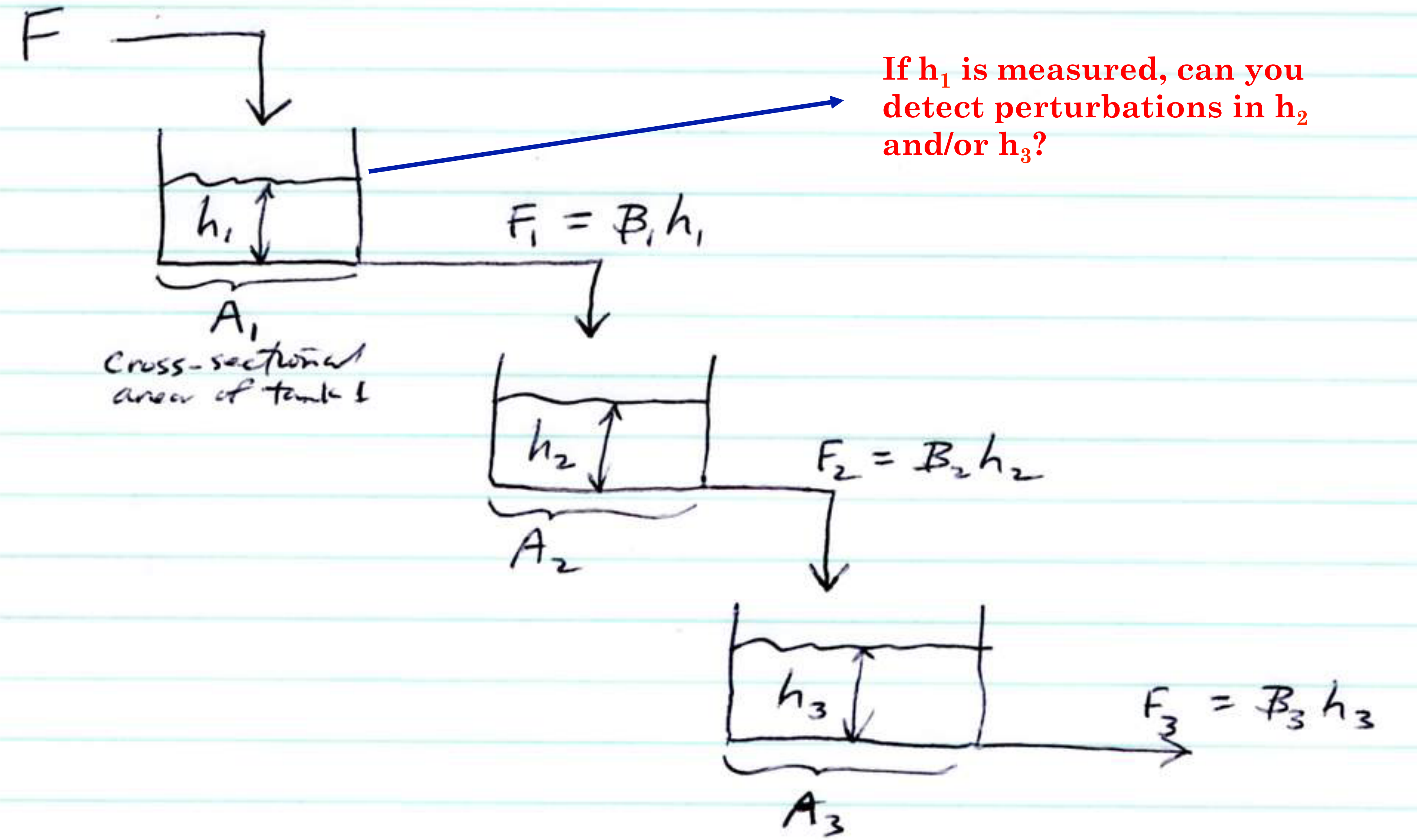
Does this make sense?



If h_1 is measured, can you detect perturbations in h_2 and/or h_3?

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Example, Discrete-Time

$$\Phi = \begin{bmatrix} 0.6065 & 0 & 0 \\ 0.3033 & 0.6065 & 0 \\ 0.0758 & 0.3033 & 0.6065 \end{bmatrix}$$

$C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$ — State 1 measured

$C = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}$ — State 2 measured

$C = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$ — State 3 measured

$$\underbrace{\begin{bmatrix} C \\ C\Phi \\ C\Phi^2 \end{bmatrix}}_{O(3\times 3)} = \begin{bmatrix} 1 & 0 & 0 \\ 0.6065 & 0 & 0 \\ 0.3679 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0.3033 & 0.6065 & 0 \\ 0.3679 & 0.3679 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0.0758 & 0.3033 & 0.6065 \\ 0.1839 & 0.3679 & 0.3679 \end{bmatrix}$$

Rank = 1

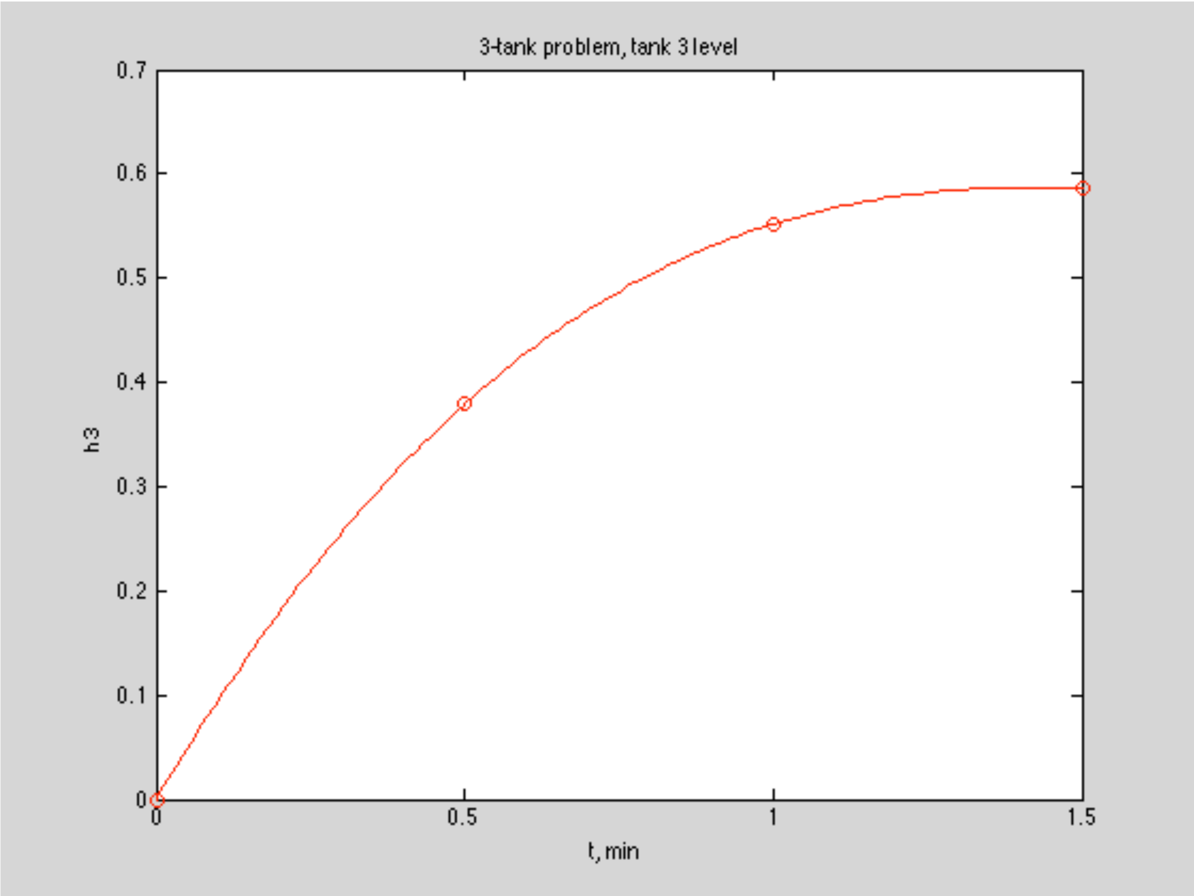
Rank = 2

Rank = 3

If state 3 (height of 3rd tank) is measured, states 1 and 2 are observable

Numerical Example

- Some perturbation in states initially, with no input changes. The following output measurements are available
  - y0 = 0 (t=0 min), y1 = 0.3791 (t=0.5 min), y2 = 0.5518 (t=1.0 min)



The observability matrix

$$\underbrace{\begin{bmatrix} C \\ C\Phi \\ C\Phi^2 \end{bmatrix}}_{O(3\times 3)} = \begin{bmatrix} 0 & 0 & 1 \\ 0.0758 & 0.3033 & 0.6065 \\ 0.1839 & 0.3679 & 0.3679 \end{bmatrix}$$

Solve for the initial condition

$$x_0 = O^{-1} \; Y$$

To find

$$x_0 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

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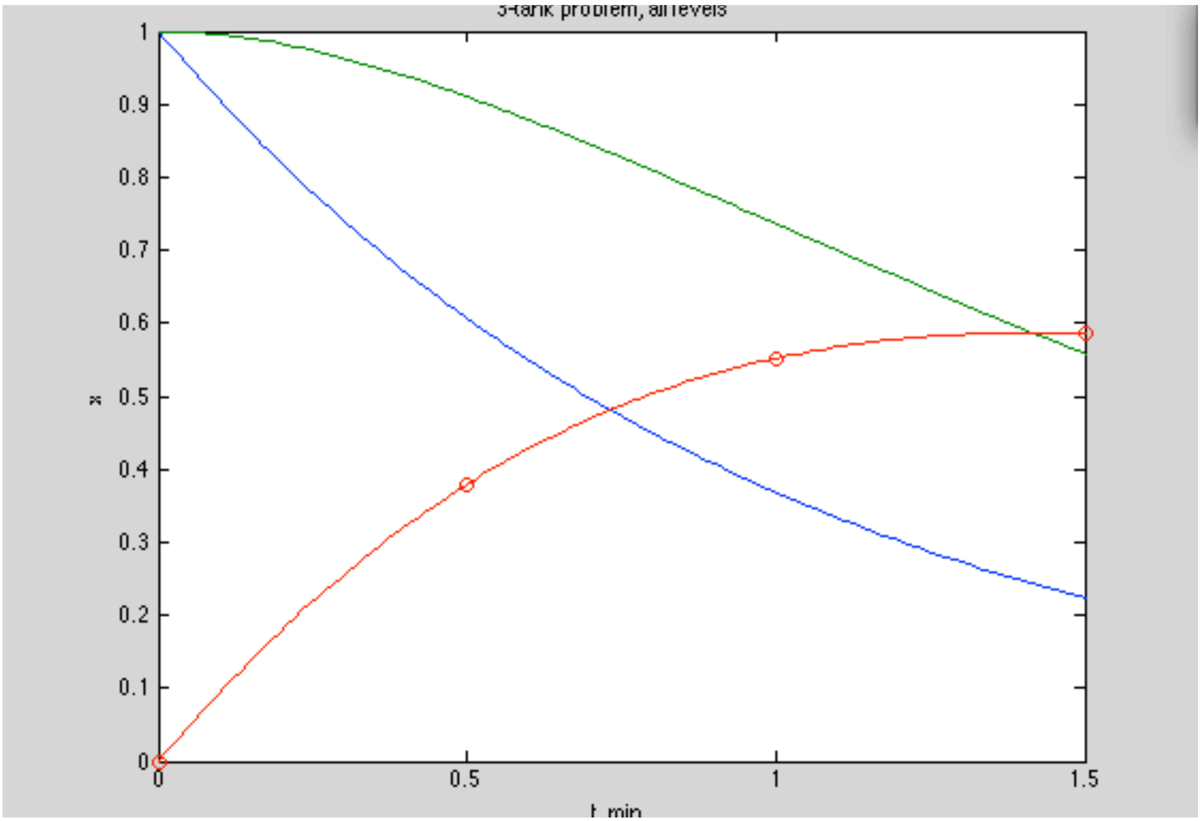
Which is consistent with the simulation



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Controllability

$$\left.\begin{aligned} x_{k+1} &= \Phi x_k + \Gamma u_k \\ y_k &= C x_k \end{aligned}\right\}$$

Discrete, state space model

For an n-state system with known initial condition, find the sequence of n manipulated input moves to reach a desired x_n

$$\begin{aligned} &x_0 \\ &x_1 = \left[\Phi x_0 + \Gamma u_0\right] \\ &x_2 = \left[\Phi x_1 + \Gamma u_1\right] = \left[\Phi\left\{\Phi x_0 + \Gamma u_0\right\} + \Gamma u_1\right] \\ &x_2 = \Phi^2 x_0 + \Phi\Gamma u_0 + \Gamma u_1 \\ &etc. \end{aligned}$$

Exercise:

Solve for the manipulated inputs u_0 → u_{n-1} to achieve x_n

Use the 3-tank example, then generalize

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Controllability Solution

$$x_1 = \Phi x_0 + \Gamma u_0$$

$$x_2 = \Phi x_1 + \Gamma u_1 = \Phi\left[\Phi x_0 + \Gamma u_0\right] + \Gamma u_1$$

$$x_2 = \Phi^2 x_0 + \Phi\Gamma u_0 + \Gamma u_1$$

$$x_3 = \Phi^3 x_0 + \Phi^2\Gamma u_0 + \Phi\Gamma u_1 + \Gamma u_2$$

$$x_3 = \Phi^3 x_0 + \begin{bmatrix}\Phi^2\Gamma & \Phi\Gamma & \Gamma\end{bmatrix}\begin{bmatrix}u_0 \\ u_1 \\ u_2\end{bmatrix}$$

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$$x_3 - \Phi^3 x_0 = \begin{bmatrix} \Phi^2\Gamma & \Phi\Gamma & \Gamma \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ u_2 \end{bmatrix}$$

$$\begin{bmatrix} u_0 \\ u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} \Phi^2\Gamma & \Phi\Gamma & \Gamma \end{bmatrix}^{-1} \begin{bmatrix} x_3 - \Phi^3 x_0 \end{bmatrix}$$

Must be invertible (non-singular)

In General

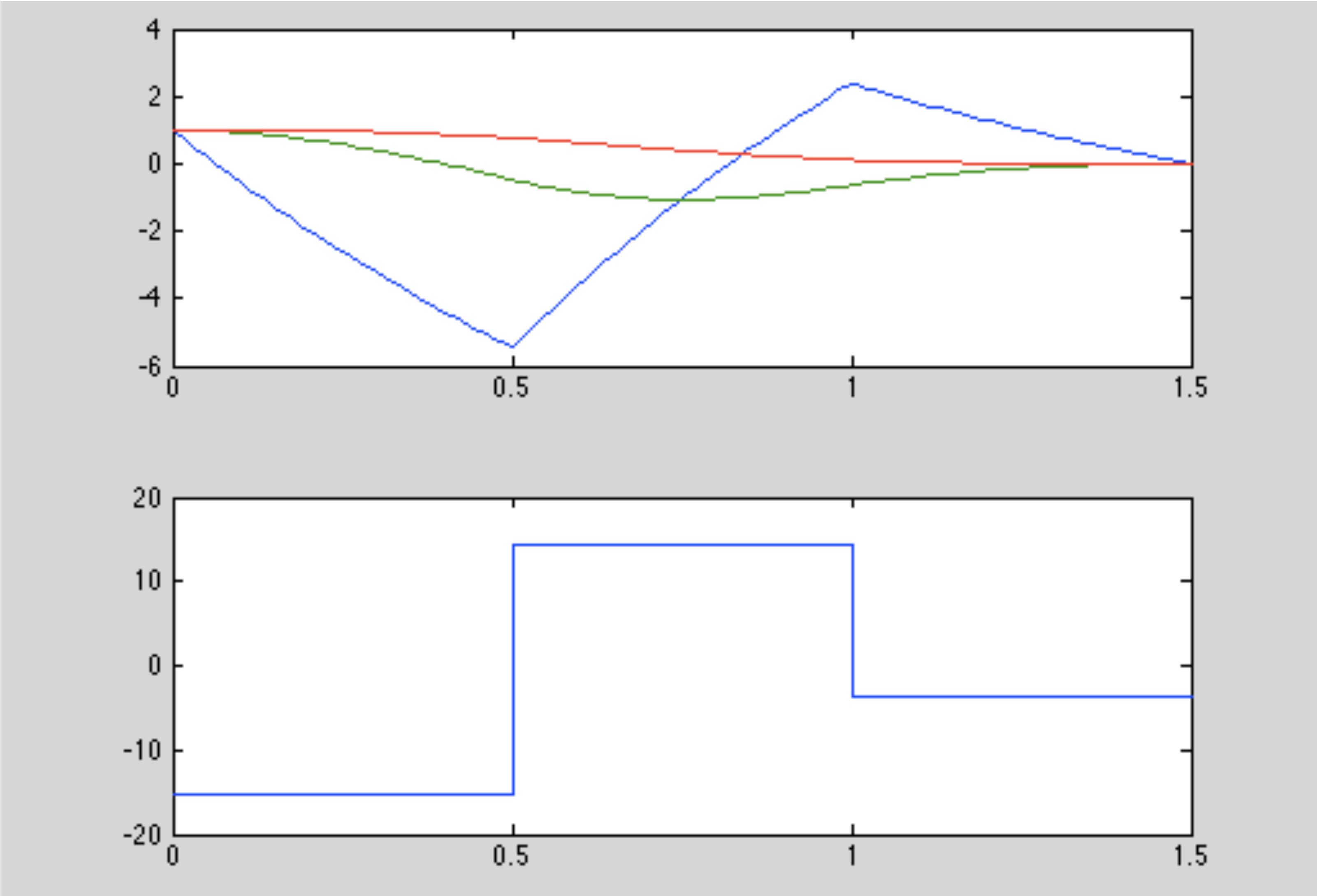
$$x_n - \Phi^n x_0 = \left[\Phi^n\Gamma \ldots \Phi\Gamma \;\; \Gamma\right]\begin{bmatrix} u_0 \\ u_{n-2} \\ u_{n-1} \end{bmatrix}$$

$$\begin{bmatrix} u_0 \\ u_{n-2} \\ u_{n-1} \end{bmatrix} = \left[\Phi^n\Gamma \;\; \ldots\Phi\Gamma \;\; \Gamma\right]^{-1}\left[x_n - \Phi^n x_0\right]$$

Must be invertible (non-singular)

Three Tank Example

$x_0 = [1;1;1]$ $x_3 = [0;0;0]$



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```
%  use the three tank example
%   first, continuous state space model
 a = [-1 0 0;1 -1 0;0 1 -1]
 b = [1;0;0]
 c = [0 0 1]
 d = 0
%
 lintank = ss(a,b,c,d)             % defines continuous state space model
% ------------------------------------------------------------------
%
% discrete time model
 delt = 0.5;                        % sample time of 0.5 minutes
 tankssz = c2d(lintank,delt,'zoh')  % create discrete state space model from continuous
%
%  extract the matrix information
%
 [phi,gamma,cmat,dmat] = ssdata(tankssz)   % cmat will equal c
%
%
 CONTmat = [phi*phi*gamma phi*gamma gamma]
%
 rank(CONTmat)
```

```
x0 = 1;1;1] ;
xdis(:,1) = x0;
x3  = [0;0;0]
u = inv(CONTmat)*(x3 - (phi^3)*x0)
time = [0 0.5 1 1.5];
tplot = [];
yplot = [];
xplot = [];
%
  for k = 1:3;
     [tdummy,xdummy] = ode45('linodepar',[time(k) time(k+1)],xdis(:,k),[],a,b,u(k));
      ndum = length(tdummy);
      xdis(:,k+1) = xdummy(ndum,:);   % plant state
      tplot = [tplot;tdummy];
      xplot = [xplot;xdummy];
  end
%
  u(k+1) = u(k)
```