Introduction to MPC

Model predictive control

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Aim of Presentation

- What is model predictive control
- Linear quadratic control, LQR (LQG)
- MPC based on step response model (DMC)
- Constraint handling
- Some examples
- Other approaches, State space MPC, GPC

What is model predictive control?

Predictor A model that predicts the output given a sequence of inputs
Filter Using the measurements to update the states
Optimizer Finding a control sequence, which minimizes a selected objective function
On-line solution of optimal control problem

Why model predictive control

- Multivariable control
- Constraint handling
- Feedforward control
- Information of predicted future events
- Nonsquare systems

Linear Quadratic Regulator

Given a linear plant without disturbance

$$x_{k+1} = Ax_k + Bu_k$$

initial condition x_{k_0} and a quadratic objective

$$J_{k_0} = x_{k_N}^T Q_{k_N} x_{k_N} + \sum_{i=k_0}^{i=N-1} \left(x_i^T Q_i x_i + u_i^T R_i u_i \right)$$

This problem has a closed solution

$$u_k = -K_k x_k$$

If $N \to \infty$ then K_k becomes constant.
Traditional LQG.

The moving horizon (or receding horizon)

MPC Receding horizon

A quadratic objective

$$\begin{aligned} J_k \; = & \; \sum_{i=1}^{P} \left(y_{k+i} - y_{k+i}^r \right)^T Q_i \left(y_{k+i} - y_{k+i}^r \right) \\ & + \sum_{i=0}^{M} \Delta u_{k+i}^T R_i \Delta u_{k+i} \\ = & \; \left(Y_{k+1} - Y_{k+1}^r \right)^T \Gamma^y \left(Y_{k+1} - Y_{k+1}^r \right) \\ & + \Delta U_k^T \Gamma^{\Delta u} \Delta U_k \end{aligned}$$

Where

$$\begin{aligned} Y_{k+1} &= \left[\, y_{k+1}^T \; y_{k+2}^T \; \cdots \; y_{k+p}^T \, \right]^T \\ \Gamma^y &= diag\{ \, Q_1 \; \cdots \; Q_p \, \} \\ \Delta U_k &= \left[\, \Delta u_{k+1}^T \; \Delta u_{k+2}^T \; \cdots \; \Delta u_{k+m+1}^T \, \right]^T \\ \Gamma^{\Delta u} &= diag\{ \, R_0 \; \cdots \; R_m \, \} \end{aligned}$$

MPC based on step response

(Cutler, Ramaker 1980) and (Garcia, Morshedi 1986)

Uses

- A step response model

Assumes

- Stable, linear and time invariant plant
- That the difference between measured and predicted output can be modeled as a step disturbance on the output

Step response

System is at rest at time k and we apply a unit step and get the response

$$[0 \; S_1^u \; S_2^u \; S_3^u \; \cdots \; S_n^u \; \cdots]$$

In the MIMO case each S_l^u is a matrix, where each element $S_{l_{ij}}^u$ is the $l-th$ step response coefficient from input j to output i.

Past moves

To capture the effect of initial condition define

$$Y_k^* = \begin{bmatrix} y_k^T \; y_{k+1}^T \; \cdots \; y_{k+n-1}^T \\ (\Delta u_{k+i} = 0, \forall i \geq 0) \end{bmatrix}^T$$

Past moves

If no move is made at the current step

$$Y_{k+1}^* = \underbrace{\begin{bmatrix} 0 & I & 0 & 0 & \cdots & 0 \\ 0 & 0 & I & \cdots & 0 & \\ \vdots & & & & & \vdots \\ 0 & \cdots & & & 0 & I \\ 0 & \cdots & & & 0 & I \end{bmatrix}}_{M} Y_k^*$$

This is the effect of past moves on future outputs.

Step response model

Assume $\Delta u_i = 0, \; \forall i \geq k+1$
Superposition gives

$$Y_{k+1}^* = MY_k^* + \begin{bmatrix} S_1^u \\ S_2^u \\ S_3^u \\ \vdots \\ S_n^u \end{bmatrix} \Delta u_k$$

A state space realization

Predictor

Must add the effect of future moves to the future output.

$$Y_{k+1} = M_p Y_k^* + \underbrace{\begin{bmatrix} S_1^u & 0 & \cdots & 0 \\ S_2^u & S_1^u & \cdots & 0 \\ \vdots & & \cdots & \vdots \\ S_p^u & S_{p-1}^u & \cdots & S_{p-m+1}^u \end{bmatrix}}_{S^u} \begin{bmatrix} \Delta u_k \\ \Delta u_{k+1} \\ \vdots \\ \Delta u_{k+m} \end{bmatrix}$$

Only p-step ahead prediction is needed.

Disturbances

Deviation between measured output and predicted output

$$d_k = \hat{y}_k - y_k$$

Assume that this deviation is constant into the future $d_{k+l} = d_k$

This is added to all the predicted outputs

$$D_{k+1} = [d_k^T \;\; d_k^T \;\; \cdots \;\; d_k^T]^T$$

Unconstrained MPC

The predictor including measured V_k and unknown D_k disturbances

$$Y_{k+1} = M_p Y_k^* + S^u \Delta U_k + S^v \Delta V_k + D_{k+1}$$

The quadratic objective is

$$\begin{aligned} J_k &= \left(Y_{k+1} - Y_{k+1}^r\right)^T \Gamma^y \left(Y_{k+1} - Y_{k+1}^r\right) \\ &+ \Delta U_k^T \Gamma^{\Delta u} \Delta U_k \end{aligned}$$

Inserting the predictor and manipulations gives

$$\begin{aligned} J_k &= \Delta U_k^T \left(S^{uT} \Gamma^y S^u + \Gamma^{\Delta u}\right) \Delta U_k \\ &+ 2 \underbrace{\left(M_p Y_k^* + S^v \Delta V_k + D_{k+1} - Y_{k+1}^r\right)^T}_{-E_{k+1}^T} \Gamma^y S^u \Delta U_k \end{aligned}$$

E_{k+1} is the predicted output deviation from the reference Y^r, if the future Δu is zero.

Solution of unconstrained MPC

The solution is found by

$$\frac{\partial J_k}{\partial \Delta U_k} = 0$$

which yields

$$\Delta U_k = \underbrace{\left(S^{uT} \Gamma^y S^u + \Gamma^{\Delta u}\right)^{-1} S^{uT} \Gamma^y}_{K_{MPC}} E_{k+1}$$

Only the first move is calculated and applied.

- The gain matrix K_{MPC} is independent of the output, input and disturbances
- The move depends on the expected deviation E_{k+1}
- $S^{uT} \Gamma^y S^u + \Gamma^{\Delta u}$ is positive definite so we have a unique minimum solution
- No on-line optimization

Eksempel 2 til Stig Strand. a og b, paapeke at paadraget gaar i metning, nodvendig med constraints

Constrained MPC

Constraints must be expressed as $C\Delta U \geq b$.

Input limitations

$$u_{k+l}^{min} \leq u_{k-1} + \sum_{i=0}^{l} \Delta u_{k+i} \leq u_{k+l}^{max}$$

Put on matrix form

$$\begin{bmatrix} -I_L \\ I_L \end{bmatrix} \Delta U_k \geq \begin{bmatrix} u_{k-1} - u_k^{max} \\ \vdots \\ u_{k-1} - u_{k+n-1}^{max} \\ -u_{k-1} + u_k^{min} \\ \vdots \\ -u_{k-1} + u_{k+n-1}^{min} \end{bmatrix}$$

Where

$$I_L = \begin{bmatrix} I & 0 & \cdots & 0 \\ I & I & \cdots & 0 \\ \vdots & & \vdots & \vdots \\ I & I & \cdots & I \end{bmatrix}$$

Constrained MPC

Can also include limits on the move of the input

$$\begin{bmatrix} -I \\ I \end{bmatrix} \Delta U_k \geq \begin{bmatrix} -\Delta U_k^{max} \\ \Delta U_k^{min} \end{bmatrix}$$

Constrained MPC

Output limitation

$$Y_{k+1}^{min} \leq Y_{k+1} \leq Y_{k+1}^{max}$$

The predictive equations can be used to put constraints on the output

$$Y_{k+1} = M_p Y_k^* + S^u \Delta U_k + S^v \Delta V_k + D_{k+1}$$

and inserted

$$\begin{bmatrix} -S^u \\ S^u \end{bmatrix} \Delta U_k \geq \begin{bmatrix} M_p Y_k^* + S^v \Delta V_k + D_{k+1} - Y_{k+1}^{max} \\ -M_p Y_k^* - S^v \Delta V_k - D_{k+1} + Y_{k+1}^{min} \end{bmatrix}$$

- Can lead to an infeasible solution set.
- Can give a very aggressive controller.

(move constraints window to later times or soften the constraints)

Quadratic programming

We have added linear inequality constraints to the input and output e.g.

$$\begin{aligned} u^{min} &\leq u \leq u^{max} \\ \Delta u^{min} &\leq \Delta u \leq \Delta u^{max} \\ y^{min} &\leq y \leq y^{max} \end{aligned}$$

Not possible to obtain a closed solution, must solve the quadratic program on-line.

The resulting QP has some features that makes it easy to solve

- Linear constraints $\Rightarrow$ a convex set.
- A Quadratic objective $\Rightarrow$ a convex function
  The gradient is linear.

The resulting QP

The resulting quadratic problem is

$$\begin{aligned} J_k &= \Delta U_k^T \left(S^{uT} \Gamma^y S^u + \Gamma^u \right) \Delta U_k \\ &\quad -2 E_{k+1}^T \Gamma^y S^u \Delta U_k \\ s.t. &\quad C \Delta U_k \geq c_{k+1} \end{aligned}$$

C and c_{k+1} are put together from the above.

Only c_{k+1} and E_{k+1} is varying.

MPC

Figur som viser strukturen pa regulatoren

Eksempel 5, multivariabel reguletor til Stig Strand, og Inledning til eksemplet til Lundstrom viser kolonna og sprangresponsene. Maa ha residual dynamikks

Residual Dynamics

How do we deal with

- unstable plant
- disturbances which are not steps at the output

Residual Dynamics

One solution (Lee, Morari, Garcia, 1994)

$$\begin{aligned} Y^*_{k+1} &= MY^*_k + S\Delta u_k + T\Delta w_k \\ y_k &= NY^*_k \\ \hat{y}_k &= y_k + v_k \end{aligned}$$

Where

$$Y^*_k = \left[\, y_k^T \;\, y_{k+1}^T \, \cdots \, y_{k+n-1}^T \;\, xu_k^T \;\, xd_k^T \,\right]^T$$

$$M = \begin{bmatrix} 0 & I & 0 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & & & \cdots & & & & \vdots \\ 0 & 0 & 0 & \cdots & 0 & I & & 0 \\ 0 & 0 & 0 & \cdots & 0 & I & C_u & C_w \\ 0 & 0 & 0 & \cdots & 0 & 0 & A_u & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & A_w \end{bmatrix}$$

$$S^T = \left[S_1^T \; S_2^T \; S_3^T \; \cdots \; S_n^T \; B_u^T \; 0\right]^T$$

$$T^T = \left[0 \; 0 \; 0 \; \cdots \; 0 \; 0 \; B_d^T\right]^T$$

Her kjem Lundstroms eksempel inn. Kommenter at MPC forblir stabil selv om systmet gar i metning.

State space MPC

(Lee, Morari, Garcia 1994), (Muske, Rawlings 1993)

The state space description

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k \\ y_k &= Cx_k \end{aligned}$$

And a quadratic objective function

$$J_k = \sum_{i=1}^{P} y_{k+i}^T Q_i y_{k+i} + \sum_{i=0}^{M} u_{k+i}^T R_i u_{k+i}$$

Same procedure, use the plant and develop predictor for future y.

The result will have the form

$$\begin{aligned} J_k &= U_k^T H U_k + 2U_k^T G x_k \\ U_k &= [u_k^T \; \cdots \; u_{k+M}^T] \end{aligned}$$

A state observer is needed.

Constraints, reference tracking and feedforward is handled in a similar way.

Generalized Predictive Control

(Clarke, Mohtadi, Tuffs 1987)

Uses a ARIMAX model

$$A(z^{-1})y_k = B(z^{-1})u_{k-1} + \frac{C(z^{-1})e_k}{\Delta}$$

$\frac{e_k}{\Delta}$ is integrated white noise.

Quadratic objective

$$J_k = E\{\sum_{i=1}^{P} [y_{k+i} - y_{k+i}^r]^2 + \sum_{i=0}^{M} \lambda_i \Delta u_i^2\}$$

Once again follow the same procedure as earlier.

Model Predictive Control

Similarities between approaches

- Linear model
- Quadratic objective
- Receding horizon
- Prediction
- A constrained QP, at each time step

Important topics not touched

- Optimal state estimation
- Stability and robustness
- Tuning

Conclusion

- A useful and proven technology

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Vaporizer

The plant is a model of a vaporizer (from a course held by Morari)

$$g(s) = \frac{2.69\,(-6s+1)\,exp(-1.5s)}{(20s+1)\,(5s+1)}$$

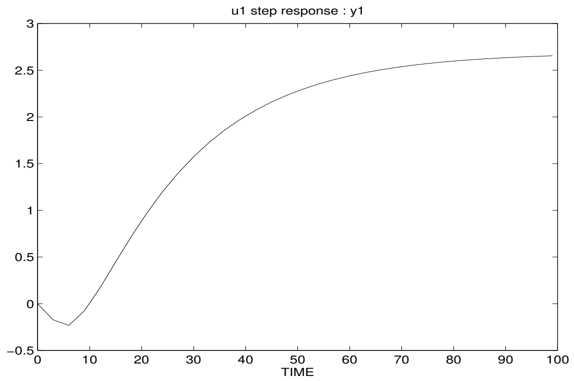

Figure 1: Step response in concentration for a step inn steam flow.

This is put in MPC format

```
delt1=0; %Zero for continious transfer function
num1=2.69*[-6 1];
den1=[100 25 1];
delay1=1.52;
g = poly2tfd(num1,den1,delt1,delay10);
```

The step response is obtained by

```
tfinal=75; %Final time
delt2=3;   %Step size
nout=1; %Stable plant
%Step function model format
plant=tfd2step(tfinal, delt2,nout, g);
plotstep(plant)
```



```
%MPC with a predicted setpoint change
model=plant;
m=3;   %Number of moves
p=25;  %Predicting horizon
%Must be longer then the invers response
ywt=1  %Weights on y-yr
uwt=.1; %Weights on moves
tend=100; i=ceil(100/delt2);
yr=ones(i,1); %Reference vector
[yp,u,ym]=cmpc(plant,model,ywt,uwt,m,p,tend,yr);
```

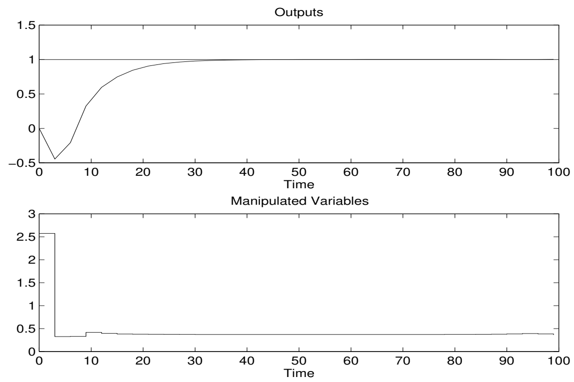

Figure 2: Simulation of vaporizer with a unpredicted setpoint change at t=0.


```
%MPC with a predicted setpoint change
model=plant;
m=3;   %Number of moves
p=25;  %Predicting horizon
ywt=1  %Weights on y-yr
uwt=.1; %Weights on moves
tend=100; i=ceil(100/delt2);
yr=zeros(i,1); %Reference vector
%A step in the reference
for j=i/2:i yr(j)=1; end
[yp,u,ym]=cmpc(plant,model,ywt,uwt,m,p,tend,yr);
```

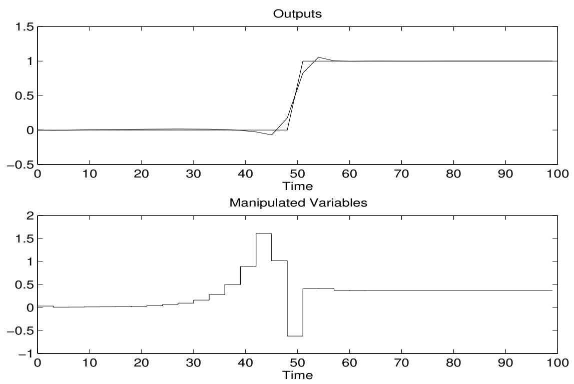

Figure 3: Simulation of vaporizer with a predicted setpoint change at t=50 known at t=0.