SOLUTION

EXAM IN COURSE 43917
MULTIVARIABLE CONTROL USING FREQUENCY DOMAIN METHODS

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Problem 1. Controllability analysis.

(a) Given

$$G(s) = \begin{bmatrix} \frac{1}{s+1} & \frac{s+3}{(s+1)(s-2)} \\ \frac{10}{s-2} & \frac{5}{s+3} \end{bmatrix}$$

The determinant is

$$\det G(s) = 5\frac{-s^2-16s-14}{(s+1)(s+3)(s-2)^2} = -5\frac{(s+0.93)(s+15.1)}{(s+1)(s+3)(s-2)^2}$$

Thus, we have 4 poles (of which 2 are unstable)

$$s_{1,2} = 2, s_3 = -1, s_4 = -3$$

and we have 2 LHP-zeros

$$s_1 = -0.93, s_2 = -15.1$$

In this case we see directly that the two unstable poles ($p_{1,2} = 2$) are associated with the offdiagonal elements; thus the output directions are

$$y_{p1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad y_{p2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

The main problem with this plant is that we need a bandwidth greater than about $2p = 4$ (rad/s) for stabilization. From the information given this should be achievable. Due to the unstable modes we would probably want to pair on the off-diagonal elements if we use decentralized control (but this needs to be checked).

(b)

$$G(s) = \left[\begin{array}{cc|cc} 1 & 2 & 2 & 2 \\ 1 & 0 & 3 & 4 \\ \hline 0 & 1 & 1 & 2 \\ 1 & 0 & 2 & 0 \end{array}\right] = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

First, evaluate the poles from

$$\det(sI - A) = s^2 - s - 2 = (s+1)(s-2) = 0$$

which has the roots $s_1 = -1$ and $s_2 = 2$. The direction for the unstable pole ($p = s_2 = 2$) is most easily evaluated from $y_p = Ct$ where t is the corresponding eigenvector of A. We have $At = 2t$ or $t_1 + 2t_2 = 2t_1, t_1 = 2t_2$ and we have (selecting $t_2 = 1$) that $t = [2 \quad 1]^T$. Thus, $y_p = Ct = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ or normalized,

$$y_p = \begin{bmatrix} 0.44 \\ 0.89 \end{bmatrix}$$

Thus the unstable pole is mainly associated with output 2.
The transfer function is $G(s) = C(sI - A)^{-1}B + D$ where

$$sI - A = \begin{bmatrix} s-1 & -2 \\ -1 & s \end{bmatrix}, \quad (sI-A)^{-1} = \frac{\text{adj}(sI-A)}{\det(sI-A)} = \frac{1}{\underbrace{s^2 - s - 2}_{(s+1)(s-2)}} \begin{bmatrix} s & 2 \\ 1 & s-1 \end{bmatrix}$$

We get

$$\begin{aligned} G(s) &= \underbrace{\frac{1}{s^2-s-2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} s & 2 \\ 1 & s-1 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ 3 & 4 \end{bmatrix}}_{\frac{1}{s^2-s-2}\begin{bmatrix} 3s-1 & 4s-2 \\ 2s+6 & 2s+8 \end{bmatrix}} + \underbrace{\begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix}}_{\frac{1}{s^2-s-2}\begin{bmatrix} s^2-s-2 & 2s^2-2s-4 \\ 2s^2-2s-4 & 0 \end{bmatrix}} & (1) \\ &= \frac{1}{s^2-s-2} \begin{bmatrix} s^2+2s-3 & 2s^2+2s-6 \\ 2s^2+2 & 2s+8 \end{bmatrix} & (2) \end{aligned}$$

and we have

$$\det G(s) = \frac{-4s^4 - 2s^3 + 20s^2 + 6s - 12}{(s+1)^2(s-2)^2} \tag{3}$$

Since the system only has two poles, the numerator must have the factor $(s+1)(s-2)$ and long division yields

$$\det G(s) = -\frac{4s^2 + 6s - 6}{(s+1)(s-2)} = -4\frac{(s-0.686)(s+2.186)}{(s+1)(s-2)}$$

and the zeros are $s_1 = 0.686$ and $s_2 = -2.186$, so there is a RHP-zero $z = 0.686$ quite close to the RHP-pole $p = 2$. To evaluate the zero direction evaluate

$$G(z) = G(0.686) = \frac{1}{-2.215} \begin{bmatrix} -1.157 & -3.686 \\ 2.941 & 9.372 \end{bmatrix}$$

which is singular as expected. The zero direction $[\, y_1 \quad y_2 \,]^T$ is given by $-1.157y_1 + 2.941y_2 = 0$, and we get $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1.157 \\ 2.941 \end{bmatrix} y_0$, and we select y_0 to normalize the length to one and get,

$$y_z = \begin{bmatrix} 0.92 \\ 0.37 \end{bmatrix}$$

This is mainly in the direction of output 1, which is good since the RHP-pole is mainly in output 2. To evaluate this more exactly, compute the angle between the RHP-pole and RHP-zero,

$$\phi = \cos^{-1} |y_z^H y_p| = \cos^{-1} 0.73 = 42.7^o$$

and we have for any controller design that $\|S\|_\infty \geq c$ and $\|T\|_\infty \geq c$ where from Eq. (6.27)

$$c = \sqrt{\sin^2 \phi + \frac{|z+p|^2}{|z-p|^2} \cos^2 \phi} = 1.68$$

(which does not seem too bad.) In conclusion, stabilization of this plant is fairly difficult, especially since the RHP-zero is closer to the origin than the RHP-pole, so the controller must most likely be unstable (see remark 5 on p. 226).

Remark 1. The algebra involved for hand calculation for this example is a bit extensive. Also, if you evaluate the zeros by evaluating the determinant of the transfer function you must apply some insight: To find the zeros you get a fourth order polynomial, see (3), but note that two of the roots are known – they are the poles – and can be eliminated, for example, by long division. It is possible to evaluate the zeros also from the generalized eigenvalue problem, see eq. (4.61) in the book, but I do not how this is done analytically.

Remark 2. Numerical calculations, using MATLAB, are of course preferred:

```
a=[1 2;1 0]; b=[2 2; 3 4]; c=[0 1; 1 0]; d=[1 2;2 0];
g = pck(a,b,c,d);
spoles(g)
szeros(g)
gp=frsp(g,-1.99i); [u,s,v]=vsvd(gp); yp = sel(u,':',1)
gz=frsp(g,-0.68i); [u,s,v]=vsvd(gz); yz = sel(u,':',2)
% Generally, it is better to use state space form:
% Compute eigenvalues of A and eigenvector, and similar for zeros
% using the generalized eigenvalue problem
```

Additional exercise. *Design a controller for this plant using MATLAB, using whatever design method you like, to see if acceptable performance can be achieved.*

(c)

$$G(s) = \frac{10}{(s+1)(s+4)} \begin{bmatrix} s+1 & 3 \\ 3 & s+1 \end{bmatrix}$$

Get

$$\det G(s) = \frac{100\overbrace{\left((s+1)^2-9\right)}^{(s+4)(s-2)}}{(s+1)^2(s+4)^2} = 100\frac{(s-2)}{(s+1)^2(s+4)}$$

So the system has 3 stable poles (at $s_{1,2} = -1$ and $s_3 = -4$), and 1 RHP-zero at $s = z = 2$. From $G(z) = \frac{10}{18}\begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix}$ the zero direction is $3y_1 + 3y_2 = 0$, which gives the normalized direction

$$y_z = \begin{bmatrix} 0.71 \\ -0.71 \end{bmatrix}$$

so the zero has equal effect in the two outputs. The RHP-zero implies that we cannot have tight control in this direction around frequency 2 [rad/s].

Are there any particular control problems for the three plants?

- We have already answered this when it comes to the effect of RHP-poles and RHP-zeros.
- Input constraints may also be a problem. I would suggest to look at $G(0)^{-1}$, and if there is any element larger than 1 then it may signal problems with references (since perfect

control requires $u = G^{-1}r$). For the three above cases, we find that in none of the cases does $G(0)^{-1}$ have elements larger than 1 in magntidue – the closest is in case 2 which has one element equal to -1.

- Another problem may be uncertainty, but I don't see any large RGA-elements or large condition number (look at the product of the largest element in G and G^{-1} to get a rough idea of the condition mumber) which may signal sensitivity to uncertainty.

(d) For the plant in (c) there are three possible disturbances,

$$g_{d1} = \frac{1}{10s+1}\begin{bmatrix}10\\10\end{bmatrix},\quad g_{d2} = \frac{1}{s+1}\begin{bmatrix}10\\10\end{bmatrix},\quad g_{d3} = \frac{1}{s+1}\begin{bmatrix}10\\-10\end{bmatrix}$$

Is acceptable disturbance rejection possible for each of these? We will look at two issues: (i) Input constraints and (ii) the possible conflict between the desire for disturbance rejection and RHP-zeros.

(i) For perfect disturbance rejection we need $u = -G^{-1}g_d d$, so if the variables are scaled appropriatly, we need the elements in the vector $G^{-1}g_d$ to be less than 1 in magnitude to avoid input constraints [1]. Let us look at steady-state, since things improve at higher frequencies (since the gain of g_{di} drops more rapidly than that of G at high frequencies). We have

$$G(0) = \frac{10}{4}\begin{bmatrix}1 & 3\\3 & 1\end{bmatrix},\quad G(0)^{-1} = \frac{1}{10}\begin{bmatrix}-0.5 & 1.5\\1.5 & -0.5\end{bmatrix}$$

and $G^{-1}(0)g_d(0)$ for the three disturbances is $\begin{bmatrix}1\\1\end{bmatrix}$, $\begin{bmatrix}1\\1\end{bmatrix}$, and $\begin{bmatrix}2\\2\end{bmatrix}$. In conclusion, input constraints may be a problem for disturbance 3.

(ii) (This is also the solution to Problem 3(e)). With feedback control $y = Sg_d d$, and we assume the variables have been scaled such that the worst-case disturbance is $|d| = 1$ and we want for acceptable control $\|y\| < 1$ at all frequencies, that is, we want $\|Sg_d\|$ to be less than 1 at all frequencies. But we have

$$\|Sg_d\|_\infty \geq \|y_z^H Sg_d\|_\infty \geq |y_z^H S(z)g_d(z)| = |y_z^H g_d(z)| \tag{4}$$

Proof of (4): The first inequality is just because the gain in any direction must be larger than the gain in a specific direction, the second inequality is the maximum modulus principle which says that the maximum in the RHP for a stable system is achieved on the $j\omega$-axis, and the final equality is because when G has a RHP-zero at z, then for internal stability $S(z)$ must have gain 1 in the direction of the RHP-zero, $y_z^H S(z) = y_z^H$, see (6.4).

From (4) it then follows that we must *at least* require $|y_z^H g_d(z)| \leq 1$ for acceptable disturbance rejection in the presence of a RHP-zero. As found earlier $z = 2$ and $y_z = \begin{bmatrix}0.71\\-0.71\end{bmatrix}$, and we find for the three disturbances

$$|y_z^H g_{d1}(z)| = 0,\ |y_z^H g_{d1}(z)| = 0,\ |y_z^H g_{d1}(z)| = 4.7$$

[1] If this is satisfied then we are OK; if not then we may start looking into the more fine details about acceptable control ($|y| < 1$ rather than $y = 0$), e.g. see eq. (6.47), which may give different results for ill-conditioned plants if disturbance rejection is actually not required in some directions).

In conclusion, the presence of the RHP-zero at $z = 2$ *is not* a problem for disturbances 1 and 2 (the output direction for disturbance 3 is orthogonal to the direction of the RHP-zero), but it *is* a problem for disturbance 3 (4.7 being much larger than 1).

In conlusion, taking both the RHP-zero and input constraints into account, we may expect serious problems for disturbance 3.

Exercise. *Again, try this out using MATLAB. Design a controller, using whatever design method you like, to see if acceptable performance can be achieved for each of the three disturbances (you may want to design a diferent controller for each disturbance).*

Problem 2. General control formulation.

Consider a stable uncertain plant

$$G_p = G(I + W_1\Delta_1) + W_2\Delta_2, \quad \|\Delta_i\|_\infty \le 1$$

for which the robust performance (RP) objective is to achieve

$$\|W_3 S_p\|_\infty = \|W_3(I + G_p K)^{-1}\|_\infty \le 1, \ \forall G_p$$

(a) The block diagram representation of the closed-loop system with all weights included is shown in Figure 1. The block Δ_P is a ficticious uncertainty block for the performance specification.

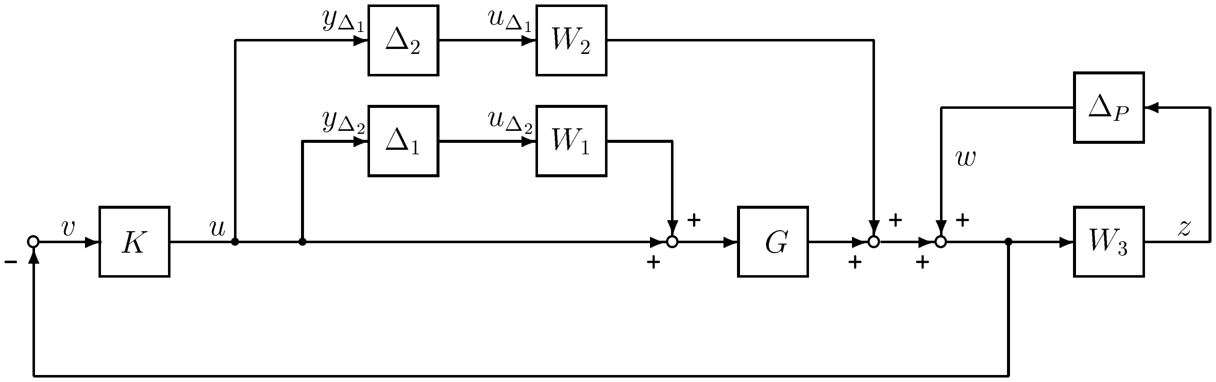

Figure 1: Block diagram of uncertain plant in Problem 2

(b) Let the overall block-diagonal uncertainty matrix be $\Delta = \text{diag}\{\Delta_1, \Delta_2\}$. The generalized plant P with uncertainty (from $[\,u_{\Delta_1} \quad u_{\Delta_2} \quad w \quad u\,]^T$ to $[\,y_{\Delta_1} \quad y_{\Delta_2} \quad z \quad v\,]^T$; see Figure 3.21 in the book) is then:

$$P = \begin{bmatrix} 0 & 0 & 0 & I \\ 0 & 0 & 0 & I \\ W_3GW_1 & W_3W_2 & W_3 & W_3G \\ -GW_1 & -W_2 & -I & -G \end{bmatrix}$$

The corresponding interconnection matrix N with the controller included ((from $[\,u_{\Delta_1} \quad u_{\Delta_2} \quad w\,]^T$ to $[\,y_{\Delta_1} \quad y_{\Delta_2} \quad z\,]^T$; see Fig. 3.22) is:

$$N = \begin{bmatrix} -KG(I+KG)^{-1}W_1 & -K(I+GK)^{-1}W_2 & -K(I+GK)^{-1} \\ -KG(I+KG)^{-1}W_1 & -K(I+GK)^{-1}W_2 & -K(I+GK)^{-1} \\ W_3G(I+KG)^{-1}W_1 & W_3(I+GK)^{-1}W_2 & W_3(I+GK)^{-1} \end{bmatrix}$$

$$= \begin{bmatrix} -T_I W_1 & -KSW_2 & -KS \\ -T_I W_1 & -KSW_2 & -KS \\ W_3 GSW_1 & W_3 SW_2 & W_3 S \end{bmatrix}$$

The matrix M for stability analysis consists of the upper left 2×2 block of N.

(c) The RS- and RP-conditions in terms of N and M are:

$$RS: \quad \mu_\Delta(M) < 1, \forall\omega$$

$$RP: \quad \mu_{\begin{bmatrix} \Delta & 0 \\ 0 & \Delta_P \end{bmatrix}}(N) < 1, \forall\omega$$

where Δ_P is a full matrix compatible with the dimensions of w and z as shown in Figure 1. In addition, we should as always check nominal stability (NS).

(d) Finally, we want to derive analytical expressions for RS and RP for the case of a SISO plant. We get

$$RS: \quad \mu(M) = |W_1 T| + |W_2 KS| < 1, \ \forall\omega$$

$$RP: \quad \mu(N) = |W_1 T| + |W_2 KS| + |W_3 S| < 1, \ \forall\omega$$

where $S = (I + GK)^{-1}$ and $T = GK(I + GK)^{-1}$.

These conditions can be derived in several ways, and we here outline two approaches. We only derive the expression for RP, since RS is a special case obtained by setting $W_3 = 0$.

(i) We here start from the condition in terms of $\mu(N)$. Since the blocks are scalar we have that

$$\mu(N) = \mu \begin{bmatrix} W_1 T & W_1 T & W_1 T \\ W_2 KS & W_2 KS & W_2 KS \\ W_3 S & W_3 S & W_3 S \end{bmatrix}$$

Proof:

1. The order of G and K etc. does not matter for SISO.
2. We can "move" G from column 1 to row 1; e.g. use (8.84).
3. We can "move" W_1 from column 1 to row 1; e.g. use (8.84).
4. We can "move" W_2 from column 2 to row 2; e.g. use (8.84).
5. The negative sign in row 1 and 2 can be absorbed into the Δ-block, e.g. use (8.83)

The final result then follows from the fact

$$\mu \begin{bmatrix} a & a & a \\ b & b & b \\ c & c & c \end{bmatrix} = |a| + |b| + |c|$$

Proof: This follows since N here is rank 1 matrix and we can write the 3×3 matrix $N\Delta = [a \quad b \quad c]^T [\Delta_1 \quad \Delta_2 \quad \Delta_3]$. We then get that $\det(I - N\Delta) = \det(I - \Delta N) = 1 - a\Delta_1 - b\Delta_2 - c\Delta_3$. To make this determinant most easily equal to zero we select the Δ_i of magnitude δ and with phase such that $1 - |a|\delta - |b|\delta - |c|\delta = 0$, and it follows that $\mu = 1/\delta = |a| + |b| + |c|$. (This provides a generalization of the derivation at the end of p. 317 in the book, and we see that it may be easily genarlized to matrices of larger size).

(ii) The RP-condition is $|W_3 S_p| = |W_3/(1+G_pK)| < 1, \forall\omega, \forall G_p$. We get

$$|\frac{W_3}{1+GK+GKW_1\Delta_1+KW_2\Delta_2}| = |\frac{W_3S}{1+W_1T\Delta_1+W_2KS\Delta_2}| < 1$$

This should be satified for any $|\Delta_1| \leq 1$ and $|\Delta_2| \leq 1$, and we see that the worst case is when we select $|\Delta_i| = 1$ and the phases of the Δ_i's such that the condition becomes

$$\frac{|W_3S|}{1-|W_1T|-|W_2KS|} < 1 \quad \Leftrightarrow \quad |W_1T| + |W_2KS| + |W_3S| < 1$$

$\square$

Problem 3. Various.

(a) Rewrite $G = (I-H\Delta)^{-1} = I + H\Delta(I-H\Delta)^{-1} = F_u(J,\Delta) = J_{22} + J_{21}\Delta(I-J_{11}\Delta)^{-1}J_{12}$, so

$$J = \begin{bmatrix} H & I \\ H & I \end{bmatrix}$$

(b) Given the "true" plant and nominal model

$$G'(s) = \frac{3e^{-0.1s}}{(2s+1)(0.1s+1)^2}, \quad G(s) = \frac{3}{2s+1}$$

The magnitude of the multiplicative difference is

$$\frac{|G'-G|}{|G|} = |G'/G| - 1 = |\frac{e^{-0.1s}}{(0.1s+1)^2} - 1|$$

This is easily evaluated numerically using MATLAB; see Figure 2:

```
% Use mu toolbox
w = logspace(-1,2,61);
delw = delay(0.1,w);        % delay command: See MATLAB source files
num=nd2sys(1, [0.01 .2 1]);
numw = frsp(num,w);
reldiff = msub( mmult(delw,numw), 1);
vplot('liv,lm',reldiff,1,':');}
```

For an analytic evaluation use the approximation $e^{-0.1s} \approx 1 - 0.1s$ (which is exact at low frequencies; we could have used a Pade approximation instead – you can try that). We get

$$\frac{|G'-G|}{|G|} = \frac{|s(0.01s+0.3)|}{|(0.1s+1)^2|}$$

The low-frequency asymptote is 0.3s which increases with a slope 1 and crosses 1 at frequency 3.33 [rad/s] (which is close to the actual value; see Figure 2). The asymptotic plot has a peak

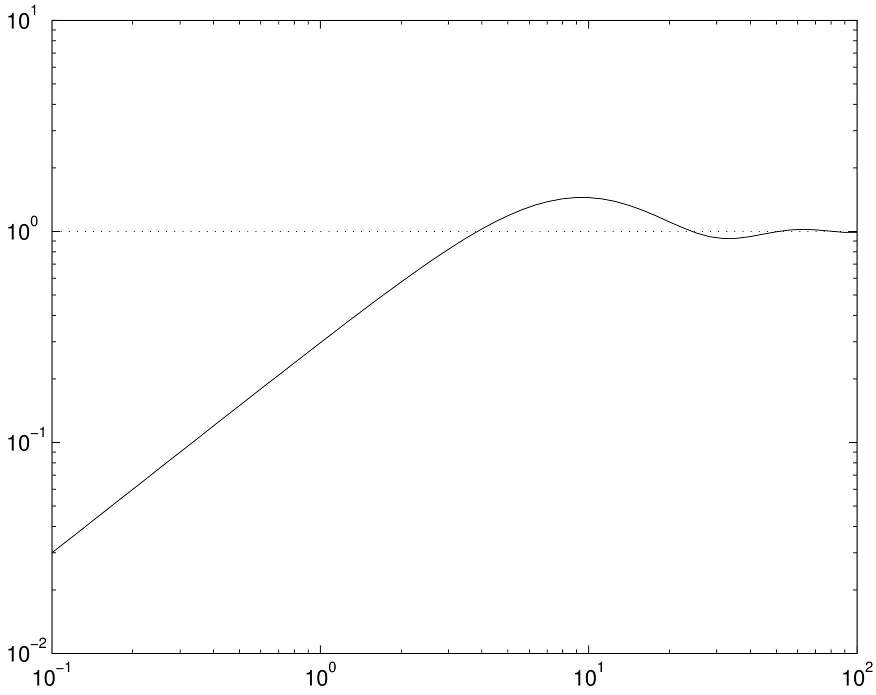

Figure 2: Magnitude of multiplicative difference computed using MATLAB

of 3 at frequency 10 (rad/s) (the actual peak is less than 2; see Figure) and approaches 1 at high frequencies.

Remark. Note that if we consider all possible delays between 0 and 0.1 and also all time constants up to 0.1, then the weight would level off at 2 at high frequencies; see Figure 7.8 on p. 262 in the book. But in this case we are not told that the delay and time constants are uncertain – we simply want to reprent the known dynamics as uncertainty – it is then correct to let the weight levell off at 1 at high frequency. In any case, this discussion is largely academic, as the most important result is the frequency where the unecertainty reaches 100%, which is at about 3.3 [rad/s].

(c) This problem shows that element-by-element uncertainty can also be written easily in the standard form, for example, in terms of additive uncertainty.

$$G_p(s) = \begin{bmatrix} (1+\Delta_{11})g_{11} & (1+\Delta_{12})g_{12} \\ (1+\Delta_{21})g_{21} & (1+\Delta_{22})g_{22} \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix}}_{G} + \underbrace{\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}}_{W_2} \underbrace{\begin{bmatrix} \Delta_{11} & 0 & 0 & \\ 0 & \Delta_{12} & 0 & 0 \\ 0 & 0 & \Delta_{21} & 0 \\ 0 & 0 & 0 & \Delta_{22} \end{bmatrix}}_{\Delta} \underbrace{\begin{bmatrix} g_{11} & 0 \\ 0 & g_{12} \\ g_{21} & 0 \\ 0 & g_{22} \end{bmatrix}}_{W_1}$$

(we could also put the g_{ij}'s into W_2 instead). To find the interconnection matrix M for RS, consider a negative feedback block diagram and evaluate the (closed-loop) transfer function from the output of Δ to its input. We find $M = -W_1K(I+GK)^{-1}W_2$ (see also p. 307).

(d) All of these are NOT true:

i. $\mu(A) = 0 \Rightarrow A = 0$
ii. $\mu(A+B) \leq \mu(A) + \mu(B)$
iii. $\rho(AB) \leq \rho(A)\bar{\sigma}(B)$

- To disprove (i), consider $\mu = \rho$ (i.e. select $\Delta = \delta I$ since the property must hold for any structure) and select $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ which has $\rho(A) = 0$. Thus, ρ and μ do *not* satisfy the positivity requirement of a norm.

- To disprove (ii), consider again $\mu = \rho$ and $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and select $B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$. Then $\rho(A+B) = 1$ which is larger than $\rho(A) + \rho(B) = 0 + 0 = 0$. Thus, ρ and μ do *not* satisfy the triangle inequality, and are therefore not norms.

- To disprove (iii), select $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, which gives $AB = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. We have $\rho(AB) = 1$ then which is larger than $\rho(A)\bar{\sigma}(B) = 0 \cdot 1 = 0$.

Remark. *The disproof of (iii) illustrates that (8.94) in the book is not generally true (since as shown it does not apply for $\Delta = \delta I$), i.e.,*

$$\text{NOT generally true}: \quad \mu_\Delta(AB) \leq \bar{\sigma}(A)\mu_\Delta(B)$$

unless it is assumed that Δ has the same structure as A. Thus, as pointed out by Skogestad and Morari (1988a), there is an error in the original paper of Doyle (1982) – please note this, because the error keeps reappearing in other papers and books.

(e) To explain and derive $|y_z^H g_d(z)| < 1$, see eq.(4) above.

(f) The task is to define and briefly explain the difference between the H_2-, H_∞- and Hankel norms of a transfer function $G(s)$.

- See the book for the definitions.

- One difference with reference to (4.119), (4.122) and (4.134) is: Consider an input and measure the output using the 2-norm ("energy"). The H_2-norm results when the input is the worst impulse combination. The H_∞-norm is for the worst input with energy less than 1. Finally, the Hankel-norm is the same as for the H_∞-norm, but we must turn off the input when we start measuring the output. Clearly, the Hankel norm is always smaller than the H_∞-norm. However, there is no general relationship between the H_2- and H_∞-norm. For example, note for the H_2-norm that the energy of an impulse is infinite; on the other hand, we have much more freedom in selecting the time dependency of the input when we evaluate the H_∞-norm.

- Another difference, with reference to (4.121) and (4.124), is that the H_2-norm meaures some average of the frequency reponse (all frequencies, all directions), whereas the H_∞-norm considers the worst frequency and worst direction.

(g) We consider a system where the disturbance enters at the plant input, $G_d = G$. The question is what form the controller should have if we want acceptable performance with minimum input usage.
This is discussed in the book on page 48 (SISO) and page 82 (MIMO). It is assumed that the system is scaled such that we have disturbances of unit magnitude and require the output to be less than 1. The answer is then that the controller should have the form $K = G^{-1}G_dU = U$ corresponding to the loop transfer function $L = G_dU = GU$. Here U is a unitary matrix (rotation and phase only, unit gain in all direction). In addition, we usually add some integral action at low frequencies, and some roll-off at high frequencies.

(h) What are the advantages and disadvantages of inverse-based controllers (decouplers)?

- Advantage decoupling: When we make a setpoint change for a given output, there is no undesirable reponse in the other outputs (no interaction).
- Disadvantages decoupling: (1) Generally not optimal for disturbance rejection (while trying to make the response decoupled we may have to sacrifice disturbance rejection). (2) Sensitive to modelling errors if the plant has large RGA-elements, and also for some other plants. (3) If the plant has RHP-zeros, then decoupling generally introduces extra RHP-zeros into the closed-loop system (increases their multiplicity because the inverse response must occur in all outputs rather than in a single direction). (4) Decoupling may give unnecessary large input signals.