Regelsysteme II

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Based on the book:

MULTIVARIABLE FEEDBACK CONTROL
Analysis and design

Sigurd Skogestad

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University of Leicester

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Contents

1 Introduction

Always keep in mind

- Power of control is limited.
- Control quality depends controller **AND** on plant/process.

Ziegler-Nichols (1943):

``In the application of automatic controllers, it is important to realize that controller and process form a unit; credit or discredit for results obtained are attributable to one as much as the other. A poor controller is often able to perform acceptably on a process which is easily controlled. The finest controller made, when applied to a miserably designed process, may not deliver the desired performance. True, on badly designed processes, advanced controllers are able to eke out better results than older models, but on these processes, there is a definite end point which can be approached by instrumentation and it falls short of perfection.''

⇒ Much of the course will be spent on input-output "controllability analysis" of the plant/process.

1.1 The control problem [1.2]

$$y = Gu + G_d d \tag{1.1}$$

y : output/controlled variable

u : input/manipulated variable

d : disturbance

r : reference/setpoint

Regulator problem : counteract d

Servo problem : let y follow r

Goal of control: make control error $e = y - r$

"small".

Major difficulties:

Model (G, G_d) inaccurate

$\Rightarrow$ RealPlant: $G_p = G + E$;

$E =$ "uncertainty" or "perturbation" (unknown)

- **Nominal stability (NS)** : system is stable with no model uncertainty.
- **Nominal Performance (NP)** : system satisfies performance specifications with no model uncertainty.
- **Robust stability (RS)** : system stable for "all" perturbed plants
- **Robust performance (RP)** : system satisfies performance specifications for all perturbed plants

1.2 Transfer functions [1.3]

$$G(s) = \frac{\beta_{n_z} s^{n_z} + \cdots + \beta_1 s + \beta_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0} \tag{1.2}$$

For multivariable systems, $G(s)$ is a matrix of transfer functions.

n = order of denominator (or pole polynomial) or *order of the system*

n_z = order of numerator (or zero polynomial)

$n - n_z$ = pole excess or *relative order*.

Definition

- A system $G(s)$ is *strictly proper* if $G(s) \to 0$ as $s \to \infty$.
- A system $G(s)$ is *semi-proper* or *bi-proper* if $G(s) \to D \neq 0$ as $s \to \infty$.
- A system $G(s)$ which is strictly proper or semi-proper is *proper*.
- A system $G(s)$ is *improper* if $G(s) \to \infty$ as $s \to \infty$.

1.3 Scaling [1.4]

Proper scaling simplifies controller design and performance analysis.

SISO :

unscaled:

$$\widehat{y} = \widehat{G}\widehat{u} + \widehat{G}_d\widehat{d}; \quad \widehat{e} = \widehat{y} - \widehat{r} \tag{1.3}$$

scaled:

$$d = \widehat{d}/\widehat{d}_{\max}, \quad u = \widehat{u}/\widehat{u}_{\max} \tag{1.4}$$

where:

- $\widehat{d}_{\max}$ — largest expected change in disturbance
- $\widehat{u}_{\max}$ — largest allowed input change

Scale $\widehat{y}$, $\widehat{e}$ and $\widehat{r}$ by:

- $\widehat{e}_{\max}$ — largest allowed control error, or
- $\widehat{r}_{\max}$ — largest expected change in reference value

Usually:

$$y = \widehat{y}/\widehat{e}_{\max}, \quad r = \widehat{r}/\widehat{e}_{\max}, \quad e = \widehat{e}/\widehat{e}_{\max} \tag{1.5}$$

MIMO :

$$d = D_d^{-1}\widehat{d}, \quad u = D_u^{-1}\widehat{u}, \quad y = D_e^{-1}\widehat{y} \tag{1.6}$$

$$e = D_e^{-1}\widehat{e}, \quad r = D_e^{-1}\widehat{r} \tag{1.7}$$

where D_e, D_u, D_d and D_r are diagonal scaling *matrices*

Substituting (1.6) and (1.7) into (1.3):

$$D_e y = \widehat{G} D_u u + \widehat{G}_d D_d d; \quad D_e e = D_e y - D_e r$$

and introducing the scaled transfer functions

$$G = D_e^{-1}\widehat{G} D_u, \quad G_d = D_e^{-1}\widehat{G}_d D_d \tag{1.8}$$

Model in terms of scaled variables:

$$y = Gu + G_d d; \quad e = y - r \tag{1.9}$$

Often also:

$$\widetilde{r} = \widehat{r}/\widehat{r}_{\max} = D_r^{-1}\widehat{r} \tag{1.10}$$

so that:

$$r = R\widetilde{r} \quad \text{where} \quad R \triangleq D_e^{-1} D_r = \widehat{r}_{\max}/\widehat{e}_{\max} \tag{1.11}$$

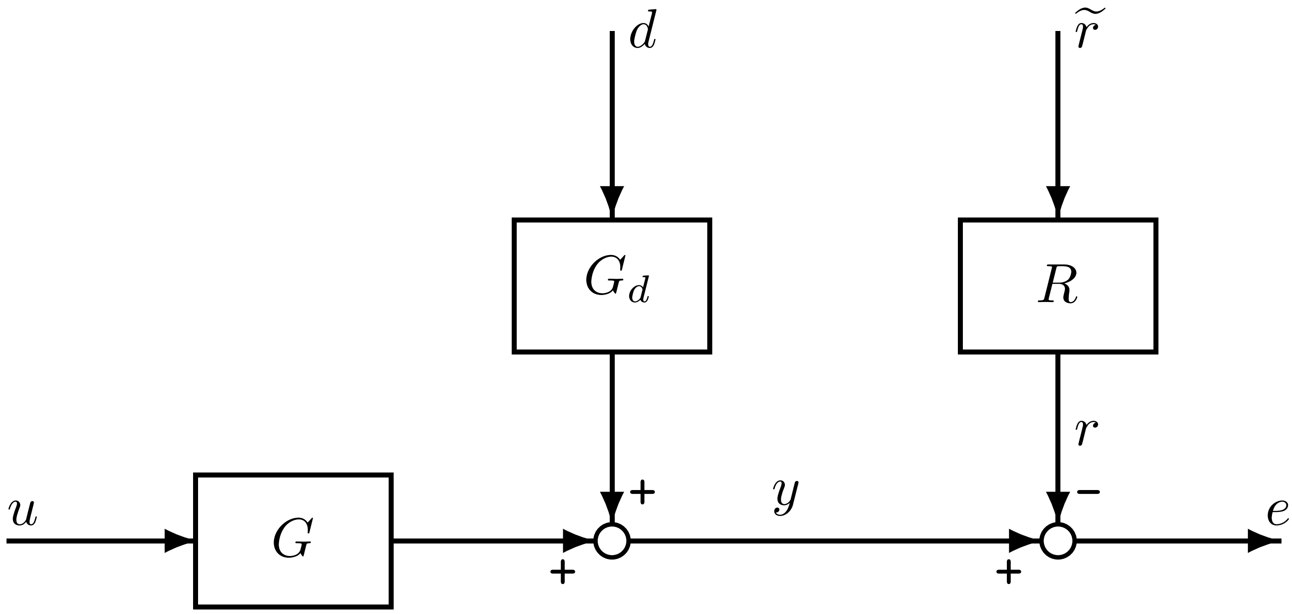


Figure 1: Model in terms of scaled variables

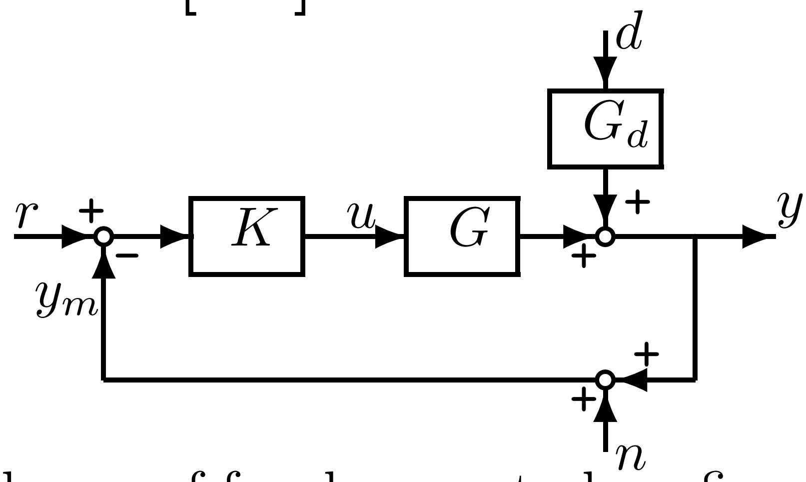
Objective:

for $|d(t)| \leq 1$ and $|\widetilde{r}(t)| \leq 1$,

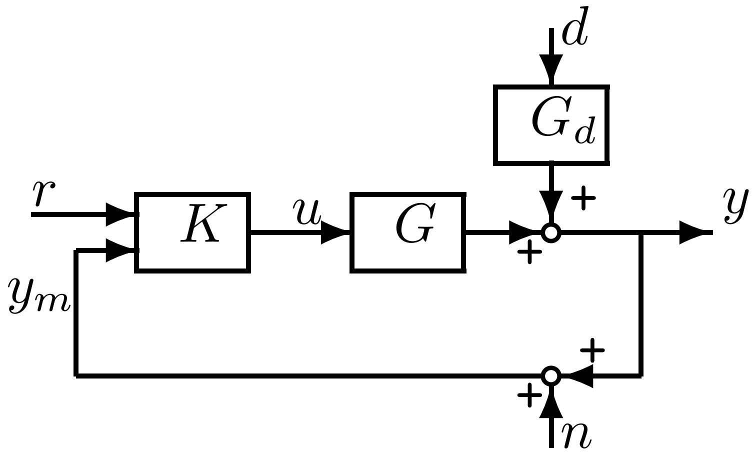
manipulate u with $|u(t)| \leq 1$

such that $|e(t)| = |y(t) - r(t)| \leq 1$.

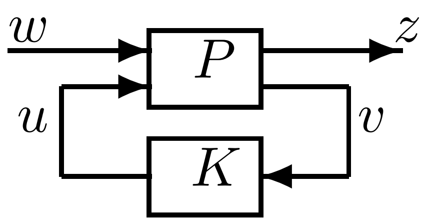
1.4 Notation [1.6]



(a) One degree-of-freedom control configuration



(b) Two degrees-of-freedom control configuration



(c) General control configuration

Figure 2: Control configurations

Table 1: Nomenclature

K controller, in whatever configuration. Sometimes broken down into parts. For example, in Figure 2(b), $K = [\, K_r \quad K_y \,]$ where K_r is a prefilter and K_y is the feedback controller.

Conventional configurations (Fig 2(a), 2(b)):

G plant model

G_d disturbance model

r reference inputs (commands, setpoints)

d disturbances (process noise)

n measurement noise

y plant outputs. (include the variables to be controlled ("primary" outputs with reference values r) and possibly additional "secondary" measurements to improve control)

y_m measured y

u control signals (manipulated plant inputs)

General configuration (Fig 2(c)):

P	generalized plant model. Includes G and G_d and the interconnection structure between the plant and the controller. May also include weighting functions.
w	exogenous inputs: commands, disturbances and noise
z	exogenous outputs; "error" signals to be minimized, e.g. $y - r$
v	controller inputs for the general configuration, e.g. commands, measured plant outputs, measured disturbances, etc. For the special case of a one degree-of-freedom controller with perfect measurements we have $v = r - y$.
u	control signals

2 Classical feedback control [2]

2.1 Feedback control [2.2]

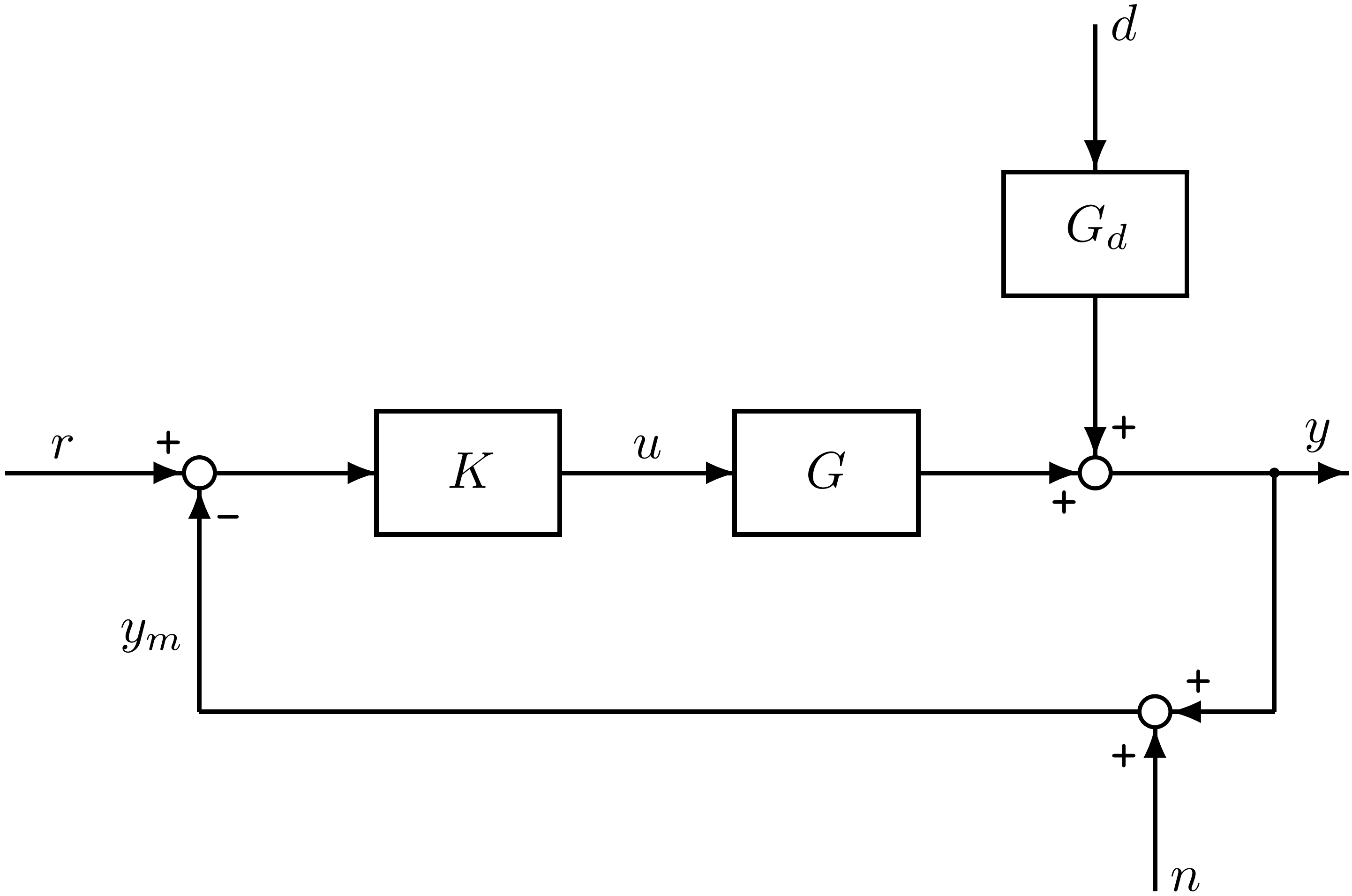


Figure 3: Block diagram of one degree-of-freedom feedback control system

$$y = GK(r - y - n) + G_d d$$

or

$$(I + GK)y = GKr + G_d d - GKn \tag{2.1}$$

Closed-loop response:

$$y = \underbrace{(I + GK)^{-1}GK}_{T}\, r \tag{2.2}$$

$$+ \underbrace{(I + GK)^{-1}}_{S}\, G_d d \tag{2.3}$$

$$- \underbrace{(I + GK)^{-1}GK}_{T}\, n \tag{2.4}$$

Control error:

$$e = y - r = -Sr + SG_d d - Tn \tag{2.5}$$

Plant input:

$$u = KSr - KSG_d d - KSn \tag{2.6}$$

Note that:

$$L = GK \tag{2.7}$$

$$S = (I + GK)^{-1} = (I + L)^{-1} \tag{2.8}$$

$$T = (I + GK)^{-1}GK = (I + L)^{-1}L \tag{2.9}$$

$$S + T = I \tag{2.10}$$

Notation :

$L = GK$	loop transfer function
$S = (I + L)^{-1}$	sensitivity function
$T = (I + L)^{-1}L$	complementary sensitivity function

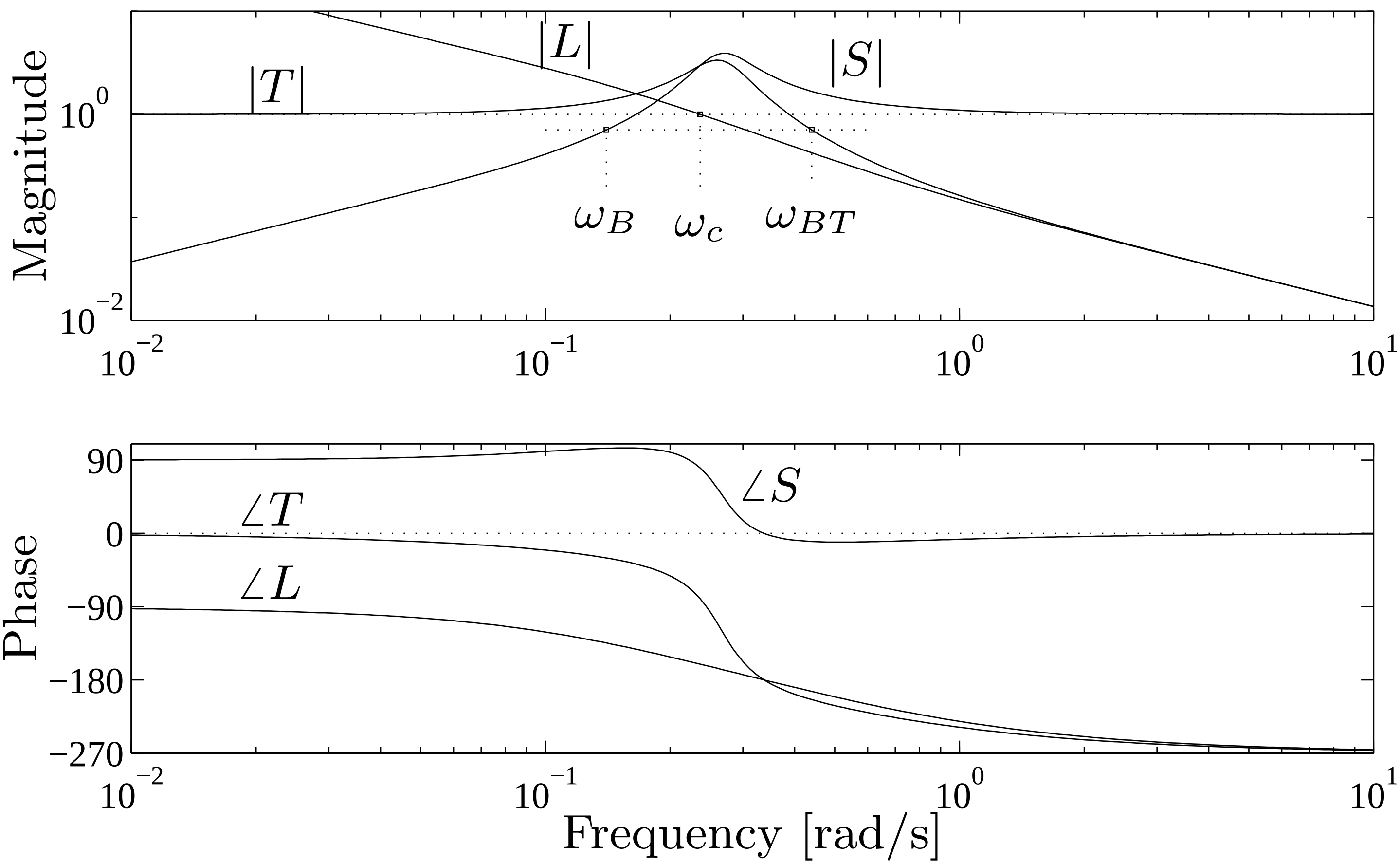


Figure 4: Bode magnitude and phase plots of $L = GK$, S and T when $G(s) = \frac{3(-2s+1)}{(5s+1)(10s+1)}$, and $K(s) = 1.136(1 + \frac{1}{12.7s})$

2.2 Closed-loop performance [2.4]

Frequency domain performance

Gain and phase margins

⇒ See RS I

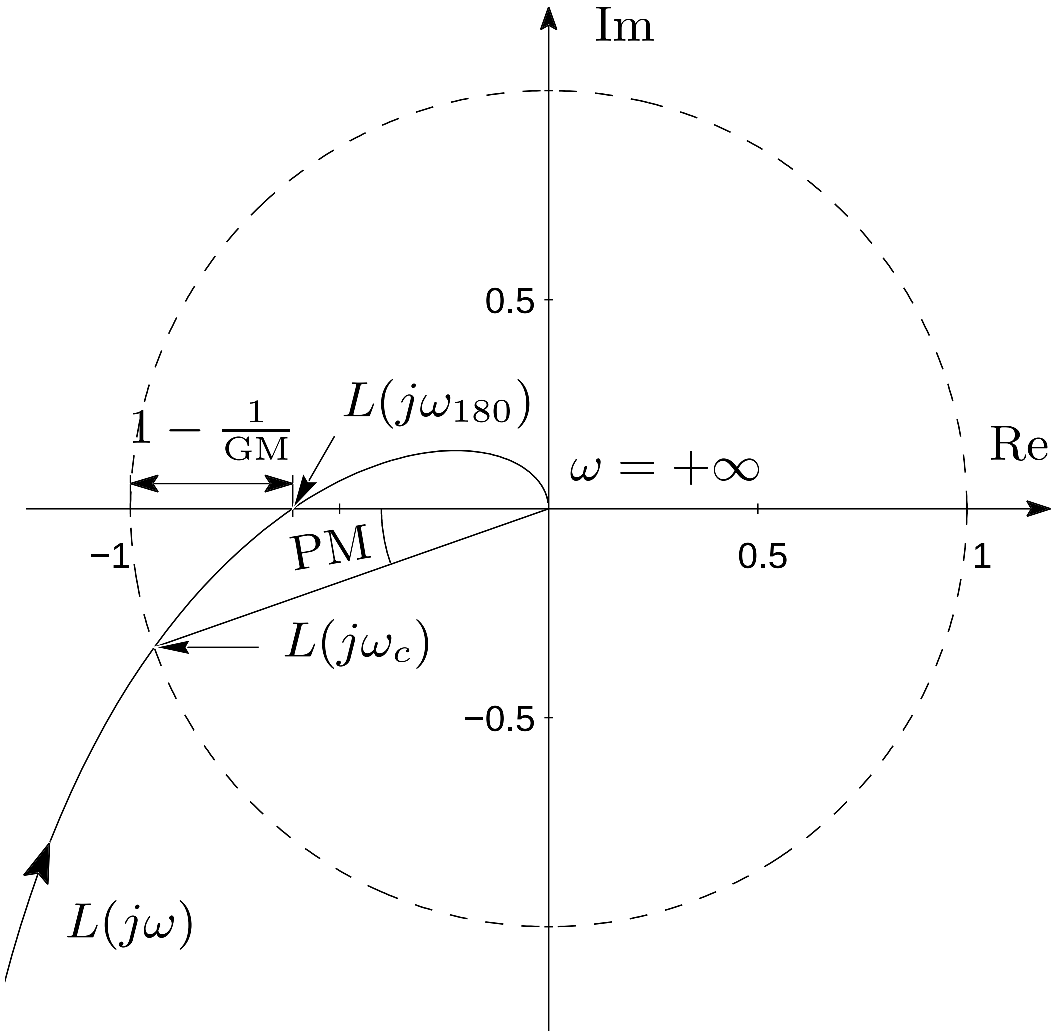


Figure 5: Typical Nyquist plot of $L(j\omega)$ for stable plant with PM and GM indicated. Closed-loop instability occurs if $L(j\omega)$ encircles the critical point -1

Maximum peak criteria

Maximum peaks of sensitivity and complementary sensitivity functions:

$$M_S \stackrel{\triangle}{=} \max_{\omega} |S(j\omega)|; \quad M_T \stackrel{\triangle}{=} \max_{\omega} |T(j\omega)| \tag{2.11}$$

Typically :

$$M_S \leq 2 \quad (6dB) \tag{2.12}$$

$$M_T < 1.25 \quad (2dB) \tag{2.13}$$

Note :

$$\text{GM} \geq \frac{M_S}{M_S - 1} \tag{2.14}$$

$$\text{PM} \geq 2 \arcsin\left(\frac{1}{2M_S}\right) \geq \frac{1}{M_S} \text{ [rad]} \tag{2.15}$$

For example, for $M_S = 2$ we are guaranteed

$$\text{GM} \geq 2 \text{ and PM} \geq 29.0°.$$

Bandwidth and crossover frequency

Bandwidth is defined as the frequency range $[\omega_1, \omega_2]$ over which control is "effective". Usually $\omega_1 = 0$, and then $\omega_2 = \omega_B$ is the bandwidth.

Definition The (closed-loop) bandwidth, ω_B, is the frequency where $|S(j\omega)|$ first crosses $1/\sqrt{2} = 0.707(\approx -3 \text{ dB})$ from below.

The bandwidth in terms of T, ω_{BT}, is the highest frequency at which $|T(j\omega)|$ crosses $1/\sqrt{2} = 0.707(\approx -3 \text{ dB})$ from above. (Usually a poor indicator of performance).

The *gain crossover frequency*, ω_c, is the frequency where $|L(j\omega_c)|$ first crosses 1 from above.For systems with PM $< 90^\circ$ we have

$$\omega_B < \omega_c < \omega_{BT} \tag{2.16}$$

2.3 Controller design [2.5]

Three main approaches:

1. **Shaping of transfer functions.**
   (a) **Loop shaping.** Classical approach in which the magnitude of the open-loop transfer function, $L(j\omega)$, is shaped.
   (b) **Shaping of closed-loop transfer functions, such as S, T and KS**
   $\Rightarrow \mathcal{H}_\infty$ optimal control
2. **The signal-based approach.** One considers a particular disturbance or reference change and tries to optimize the closed-loop response
   $\Rightarrow$ Linear Quadratic Gaussian (LQG) control.
3. **Numerical optimization.** Multi-objective optimization to optimize directly the true objectives, such as rise times, stability margins, etc. Computationally difficult.

2.4 Loop shaping [2.6]

Shaping of open loop transfer function $L(j\omega)$:

$$e = -\underbrace{(I+L)^{-1}}_{S} r + \underbrace{(I+L)^{-1}}_{S} G_d d - \underbrace{(I+L)^{-1} L}_{T} n \tag{2.17}$$

Fundamental trade-offs:

1. Good disturbance rejection: L large.
2. Good command following: L large.
3. Mitigation of measurement noise on plant outputs: L small.
4. Small magnitude of input signals: K small and L small.

Fundamentals of loop-shaping design

Specifications for desired loop transfer function:

1. Gain crossover frequency, ω_c, where $|L(j\omega_c)| = 1$.
2. The shape of $L(j\omega)$, e.g. slope of $|L(j\omega)|$ in certain frequency ranges:
   $N = \frac{d\,ln\,|L|}{d\,ln\,w}$
   Typically, a slope $N = -1$ (-20 dB/decade) around crossover, and a larger roll-off at higher frequencies. The desired slope at lower frequencies depends on the nature of the disturbance or reference signal.
3. The system type, defined as the number of pure integrators in $L(s)$.

Note:

1. *for offset for tracking $L(s)$ must contain at least one integrator for each integrator in $r(s)$.*

2. *slope and phase are dependent. For example:*

$$\angle < \frac{1}{s^n} = -n\,\frac{\pi}{2}$$

2.4.1 Inverse-based controller [2.6.3]

Note: $L(s)$ must contain all RHP-zeros of $G(s)$.
Idea for minimum phase plant:

$$L(s) = \frac{\omega_c}{s} \tag{2.18}$$

$$K(s) = \frac{\omega_c}{s} G^{-1}(s) \tag{2.19}$$

i.e. controller inverts plant and adds integrator ($1/s$).

BUT:

this is *not* generally desirable, unless references and disturbances affect the outputs as steps.

Example: Disturbance process.

$$\boxed{G(s) = \frac{200}{10s+1}\frac{1}{(0.05s+1)^2}, \quad G_d(s) = \frac{100}{10s+1}} \tag{2.20}$$

Objectives are:

1. Command tracking: rise time (to reach 90% of the final value) less than 0.3 s and overshoot less than 5%.

2. Disturbance rejection: response to unit step disturbance should stay within the range $[-1, 1]$ at all times, and should return to 0 as quickly as possible ($|y(t)|$ should at least be less than 0.1 after 3 s).

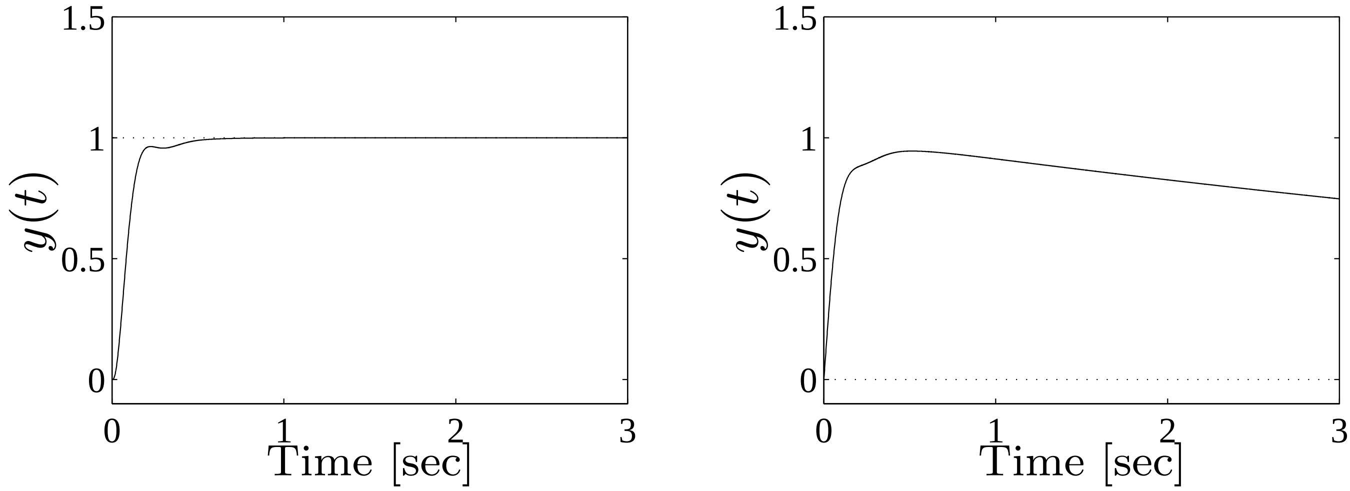
3. Input constraints: $u(t)$ should remain within $[-1, 1]$ at all times.

Analysis. $|G_d(j\omega)|$ remains larger than 1 up to $\omega_d \approx 10$ rad/s $\Rightarrow$ $\underline{\omega_c \approx 10 \text{ rad/s}}$.

Inverse-based controller design.

$$
\begin{aligned}
K_0(s) &= \frac{\omega_c}{s}\frac{10s+1}{200}(0.05s+1)^2 \\
&\approx \frac{\omega_c}{s}\frac{10s+1}{200}\frac{0.1s+1}{0.01s+1},
\end{aligned}
$$

$$
L_0(s) = \frac{\omega_c}{s}\frac{0.1s+1}{(0.05s+1)^2(0.01s+1)},\ \omega_c = 10 \quad (2.21)
$$



(a) Tracking response (b) Disturbance response

Figure 6: Responses with “inverse-based” controller $K_0(s)$ for the disturbance process. Note poor disturbance responce

2.4.2 Loop shaping for disturbance rejection [2.6.4]

$$e = y = SG_d d, \tag{2.22}$$

to achieve $|e(\omega)| \leq 1$ for $|d(\omega)| = 1$ (the worst-case disturbance) we require $|SG_d(j\omega)| < 1, \forall\omega$, or

$$|1 + L| \geq |G_d| \quad \forall\omega \tag{2.23}$$

or approximately:

$$|L| \geq |G_d| \quad \forall\omega \tag{2.24}$$

Initial guess:

$$|L_{\min}| \approx |G_d| \tag{2.25}$$

or:

$$|K_{\min}| \approx |G^{-1}G_d| \tag{2.26}$$

Controller contains the model of the disturbance.

To improve low-frequency performance

$$|K| = |\frac{s + \omega_I}{s}||G^{-1}G_d| \tag{2.27}$$

Summary:

- Controller contains the dynamics (G_d) of the disturbance and inverts the dynamics (G) of the inputs.
- For disturbance at plant output, $G_d = 1$, we get $|K_{\min}| = |G^{-1}|$.
- For disturbances at plant input we have $G_d = G$ and we get $|K_{\min}| = 1$.

Loop-shape $L(s)$ may be modified as follows:

1. Around crossover make slope N of $|L|$ to be about -1 for transient behaviour with acceptable gain and phase margins.
2. Increase the loop gain at low frequencies to improve the settling time and reduce the steady-state offset $\rightarrow$ add an integrator
3. Let $L(s)$ roll off faster at higher frequencies (beyond the bandwidth) in order to reduce the use of manipulated inputs, to make the controller realizable and to reduce the effects of noise.

Example: Loop-shaping design for the disturbance process.

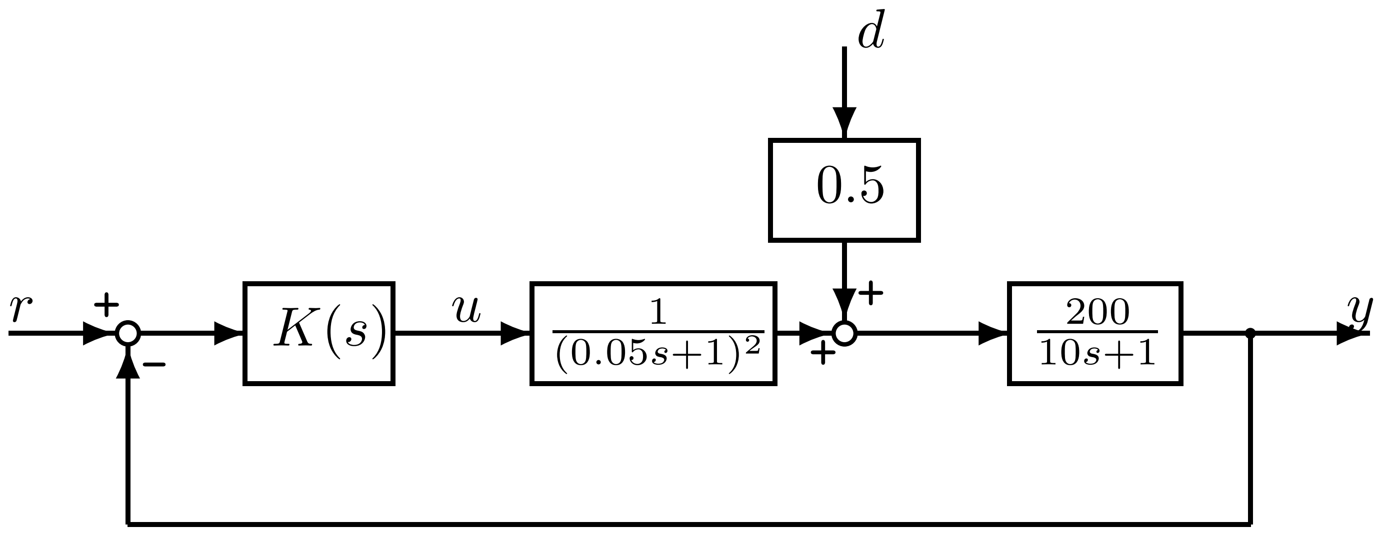
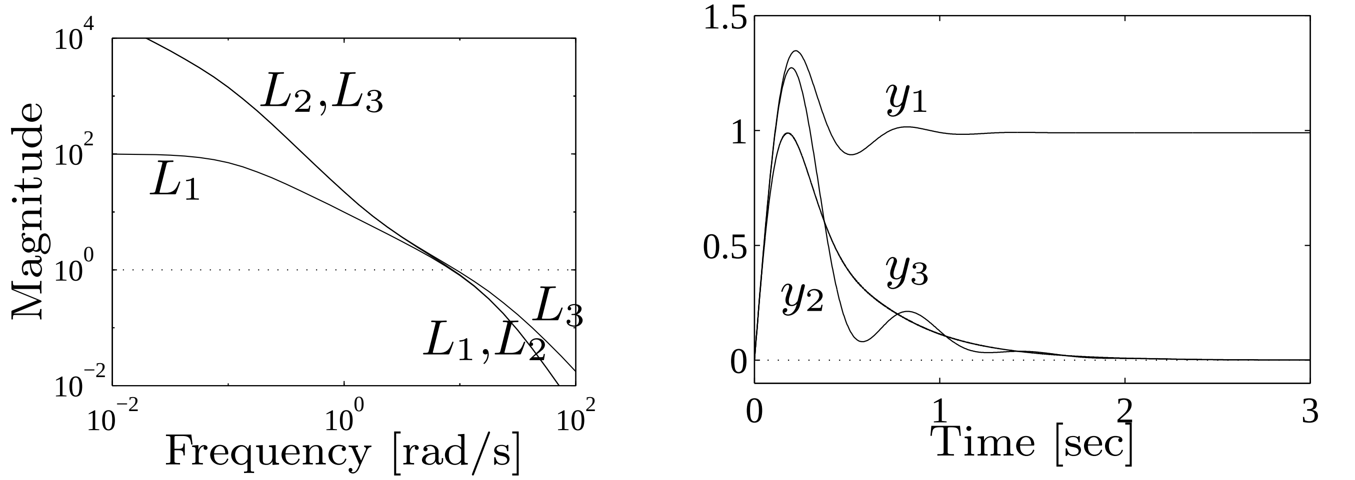


Figure 7: Block diagram representation of the disturbance process in (2.20)



(a) Loop gains

(b) Disturbance responses

Figure 8: Loop shapes and disturbance responses for controllers K_1, K_2 and K_3 for the disturbance process

Step 1. Initial design.
$K(s) = G^{-1}G_d = 0.5(0.05s+1)^2$.

Make proper:

$$K_1(s) = 0.5 \tag{2.28}$$

$\Longrightarrow$ offset!

Step 2. More gain at low frequency. To get integral action multiply the controller by the term $\frac{s+\omega_I}{s}$. For $\omega_I = 0.2\omega_c$ the phase contribution from $\frac{s+\omega_I}{s}$ is $\arctan(1/0.2) - 90^\circ = -11^\circ$ at ω_c. For $\omega_c \approx 10$ rad/s, select the following controller

$$K_2(s) = 0.5\frac{s+2}{s} \tag{2.29}$$

$\Longrightarrow$ response exceeds 1, oscillatory, small phase margin

Step 3. High-frequency correction. Supplement with "derivative action" by multiplying $K_2(s)$ by a lead-lag term effective over one decade starting at 20 rad/s:

$$K_3(s) = 0.5\frac{s+2}{s}\,\frac{0.05s+1}{0.005s+1} \tag{2.30}$$

$\Longrightarrow$ poor reference tracking (simulation)

2.4.3 *Two degrees of freedom design [2.6.5]

In order to meet both regulator and tracking performance use K_r (= "prefilter"):

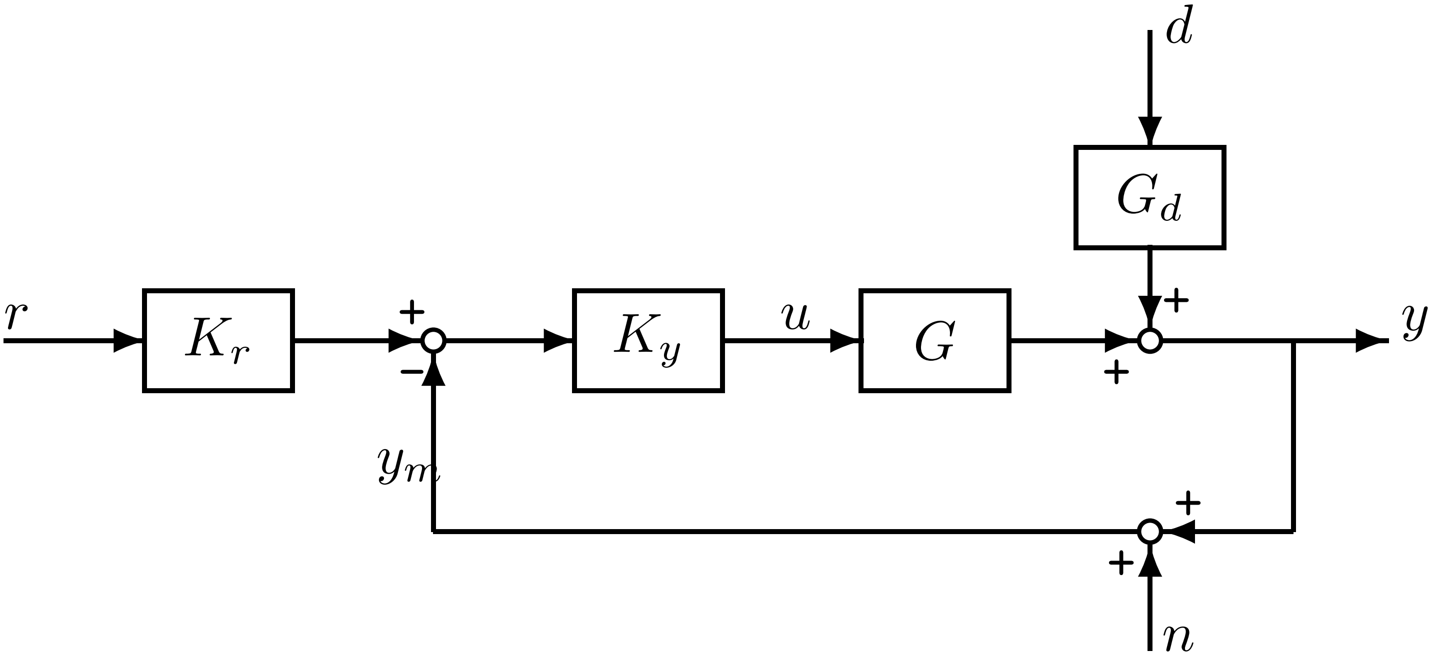


Figure 9: Two degrees-of-freedom controller

Idea:

- Design K_y
- $T = L(I + L)^{-1}$ with $L = GK_y$
- Desired $y = T_{ref}r$

$$\Longrightarrow K_r = T^{-1}T_{ref} \tag{2.31}$$

Remark:

Practical choice of prefilter is the lead-lag network

$$K_r(s) = \frac{\tau_{\text{lead}} s + 1}{\tau_{\text{lag}} s + 1} \tag{2.32}$$

$\tau_{\text{lead}} > \tau_{\text{lag}}$ to speed up the response, and $\tau_{\text{lead}} < \tau_{\text{lag}}$ to slow down the response.

Example Two degrees-of-freedom design for the disturbance process.

$K_y = K_3$. Approximate response by inpection of y_3:

$T(s) \approx \frac{1.5}{0.1s+1} - \frac{0.5}{0.5s+1} = \frac{(0.7s+1)}{(0.1s+1)(0.5s+1)}$

which yields:

$K_r(s) = \frac{0.5s+1}{0.7s+1}$.

By closed-loop simulations:

$$K_{r3}(s) = \frac{0.5s+1}{0.65s+1} \cdot \frac{1}{0.03s+1} \tag{2.33}$$

where $1/(0.03s+1)$ included to avoid initial peaking of input signal $u(t)$ above 1.

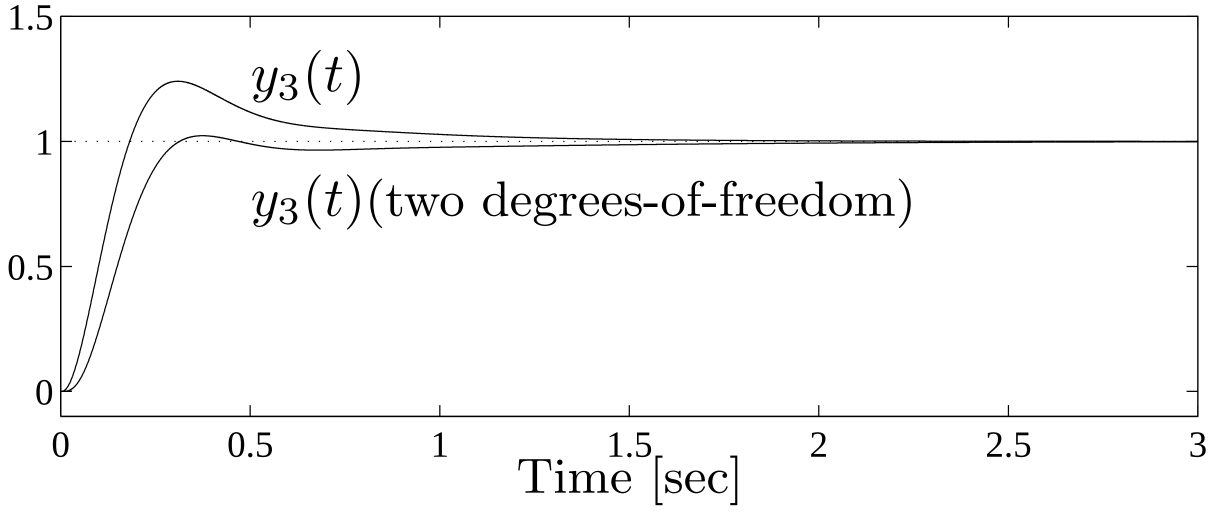


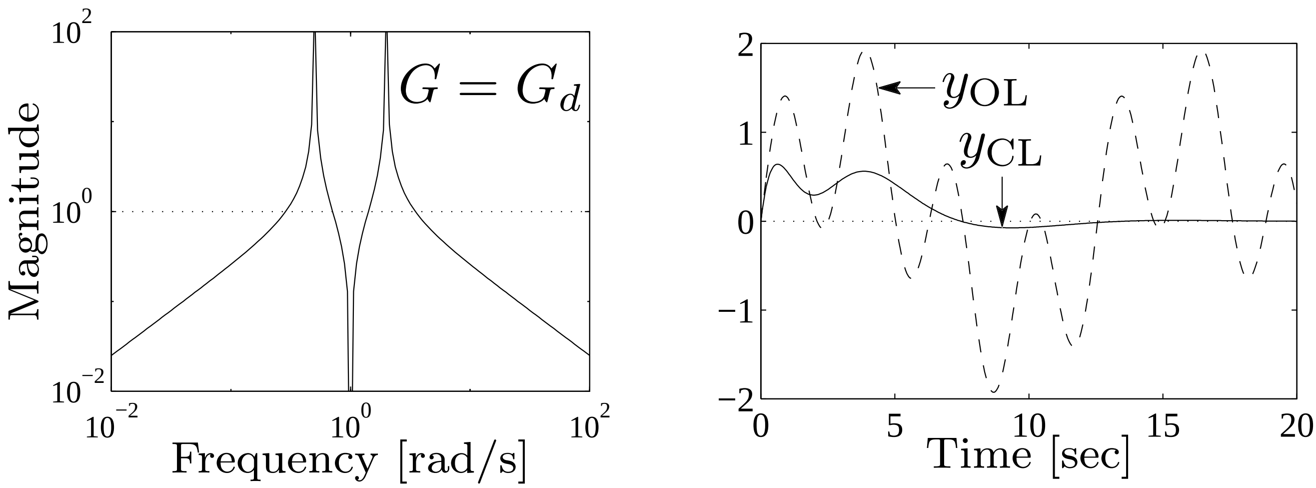
Figure 10: Tracking responses with the one degree-of-freedom controller (K_3) and the two degrees-of-freedom controller (K_3, K_{r3}) for the disturbance process

Example: Loop shaping for a flexible structure.

$$G(s) = G_d(s) = \frac{2.5s(s^2+1)}{(s^2+0.5^2)(s^2+2^2)} \tag{2.34}$$

$|K_{\min}(j\omega)| = |G^{-1}G_d| = 1 \Rightarrow$

$$K(s) = 1 \tag{2.35}$$



(a) Magnitude plot of $|G| = |G_d|$

(b) Open-loop and closed-loop disturbance responses with $K = 1$

Figure 11: Flexible structure in (2.34)

2.5 Closed-loop shaping [2.7]

Why ?

We are interested in S and T:

$$
\begin{aligned}
|L(j\omega)| \gg 1 \quad &\Rightarrow \quad S \approx L^{-1}; \quad T \approx 1 \\
|L(j\omega)| \ll 1 \quad &\Rightarrow \quad S \approx 1; \qquad T \approx L
\end{aligned}
$$

but in the crossover region where $|L(j\omega)|$ is close to 1, one cannot infer anything about S and T from $|L(j\omega)|$.

Alternative:

Directly shape the magnitudes of closed-loop $S(s)$ and $T(s)$.

The term $\mathcal{H}_\infty$

The $\mathcal{H}_\infty$ norm of a stable scalar transfer function $f(s)$ is simply the peak value of $|f(j\omega)|$ as a function of frequency, that is,

$$\|f(s)\|_\infty \triangleq \max_\omega |f(j\omega)| \qquad (2.36)$$

The symbol ∞ comes from:

$$\max_\omega |f(j\omega)| = \lim_{p\to\infty} \left(\int_{-\infty}^{\infty} |f(j\omega)|^p d\omega \right)^{1/p}$$

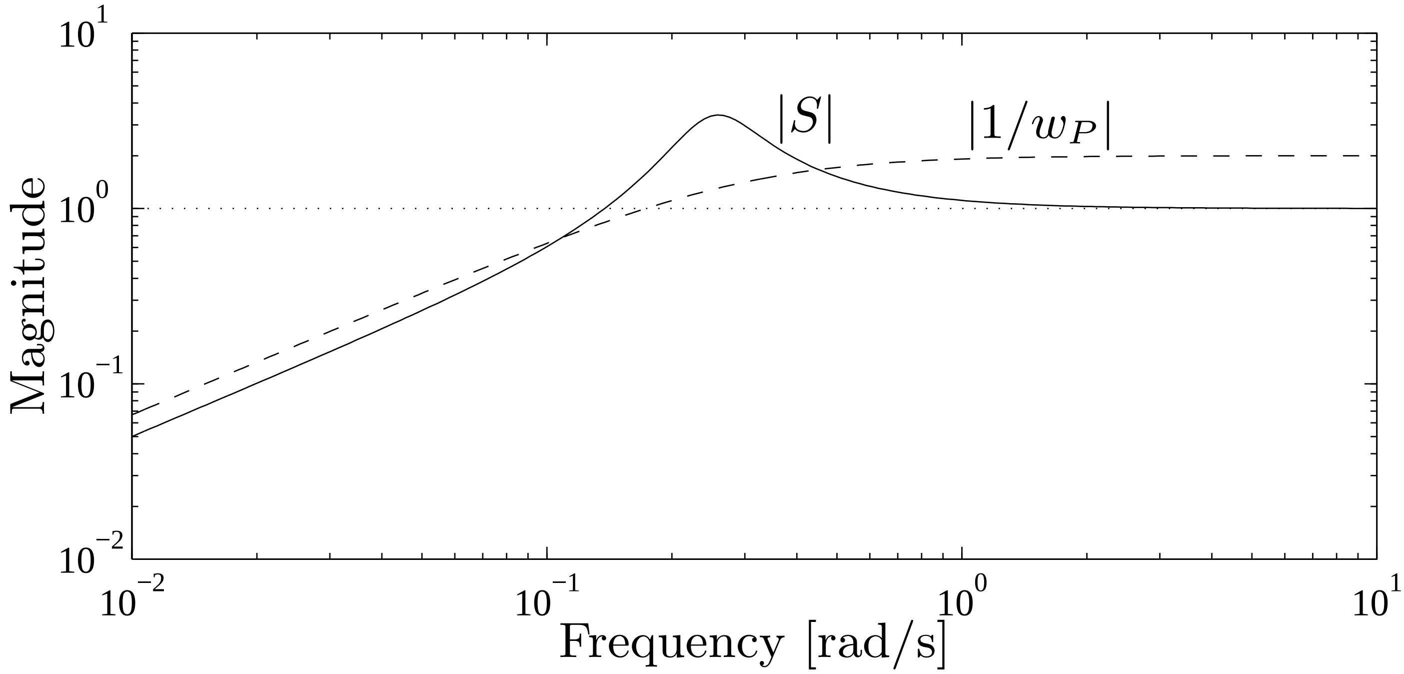
The symbol $\mathcal{H}$ stands for "Hardy space", and $\mathcal{H}_\infty$ is the set of transfer functions with bounded ∞-norm, which is simply the set of *stable and proper* transfer functions.

2.5.1 Weighted sensitivity [2.7.2]

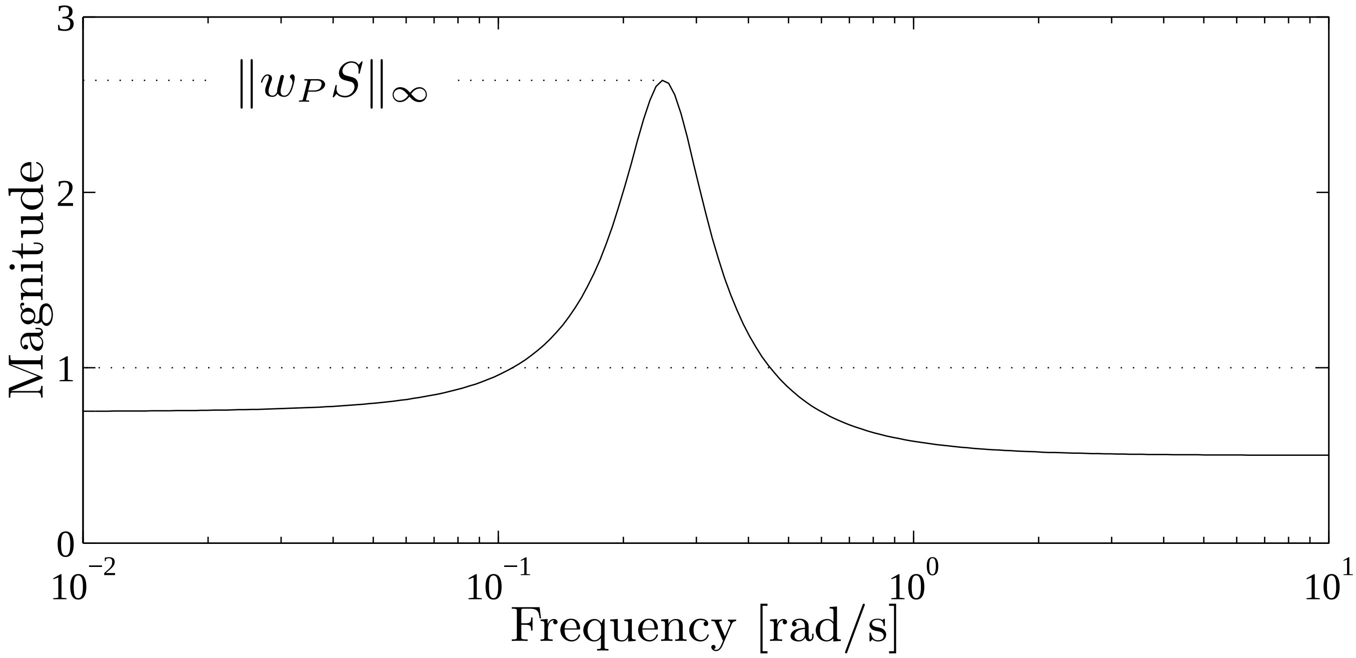
Typical specifications in terms of S:

1. Minimum bandwidth frequency ω_B^*.
2. Maximum tracking error at selected frequencies.
3. System type, or alternatively the maximum steady-state tracking error, A.
4. Shape of S over selected frequency ranges.
5. Maximum peak magnitude of S, $\|S(j\omega)\|_\infty \le M$.

Specifications may be captured by an upper bound, $1/|w_P(s)|$, on $\|S\|$.



(a) Sensitivity S and performance weight w_P



(b) Weighted sensitivity $w_P S$

Figure 12: Case where $|S|$ exceeds its bound $1/|w_P|$, resulting in $\|w_P S\|_\infty > 1$

$$|S(j\omega)| < 1/|w_P(j\omega)|, \ \forall\omega \tag{2.37}$$

$$\Leftrightarrow \quad |w_P S| < 1, \ \forall\omega \quad \Leftrightarrow \quad \boxed{\|w_P S\|_\infty < 1} \tag{2.38}$$

Typical performance weight:

$$\boxed{w_P(s) = \frac{s/M + \omega_B^*}{s + \omega_B^* A}} \tag{2.39}$$

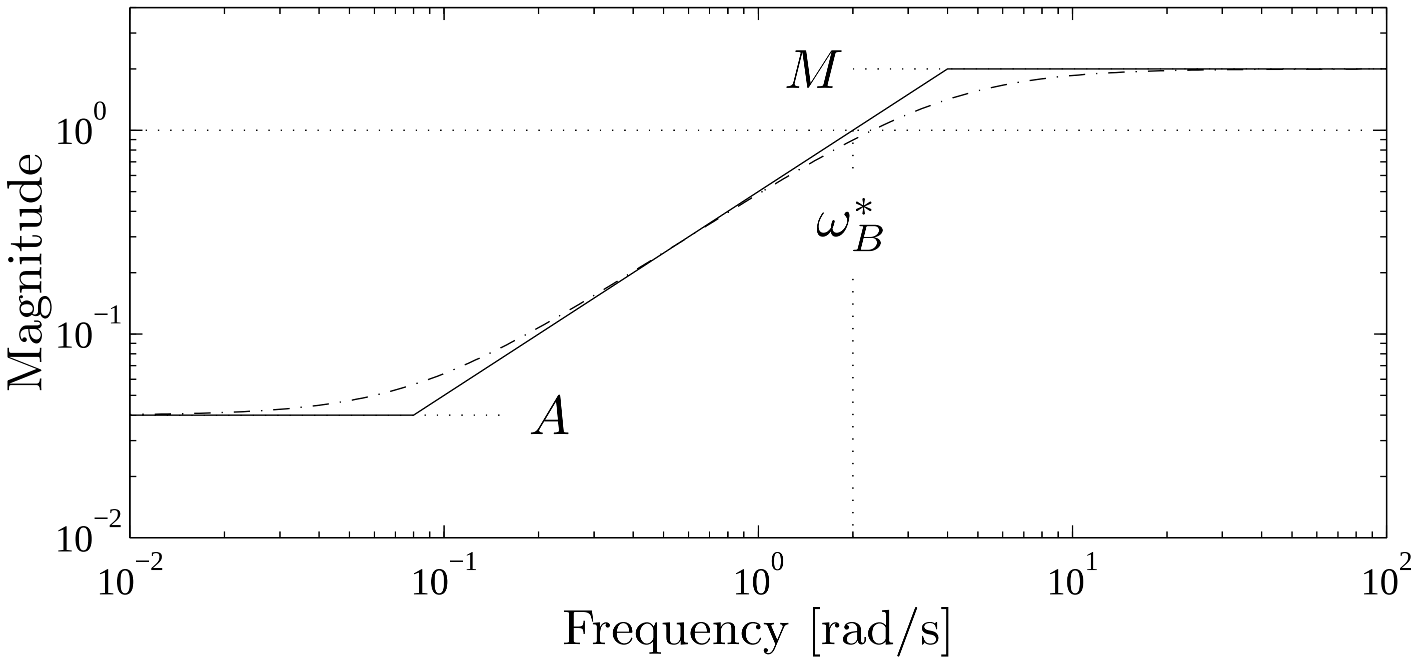


Figure 13: Inverse of performance weight. Exact and asymptotic plot of $1/|w_P(j\omega)|$ in (2.39)

To get a steeper slope for L (and S) below the bandwidth:

$$\boxed{w_P(s) = \frac{(s/M^{1/2} + \omega_B^*)^2}{(s + \omega_B^* A^{1/2})^2}} \tag{2.40}$$

2.5.2 *Mixed sensitivity [2.7.3]

In order to enforce specifications on other transfer functions:

$$\|N\|_\infty = \max_\omega \bar{\sigma}(N(j\omega)) < 1; \quad N = \begin{bmatrix} w_P S \\ w_T T \\ w_u K S \end{bmatrix} \tag{2.41}$$

N is a vector and the maximun singular value $\bar{\sigma}(N)$ is the usual Euclidean vector norm:

$$\bar{\sigma}(N) = \sqrt{|w_P S|^2 + |w_T T|^2 + |w_u K S|^2} \tag{2.42}$$

The $\mathcal{H}_\infty$ optimal controller is obtained from

$$\min_K \|N(K)\|_\infty \tag{2.43}$$

Example: $\mathcal{H}_\infty$ mixed sensitivity design for the disturbance process.

Consider the plant in (2.20), and an $\mathcal{H}_\infty$ mixed sensitivity S/KS design in which

$$N = \begin{bmatrix} w_P S \\ w_u KS \end{bmatrix} \tag{2.44}$$

Selected $w_u = 1$ and

$$w_{P1}(s) = \frac{s/M + \omega_B^*}{s + \omega_B^* A}; \quad M = 1.5,\ \omega_B^* = 10, \quad A = 10^{-4} \tag{2.45}$$

$\Longrightarrow$ poor disturbance response

To get higher gains at low frequencies:

$$w_{P2}(s) = \frac{(s/M^{1/2} + \omega_B^*)^2}{(s + \omega_B^* A^{1/2})^2}, \quad M = 1.5, \omega_B^* = 10, A = 10^{-4} \tag{2.46}$$

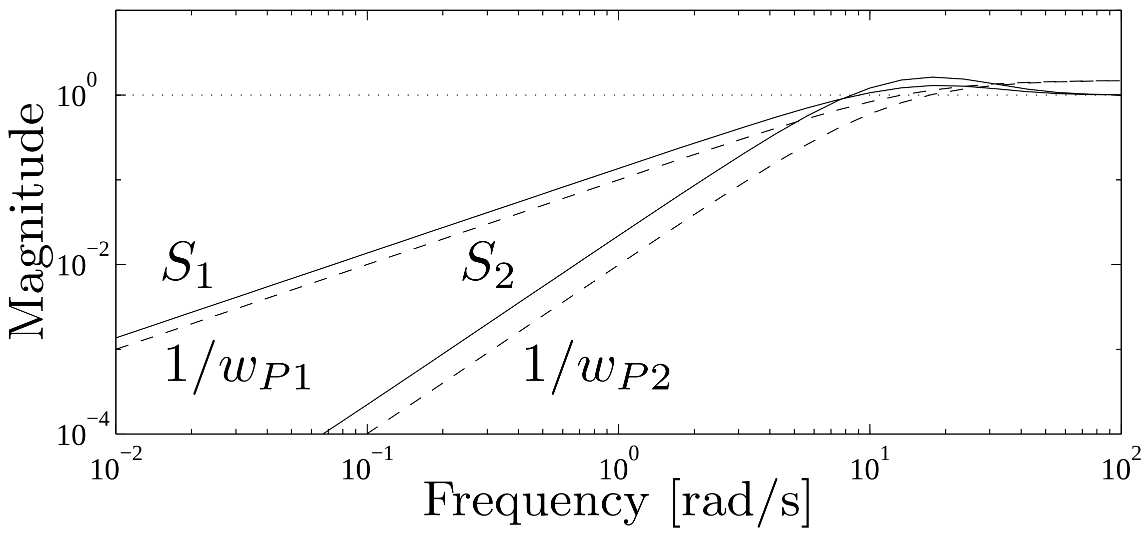
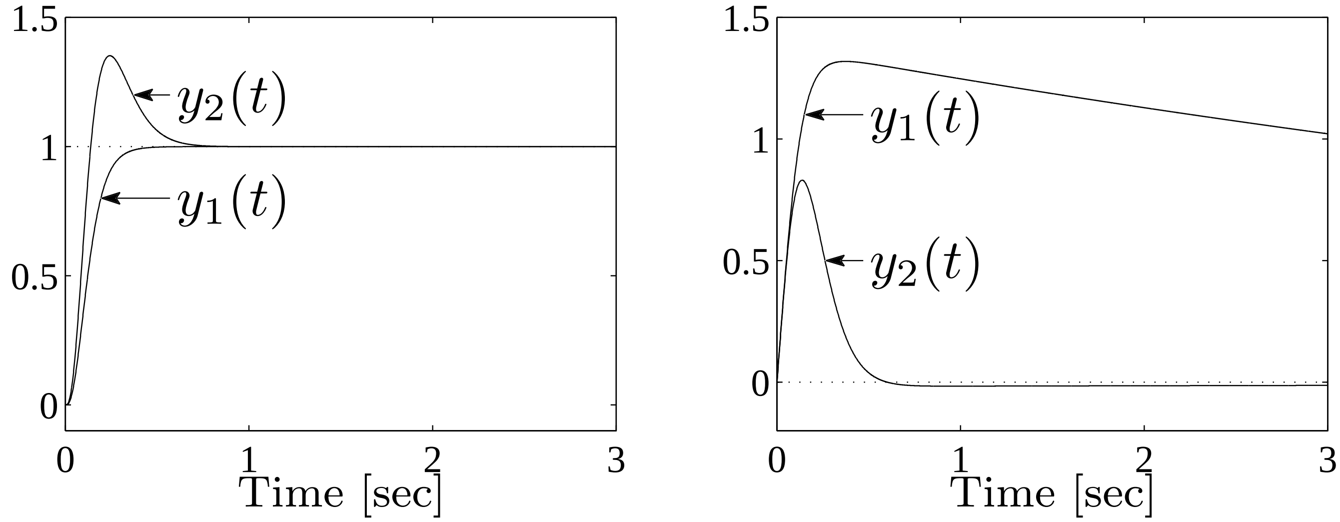


Figure 14: Inverse of performance weight (dashed line) and resulting sensitivity function (solid line) for two $\mathcal{H}_\infty$ designs (1 and 2) for the disturbance process



(a) Tracking response

(b) Disturbance response

Figure 15: Closed-loop step responses for two alternative $\mathcal{H}_\infty$ designs (1 and 2) for the disturbance process

3 PERFORMANCE LIMITATIONS IN SISO SYSTEMS [5]

3.1 Input-Output Controllability [5.1]

“Control” is not only controller design and stability analysis. Three important questions:

I. How well can the plant be controlled?

II. What control structure should be used?
(What variables should we measure, which variables should we manipulate, and how are these variables best paired together?)

III. How might the process be changed to improve control?

Definition 1 (Input-output) controllability *is the ability to achieve acceptable control performance; that is, to keep the outputs (y) within specified bounds from their references (r), in spite of unknown but bounded variations, such as disturbances (d) and plant changes, using available inputs (u) and available measurements (y_m or d_m).*

Note: controllability is independent of the controller, and is a property of the plant (or process) alone.

It can only be affected by:

- changing the apparatus itself, e.g. type, size, etc.
- relocating sensors and actuators
- adding new equipment to dampen disturbances
- adding extra sensors
- adding extra actuators

3.1.1 Scaling and performance [5.1.2]

We assume that the variables and models have been scaled so that for acceptable performance:

- Output $y(t)$ between $r-1$ and $r+1$ for any disturbance $d(t)$ between -1 and 1 and any reference $r(t)$ between $-R$ and R, using an input $u(t)$ within -1 to 1.

or frequency-by-frequency.

- $|e(\omega)| \leq 1$, for any disturbance $|d(\omega)| \leq 1$ and any reference $|r(\omega)| \leq R(\omega)$, using an input $|u(\omega)| \leq 1$.

Usually for simplicity:

$$\begin{aligned} R(\omega) &= R \qquad && \omega \leq \omega_r \\ R(\omega) &= 0 \qquad && \omega > \omega_r \end{aligned} \tag{3.1}$$

Because:

$$e = y - r = Gu + G_d d - R\widetilde{r} \tag{3.2}$$

we can apply results for disturbances also to references by replacing G_d by $-R$.

3.2 Perfect control & plant inversion [5.2]

$$y = Gu + G_d d \tag{3.3}$$

For "perfect control", i.e. $y = r$ (not realizable) we have feedforward controller:

$$u = G^{-1}r - G^{-1}G_d d \tag{3.4}$$

With feedback control $u = K(r - y)$ we have:

$$u = KSr - KSG_d d$$

or since $T = GKS$,

$$u = G^{-1}Tr - G^{-1}TG_d d \tag{3.5}$$

Where feedback is effective ($T \approx I$) feedback input in (3.5) is the same as perfect control input in (3.4) $\Longrightarrow$ High gain feedback generates an inverse of G even though K may be very simple.

As consequence perfect control *cannot* be achieved if

- G contains RHP-zeros (since then G^{-1} is unstable)
- G contains time delay (since then G^{-1} contains a prediction)
- G has more poles than zeros (since then G^{-1} is unrealizable)

For feedforward control perfect control *cannot* be achieved if

- G is uncertain (since then G^{-1} cannot be obtained exactly)

Because of input constraints perfect control *cannot* be achieved if

- $|G^{-1}G_d|$ is large
- $|G^{-1}R|$ is large

3.3 Constraints on S and T [5.3]

3.3.1 S plus T is one [5.3.1]

$$S + T = 1 \tag{3.6}$$

$\Longrightarrow$ at any frequency $|S(j\omega)| \geq 0.5$ or $|T(j\omega)| \geq 0.5$

3.3.2 The waterbed effects (sensitivity integrals) [5.3.2]

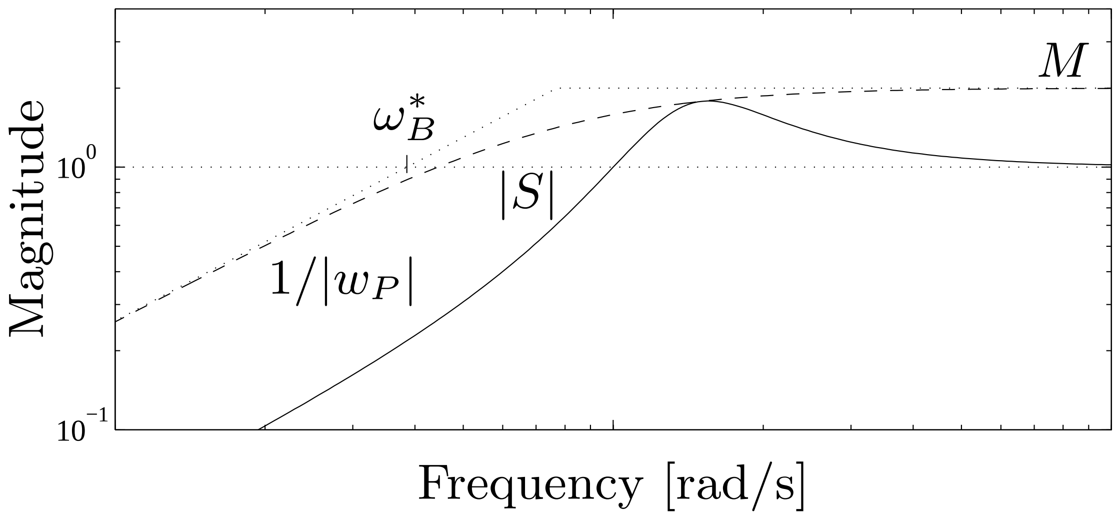


Figure 16: Plot of typical sensitivity, $|S|$, with upper bound $1/|w_P|$

Note: $|S|$ has peak greater than 1; we will show that this is unavoidable in practice.

Pole excess of two: First waterbed formula

Idea:

When $L(s) =$ has a relative degree of two or more, then for some ω the distance between L and -1 is less than one.

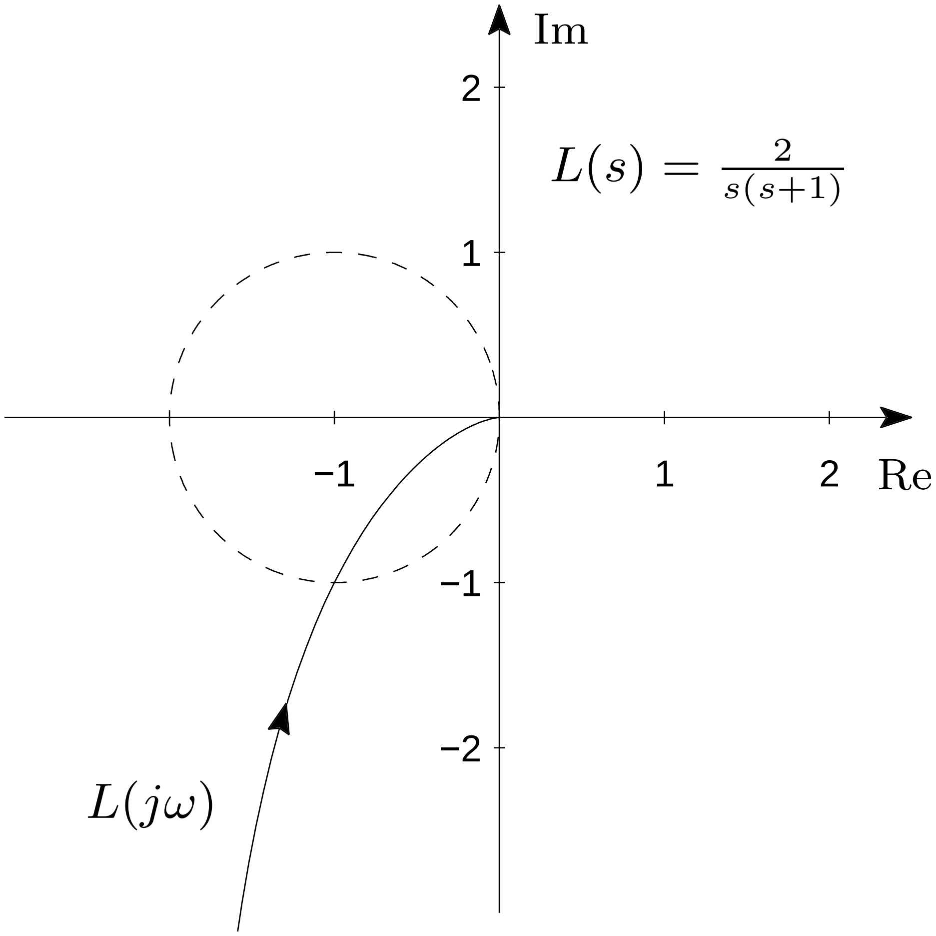


Figure 17: $|S| > 1$ whenever the Nyquist plot of L is inside the circle

Theorem 1 Bode Sensitivity Integral.

Suppose that the open-loop transfer function $L(s)$ is rational and has at least two more poles than zeros (relative degree of two or more).

Suppose also that $L(s)$ has N_p RHP-poles at locations p_i.

Then for closed-loop stability the sensitivity function must satisfy

$$\int_0^\infty \ln|S(j\omega)|d\omega = \pi \cdot \sum_{i=1}^{N_p} Re(p_i) \qquad (3.7)$$

where $Re(p_i)$ denotes the real part of p_i.

RHP-zeros: Second waterbed formula

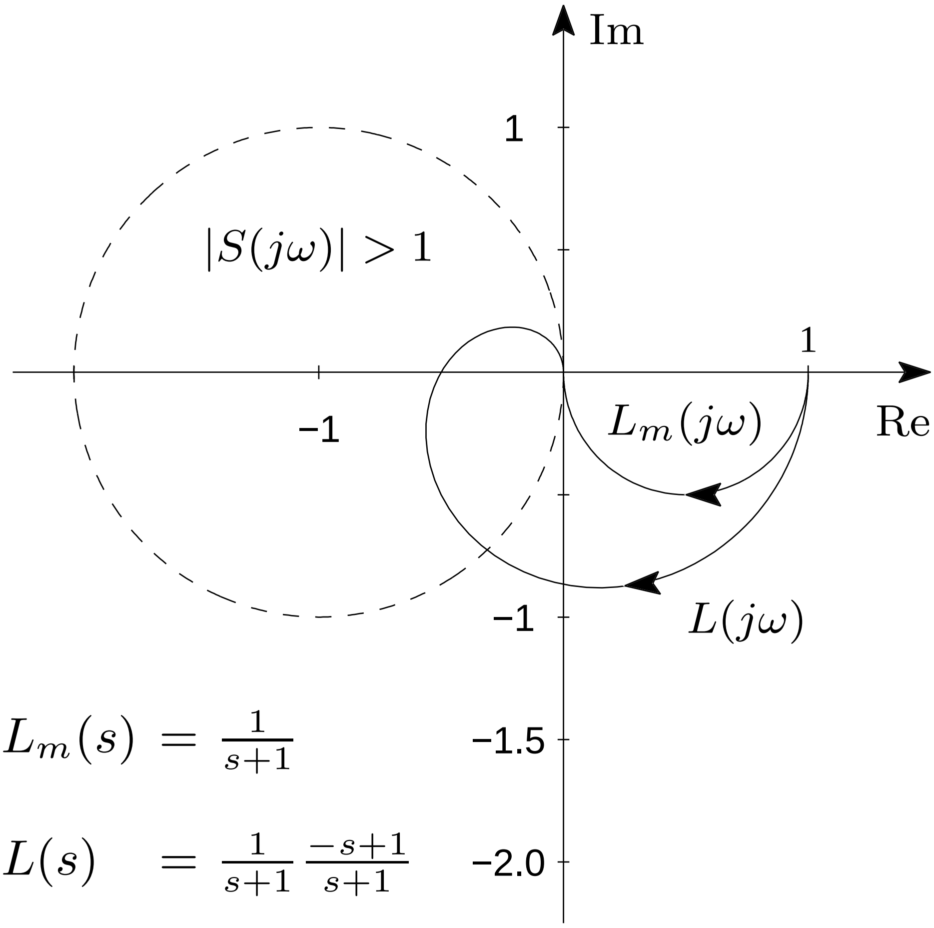


Figure 18: Additional phase lag contributed by RHP-zero causes $|S| > 1$

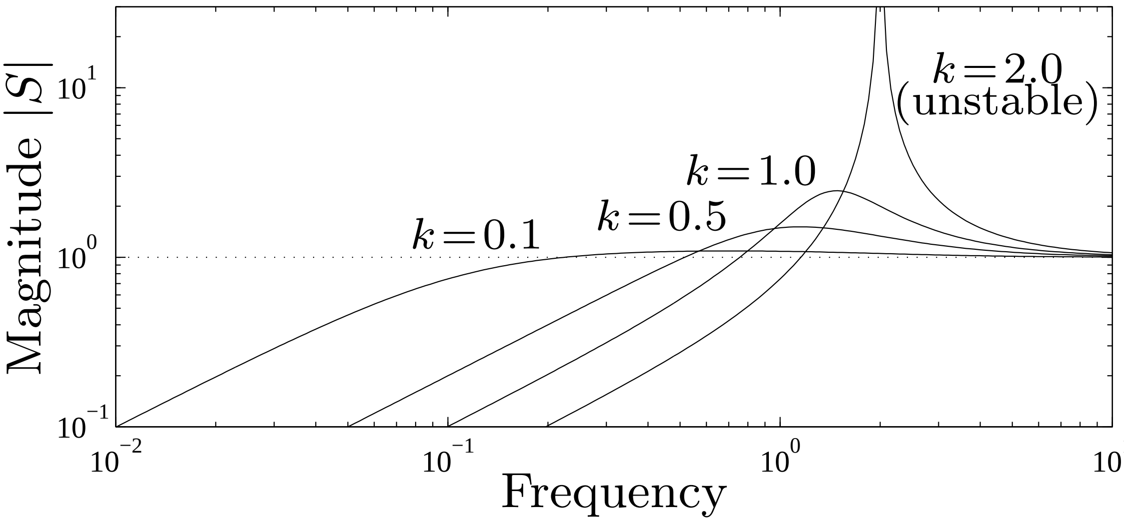


Figure 19: Effect of increased controller gain on $|S|$ for system with RHP-zero at $z = 2$, $L(s) = \frac{k}{s}\frac{2-s}{2+s}$

Theorem 2 Weighted sensitivity integral.
Suppose that $L(s)$ has a single real RHP-zero z and has N_p RHP-poles, p_i. Then for closed-loop stability the sensitivity function must satisfy

$$\int_0^{\infty} \ln |S(j\omega)| \cdot w(z,\omega) d\omega = \pi \cdot \ln \prod_{i=1}^{N_p} \left| \frac{p_i + z}{p_i - z} \right| \quad (3.8)$$

where:

$$w(z,\omega) = \frac{2z}{z^2 + \omega^2} = \frac{2}{z} \frac{1}{1 + (\omega/z)^2} \quad (3.9)$$

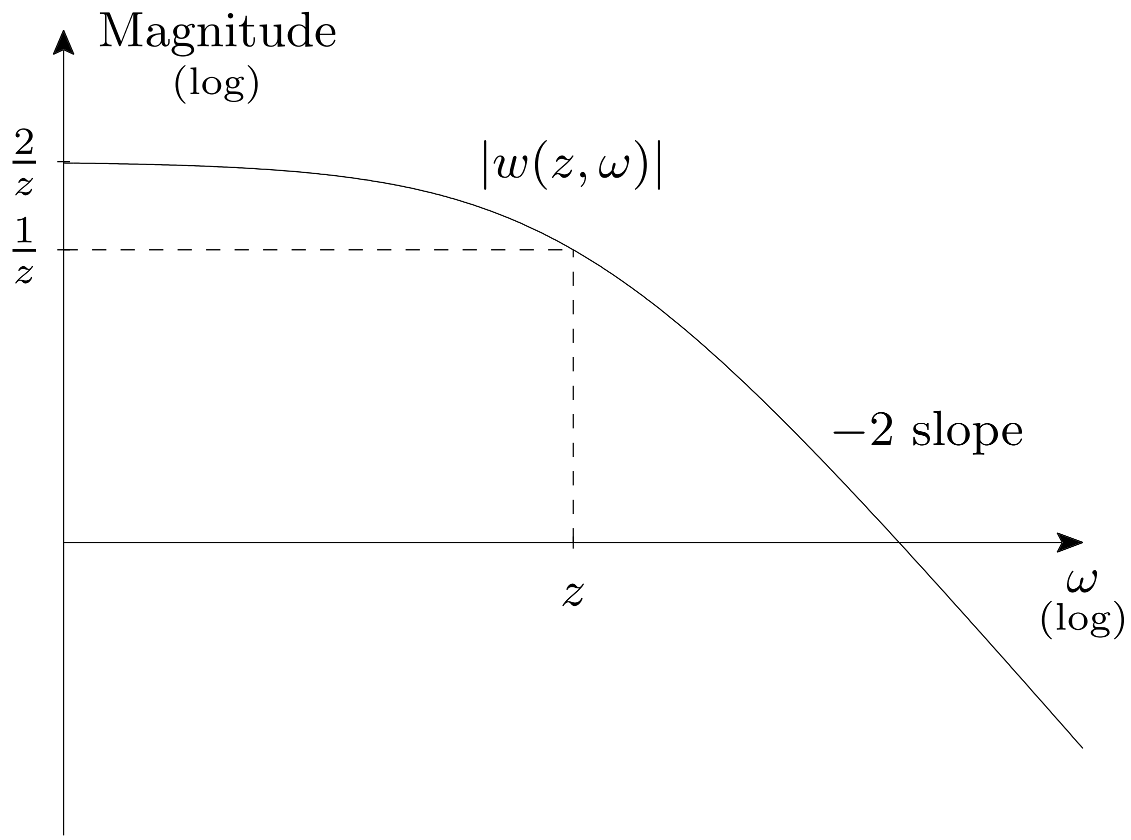


Figure 20: Plot of weight $w(z, \omega)$ for case with real zero at $s = z$

Weight $w(z, \omega)$ "cuts off" contribution of $ln|S|$ at frequencies $\omega > z$. Thus, for a stable plant:

$$\int_0^z \ln|S(j\omega)|d\omega \approx 0 \quad (\text{ for } |S| \approx 1 \text{ at } \omega > z) \quad (3.10)$$

The waterbed is finite, and a large peak for $|S|$ is unavoidable when we reduce $|S|$ at low frequencies (Figure 19).

Note also that when $p_i \to z$ then $\frac{p_i+z}{p_i-z} \to \infty$.

3.3.3 Interpolation constraints from internal stability [5.3.3]

If p is a RHP-pole of $L(s)$ then

$$T(p) = 1, \quad S(p) = 0 \tag{3.11}$$

Similarly, if z is a RHP-zero of $L(s)$ then

$$T(z) = 0, \quad S(z) = 1 \tag{3.12}$$

3.3.4 Sensitivity peaks [5.3.4]

Maximum modulus principle. *Suppose $f(s)$ is stable (i.e. $f(s)$ is analytic in the complex RHP). Then the maximum value of $|f(s)|$ for s in the right-half plane is attained on the region's boundary, i.e. somewhere along the $j\omega$-axis. Hence, we have for a stable $f(s)$*

$$\|f(j\omega)\|_\infty = \max_\omega |f(j\omega)| \geq |f(s_0)| \quad \forall s_0 \in \mathrm{RHP} \tag{3.13}$$

The results below follow from (3.13) with
$f(s) = w_P(s)S(s)$
$f(s) = w_T(s)T(s)$
for weighted sensitivity and weighted complementary sensitivity.

Theorem 3 Weighted sensitivity peak.

Suppose that $G(s)$ has a RHP-zero z and let $w_P(s)$ be any stable weight function.

Then for closed-loop stability the weighted sensitivity function must satisfy

$$\|w_P S\|_\infty \geq |w_P(z)| \tag{3.14}$$

Theorem 4 Weighted complementary sensitivity peak.

Suppose that $G(s)$ has a RHP-pole p and let $w_T(s)$ be any stable weight function.

Then for closed-loop stability the weighted complementary sensitivity function must satisfy

$$\|w_T T\|_\infty \geq |w_T(p)| \tag{3.15}$$

Theorem 5 Combined RHP-poles and RHP-zeros.

Suppose that $G(s)$ has N_z RHP-zeros z_j, and N_p RHP-poles p_i.

Then for closed-loop stability the weighted sensitivity function must satisfy for each *RHP-zero z_j*

$$\|w_P S\|_\infty \geq c_{1j}|w_P(z_j)|, \quad c_{1j} = \prod_{i=1}^{N_p} \frac{|z_j + \bar{p}_i|}{|z_j - p_i|} \geq 1 \tag{3.16}$$

and the weighted complementary sensitivity function must satisfy for each *RHP-pole p_i*

$$\|w_T T\|_\infty \geq c_{2i}|w_T(p_i)|, \quad c_{2i} = \prod_{j=1}^{N_z} \frac{|\bar{z}_j + p_i|}{|z_j - p_i|} \geq 1 \tag{3.17}$$

For $w_P = w_T = 1$:

$$\|S\|_\infty \geq \max_j c_{1j}, \quad \|T\|_\infty \geq \max_i c_{2i} \tag{3.18}$$

$\Longrightarrow$ Large peaks for S and T are unavoidable if a RHP-zero and a RHP-pole are close to each other.

3.3.5 Bandwidth limitation II [5.6.4]

Performance requirement:

$$|S(j\omega)| < 1/|w_P(j\omega)| \quad \forall\omega \quad \Leftrightarrow \quad \|w_P S\|_\infty < 1 \tag{3.19}$$

However, from (3.14) we have that
$\|w_P S\|_\infty \geq |w_P(z)|$,

so the weight must satisfy

$$\boxed{|w_P(z)| < 1} \tag{3.20}$$

For performance weight

$$w_P(s) = \frac{s/M + \omega_B^*}{s + \omega_B^* A} \tag{3.21}$$

and a real zero at z we get:

$$\omega_B^*(1 - A) < z\left(1 - \frac{1}{M}\right) \tag{3.22}$$

e.g. $A = 0, M = 2$:

$$\omega_B^* < \frac{z}{2}$$

3.4 Limitations imposed by RHP-poles [5.8]

Specification:

$$|T(j\omega)| < 1/|w_T(j\omega)| \quad \forall\omega \quad \Leftrightarrow \quad \|w_T T\|_\infty < 1 \tag{3.23}$$

However, from (3.15) we have that:

$$\|w_T T\|_\infty \geq |w_T(p)| \tag{3.24}$$

so the weight must satisfy

$$\boxed{|w_T(p)| < 1} \tag{3.25}$$

For:

$$w_T(s) = \frac{s}{\omega_{BT}^*} + \frac{1}{M_T} \tag{3.26}$$

we get:

$$\boxed{\omega_{BT}^* > p\frac{M_T}{M_T - 1}} \tag{3.27}$$

e.g. $M_T = 2$:

$$\omega_{BT}^* > 2p$$

3.5 Combined RHP-poles and RHP-zeros [5.9]

RHP-zero:

$$\omega_c < z/2$$

RHP-pole:

$$\omega_c > 2p$$

RHP-pole and RHP-zero:

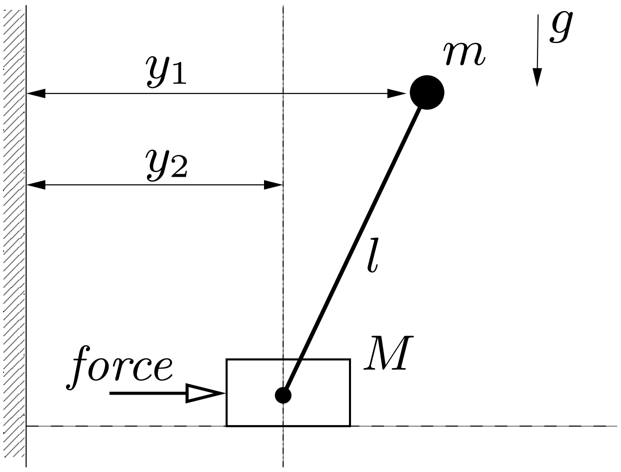
$z > 4p$ for acceptable performance and robustness.

Sensitivity peaks.

From Theorem 5 for a plant with a single real RHP-pole p and a single real RHP-zero z, we always have:

$$\boxed{\|S\|_\infty \geq c, \ \|T\|_\infty \geq c, \quad c = \frac{|z+p|}{|z-p|}} \tag{3.28}$$

Example 1 Balancing a rod. *The objective is to keep the rod upright by movement of the cart, based on observing the rod either at its far end (output y_1) or the cart position (output y_2).*



l [m] = length of rod

m [kg] = mass of rod

M [kg] = mass of hand

$g \approx 10\ m/s^2$ = acceleration due to gravity.

The linearized transfer functions for the two cases are

$$G_1(s) = \frac{-g}{s^2\left(Mls^2 - (M+m)g\right)};$$

$$G_2(s) = \frac{ls^2 - g}{s^2\left(Mls^2 - (M+m)g\right)}$$

Poles: $p = 0, 0, \pm\sqrt{\frac{(M+m)g}{Ml}}$. *For output* $y_1(G_1(s))$ *stabilization requires minimum bandwidth (3.27). For output* $y_2(G_2(s))$ *zero at* $z = \sqrt{\frac{g}{l}}$

- *For light rod* $m << M$, *pole* $\approx$ *zero* $\rightarrow$ *"impossible" to stabilize*
- *For heavy rod (m large) difficult to stabilize because* $p > z$

Example: $m/M = 0.1 \Rightarrow \|S\|_\infty \geq 42\ ;\ \|T\|_\infty \geq 42 \Rightarrow$ *poor control*

3.6 * Ideal Integral Square Error (ISE) optimal control [5.4]

$$\text{ISE} = \int_0^\infty |y(t) - r(t)|^2 dt \qquad (3.29)$$

the "ideal" response $y = Tr$ when $r(t)$ is a *unit step* is:

$$T(s) = \prod_i \frac{-s + z_j}{s + \bar{z}_j} e^{-\theta s} \qquad (3.30)$$

where $\bar{z}_j$ is the complex conjugate of z_j.

Optimal ISE for three simple stable plants are:

1. with a delay θ:

$\text{ISE} = \theta$

2. with a RHP-zero z:

$\text{ISE} = 2/z$

3. with complex RHP-zeros $z = x \pm jy$:

$\text{ISE} = 4x/(x^2 + y^2)$

3.6.1 * Limitations imposed by time delays [5.5]

Ideal for plant with delay:

$$S = 1 - T = 1 - e^{-\theta s} \tag{3.31}$$

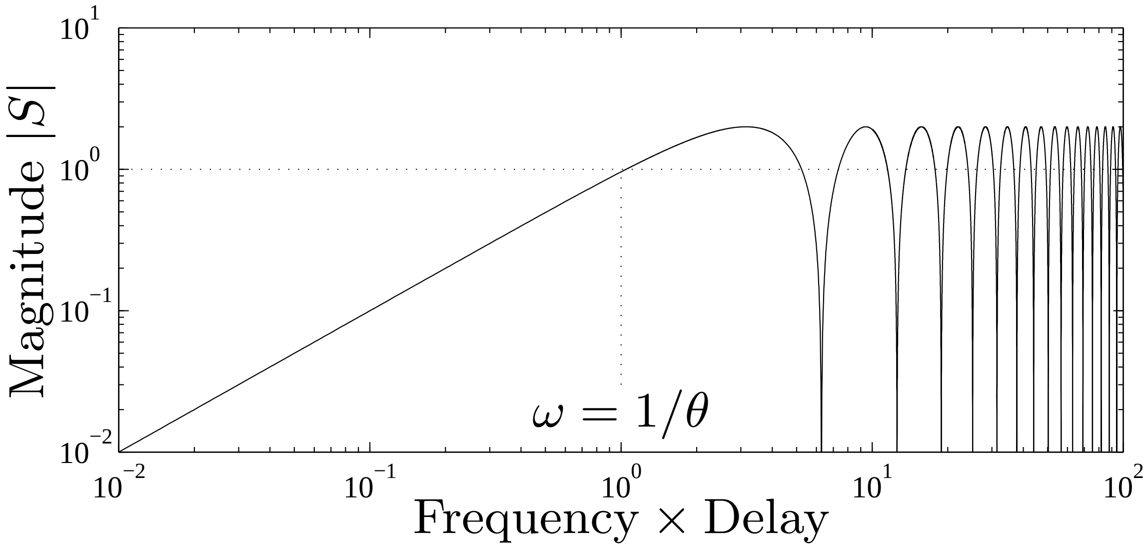


Figure 21: “Ideal” sensitivity function (3.31) for a plant with delay

$|S(j\omega)|$ in Figure 21 crosses 1 at $\frac{\pi}{3}\frac{1}{\theta} = 1.05/\theta$.

Because here $|S| = 1/|L|$, we have:

$$\omega_c < 1/\theta \tag{3.32}$$

3.6.2 * Limitations imposed by RHP-zeros [5.6]

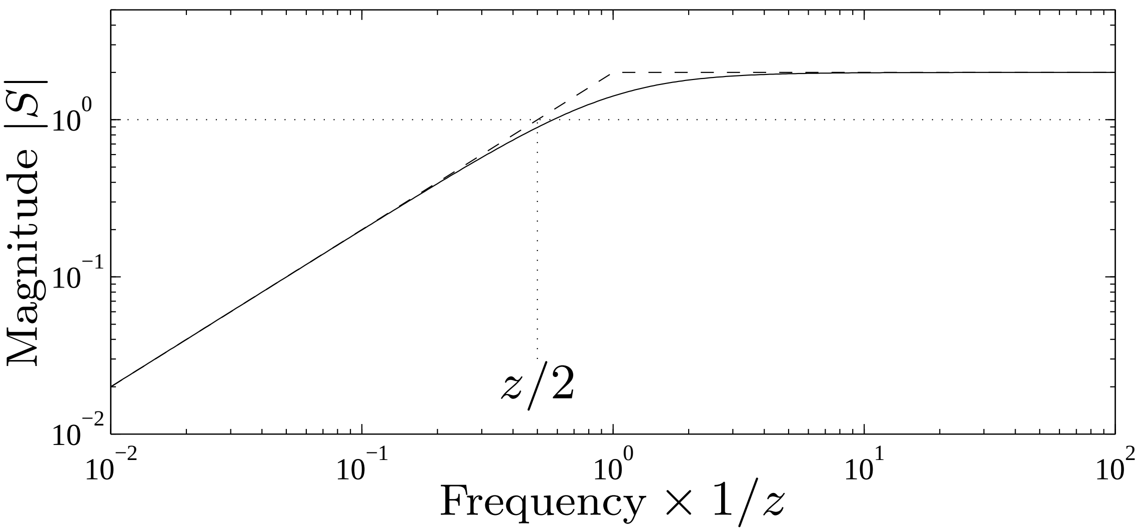
RHP-zeros typically appear when we have competing effects of slow and fast dynamics:

$$G(s) = \frac{1}{s+1} - \frac{2}{s+10} = \frac{-s+8}{(s+1)(s+10)}$$

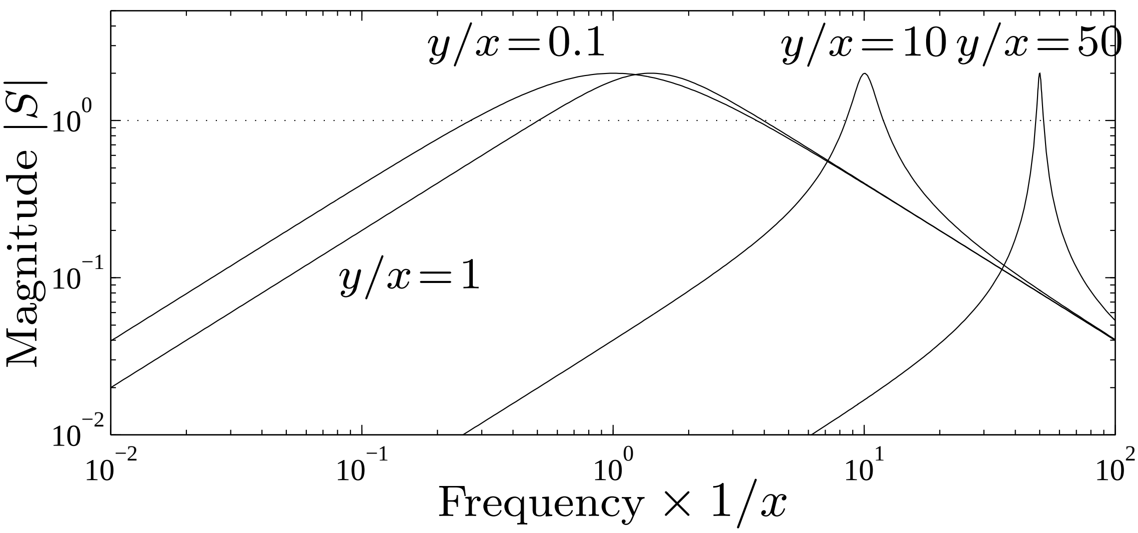
(a) Inverse response [5.6.1]

For a stable plant with n_z RHP-zeros, it may be proven that the output in response to a step change in the input will cross zero (its original value) n_z times, that is, we have *inverse response* behaviour.

(b) Bandwidth limitation I [5.6.3]



(a) Real RHP-zero



(b) Complex pair of RHP-zeros, $z = x \pm jy$

Figure 22: "Ideal" sensitivity functions for plants with RHP-zeros

For a single *real RHP-zero* the “ideal”, i.e. ISE optimal, sensitivity function is

$$S = 1 - T = \frac{2s}{s+z} \tag{3.33}$$

From Figure 22(a):

$$\omega_B \approx \omega_c < \frac{z}{2} \tag{3.34}$$

3.7 * Non-causal controllers [5.7]

Perfect control can be achieved for a plant with a time delay or RHP-zero if we use a non-causal controller, i.e. a controller which uses information about the future. (relevant for servo problems, e.g. in robotics and for batch processing.)

$$G(s) = \frac{-s+z}{s+z}; \quad z > 0 \tag{3.35}$$

$$r(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}$$

Stable *non-causal controller* generates the input

$$u(t) = \begin{cases} 2e^{zt} & t < 0 \\ 1 & t \geq 0 \end{cases}$$

(See (Figure 23))

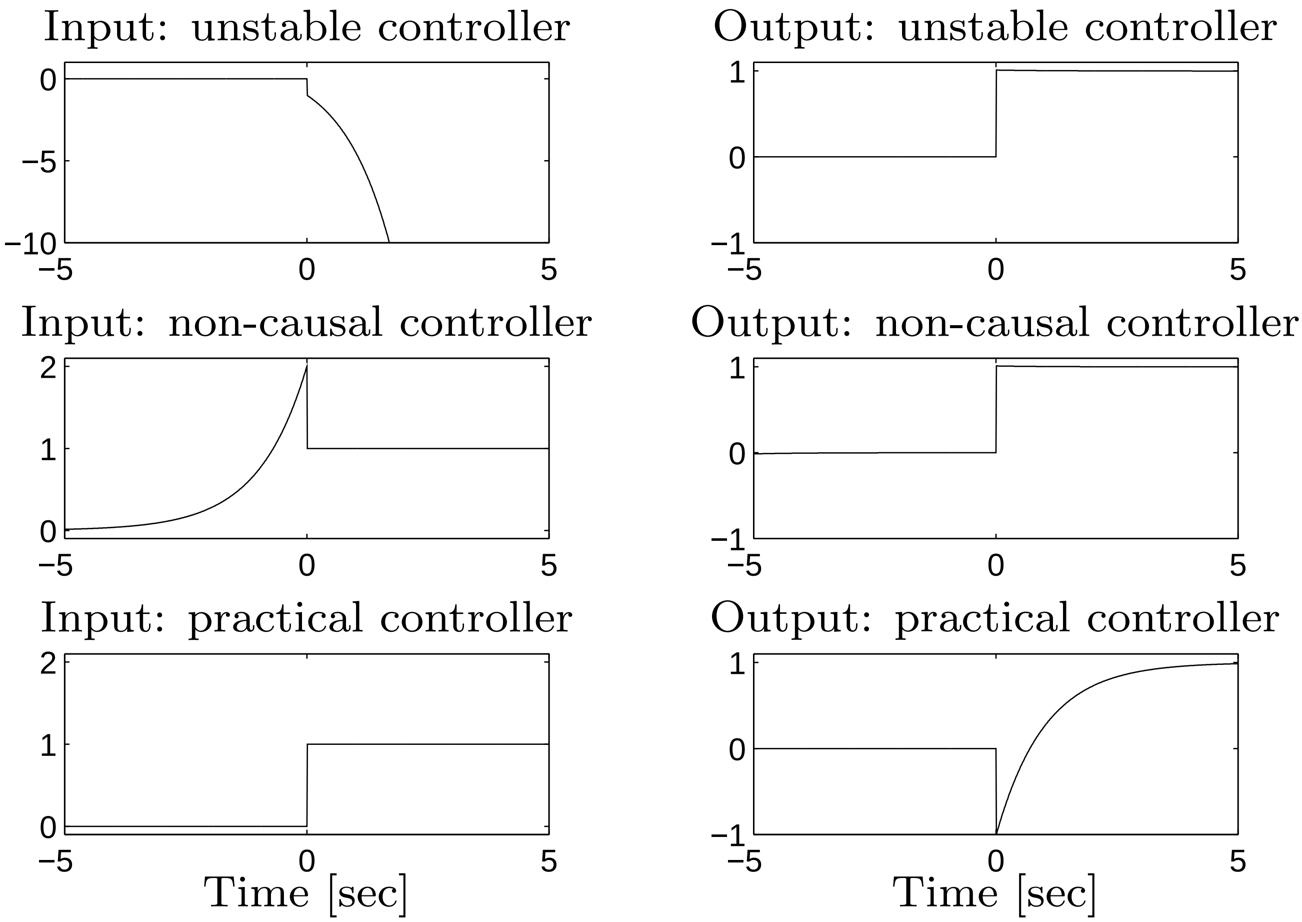


Figure 23: Feedforward control of plant with RHP-zero

3.8 Limitations imposed by input constraints [5.11]

The input required to achieve perfect control ($e = 0$) is

$$u = G^{-1}r - G^{-1}G_d d \tag{3.36}$$

Disturbance rejection. $r = 0$, $|d(\omega)| = 1$; $|u(\omega)| < 1$ implies

$$|G^{-1}(j\omega)G_d(j\omega)| < 1 \quad \forall\omega \tag{3.37}$$

Command tracking. $d = 0$, $|r(\omega)| = R\forall\omega < \omega_r$ $|u(\omega)| < 1$ implies:

$$|G^{-1}(j\omega)R| < 1 \quad \forall\omega \leq \omega_r \tag{3.38}$$

For *acceptable control* (namely $|e(j\omega)| < 1$) requirements change to:

$$\boxed{|G| > |G_d| - 1} \quad \text{at frequencies where} \quad |G_d| > 1 \tag{3.39}$$

$$\boxed{|G| > |R| - 1 < 1} \quad \forall\omega \leq \omega_r \tag{3.40}$$

3.9 Summary: Controllability analysis with feedback control [5.14]

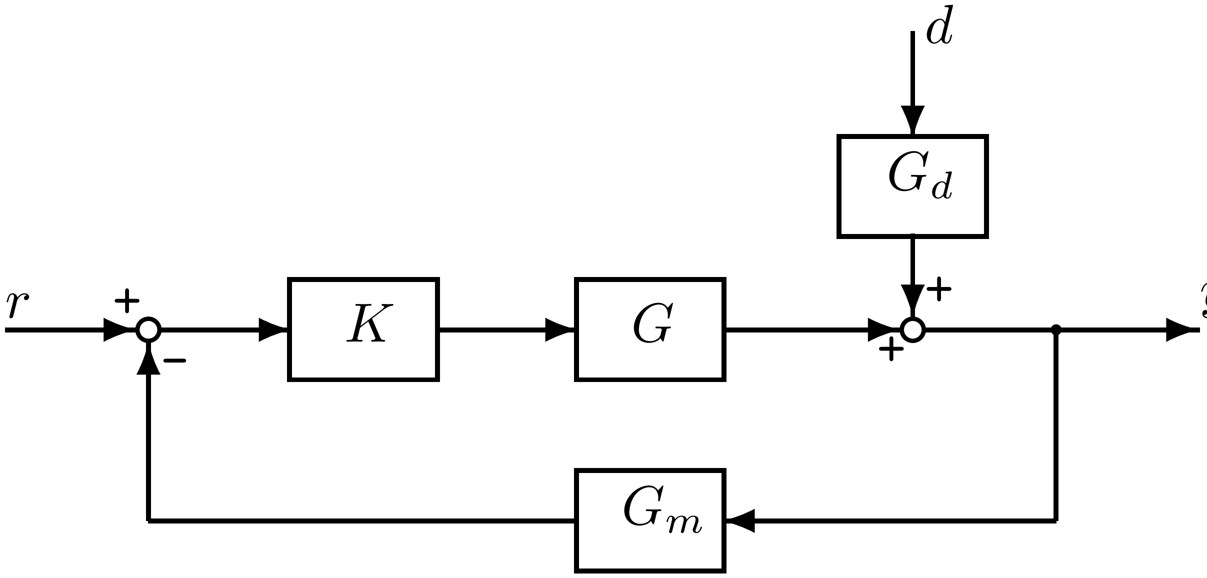


Figure 24: Feedback control system

$$y = G(s)u + G_d(s)d; \quad y_m = G_m(s)y \qquad (3.41)$$

$G_m(0) = 1$ (perfect steady-state measurement);

d, u, y and r are assumed to be scaled;

ω_c = gain crossover frequency (frequency where $|L(j\omega)|$ crosses 1 from above);

ω_d = frequency where $|G_d(j\omega_d)|$ first crosses 1 from above.

The following rules apply:

Rule 1. Speed of response to reject disturbances. *We require* $\omega_c > \omega_d$. *More specifically,* $|S(j\omega)| \leq |1/G_d(j\omega)| \ \forall\omega$.

Rule 2. Speed of response to track reference changes. *We require* $|S(j\omega)| \leq 1/R$ *up to the frequency* ω_r *where tracking is required.*

Rule 3. Input constraints arising from disturbances. *For acceptable control (*$|e| < 1$*) we require* $|G(j\omega)| > |G_d(j\omega)| - 1$ *at frequencies where* $|G_d(j\omega)| > 1$.

Rule 4. Input constraints arising from setpoints. *We require* $|G(j\omega)| > R - 1$ *up to the frequency* ω_r *where tracking is required.* (See (3.40)).

Rule 5. Time delay θ in $G(s)G_m(s)$. *We approximately require $\omega_c < 1/\theta$.* (See (3.32)).

Rule 6. Tight control at low frequencies with a RHP-zero z in $G(s)G_m(s)$. *For a real RHP-zero we require $\omega_c < z/2$.* (See (3.34)).

Rule 7. Phase lag constraint. *We require in most practical cases (e.g. with PID control): $\omega_c < \omega_u$. Here the* ultimate frequency *ω_u is where $\angle GG_m(j\omega_u) = -180°$.*

Rule 8. Real open-loop unstable pole in $G(s)$ at $s = p$. *We need high feedback gains to stabilize the system and require $\omega_c > 2p$.*

In addition, for unstable plants we need $|G| > |G_d|$ up to the frequency p (which may be larger than ω_d where $|G_d| = 1|$). Otherwise, the input may saturate when there are disturbances, and the plant cannot be stabilized.

3.10 Applications of controllability analysis [5.16]

3.10.1 First-order delay process [5.16.1]

Problem statement.

$$G(s) = k\frac{e^{-\theta s}}{1+\tau s}; \quad G_d(s) = k_d\frac{e^{-\theta_d s}}{1+\tau_d s}; \quad |k_d| > 1 \tag{3.42}$$

Also: measurement delays θ_m, θ_{md}

Specification: $|e| < 1$ for $|u| < 1, |d| < 1$.

i) feedback control only

ii) feedforward control only

Give quantitative relationships between the parameters which should be satisfied to achieve controllability.

Solution. For $|u| < 1$ we must from Rule 3 require $|G(j\omega)| > |G_d(j\omega)| \ \forall \omega < \omega_d$. For both feedback and feedforward

$$\boxed{k > k_d; \quad k/\tau > k_d/\tau_d} \tag{3.43}$$

(i) **Feedback control**. From Rule 1 for $|e| < 1$ with disturbances

$$\omega_d \approx k_d/\tau_d < \omega_c \tag{3.44}$$

On the other hand, from Rule 5 we require for stability and performance

$$\omega_c < 1/\theta_{tot} \tag{3.45}$$

where $\theta_{tot} = \theta + \theta_m$ is the total delay around the loop. (3.44) and (3.45) yield the following requirement for controllability

$$\boxed{\text{Feedback:} \qquad \theta + \theta_m < \tau_d/k_d} \tag{3.46}$$

(ii) **Feedforward control.** For $|e| < 1$ we need

$$\boxed{\text{Feedforward:} \qquad \theta + \theta_{md} - \theta_d < \tau_d/k_d} \tag{3.47}$$

3.10.2 Application: Room heating [5.16.2]

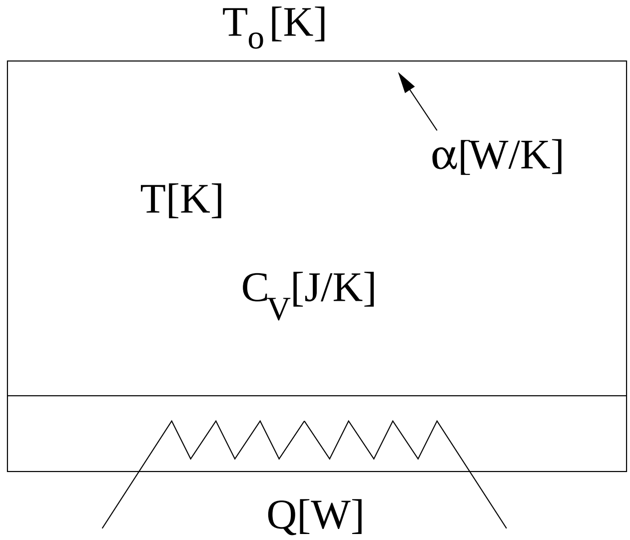


Figure 25: Room heating process

1. **Physical model.** Heat input Q, room temperature T (within $\pm 1K$), outdoor temperature T_o.

   Energy balance:

   $$\frac{d}{dt}(C_V T) = Q + \alpha(T_o - T) \qquad (3.48)$$

2. **Operating point.** Heat input Q^* is $2000W$, difference between indoor and outdoor temperatures $T^* - T_o^*$ is 20 K. The steady-state energy balance yields $\alpha^* = 2000/20 = 100W/K$. We assume $C_V = 100kJ/K$.

3. Linear model in deviation variables.

$$\begin{aligned} \delta T(t) &= T(t) - T^*; \\ \delta Q(t) &= Q(t) - Q^*; \\ \delta T_o(t) &= T_o(t) - T_o^* \end{aligned}$$

yields

$$C_V \frac{d}{dt}\delta T(t) = \delta Q(t) + \alpha(\delta T_o(t) - \delta T(t)) \qquad (3.49)$$

On taking Laplace transforms in (3.49), assuming $\delta T(t) = 0$ at $t = 0$ and rearranging we get

$$\delta T(s) = \frac{1}{\tau s + 1}\left(\frac{1}{\alpha}\delta Q(s) + \delta T_o(s)\right); \quad \tau = \frac{C_V}{\alpha} \qquad (3.50)$$

The time constant for this example is $\tau = 100 \cdot 10^3/100 = 1000s \approx 17min$

4. Linear model in scaled variables.

Introduce the following scaled variables

$$y(s) = \frac{\delta T(s)}{\delta T_{max}} \tag{3.51}$$

$$u(s) = \frac{\delta Q(s)}{\delta Q_{max}} \tag{3.52}$$

$$d(s) = \frac{\delta T_o(s)}{\delta T_{o,max}} \tag{3.53}$$

Acceptable variations in room temperature T are $\pm 1K$, i.e. $\delta T_{max} = \delta e_{max} = 1K$. The heat input can vary between $0W$ and $6000W$, since its nominal value is $2000W$ we have $\delta Q_{max} = 2000W$.

Expected variation in temperature are $\pm 10K$, i.e. $\delta T_{o,max} = 10K$.

The model becomes

$$G(s) = \frac{1}{\tau s + 1} \frac{\delta Q_{max}}{\delta T_{max}} \frac{1}{\alpha} = \frac{20}{1000s + 1} \tag{3.54}$$

$$G_d(s) = \frac{1}{\tau s + 1} \frac{\delta T_{o,max}}{\delta T_{max}} = \frac{10}{1000s + 1} \tag{3.55}$$

Measurement delay for temperature (y) be $\theta_m = 100s$.

Problem statement.

1. Is the plant controllable with respect to disturbances?

2. Is the plant controllable with respect to setpoint changes of magnitude $R = 3$ (± 3 K) when the desired response time for setpoint changes is $\tau_r = 1000$ s (17 min) ?

Solution.

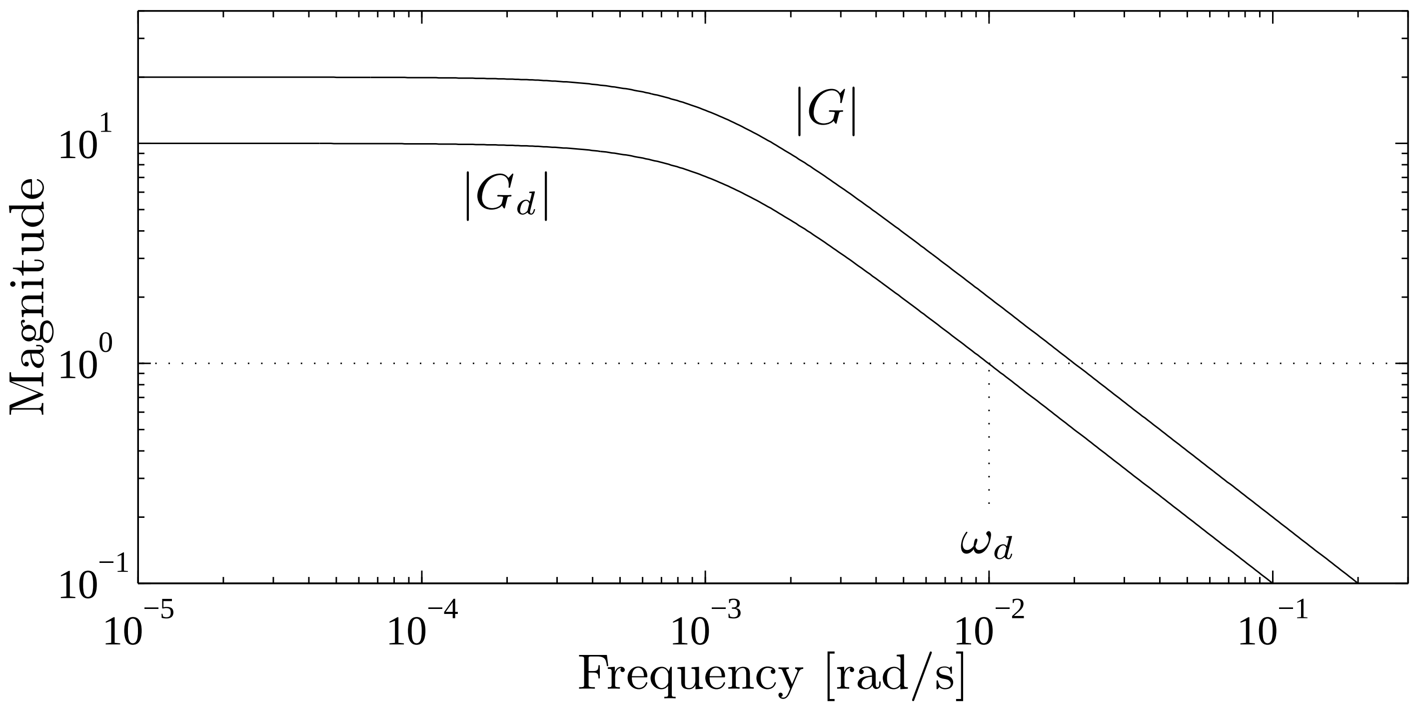
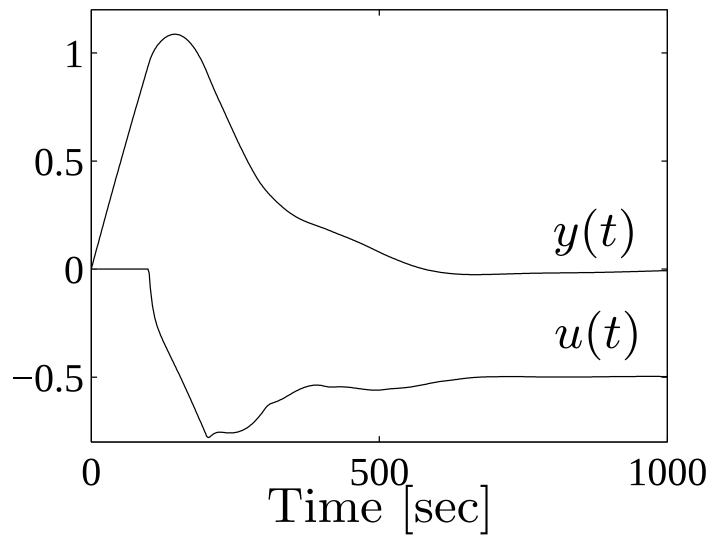


Figure 26: Frequency responses for room heating example

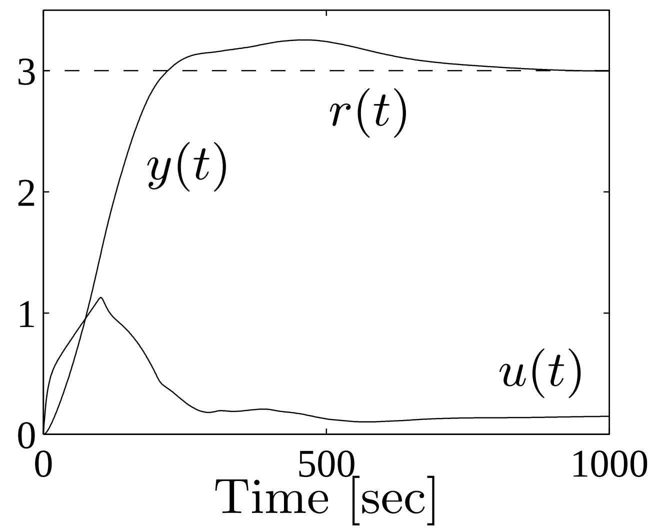
1. Disturbances. From Rule 1 feedback control is necessary up to the frequency $\omega_d = 10/1000 = 0.01$ rad/s, where $|G_d|$ crosses 1 in magnitude ($\omega_c > \omega_d$). This is exactly the same frequency as the upper bound given by the delay, $1/\theta = 0.01$ rad/s ($\omega_c < 1/\theta$). Therefore the system is barely controllable for this disturbance. From Rule 3 no problems with input constraints since $|G| > |G_d|$ at all frequencies. These conclusions are supported by the closed-loop simulation in Figure 27(a) using a PID-controller with $K_c = 0.4$ (scaled variables), $\tau_I = 200$ s and $\tau_D = 60$ s.

2. Setpoints. The plant is controllable with respect to the desired setpoint changes.

1. The delay (100 s) is much smaller than the desired response time of 1000 s
2. $|G(j\omega)| \geq R = 3$ up to about $\omega_1 = 0.007$ [rad/s] which is seven times higher than the required $\omega_r = 1/\tau_r = 0.001$ [rad/s]. This means that input constraints pose no problem. In fact, we achieve response times of about $1/\omega_1 = 150$ s without reaching the input constraints. See Figure 27(b) for a desired setpoint change $3/(150s + 1)$ using the same PID controller as above.



(a) Step disturbance in outdoor temperature



(b) Setpoint change $3/(150s+1)$

Figure 27: PID feedback control of room heating example

3.10.3 * Application: Neutralization process [5.16.3]

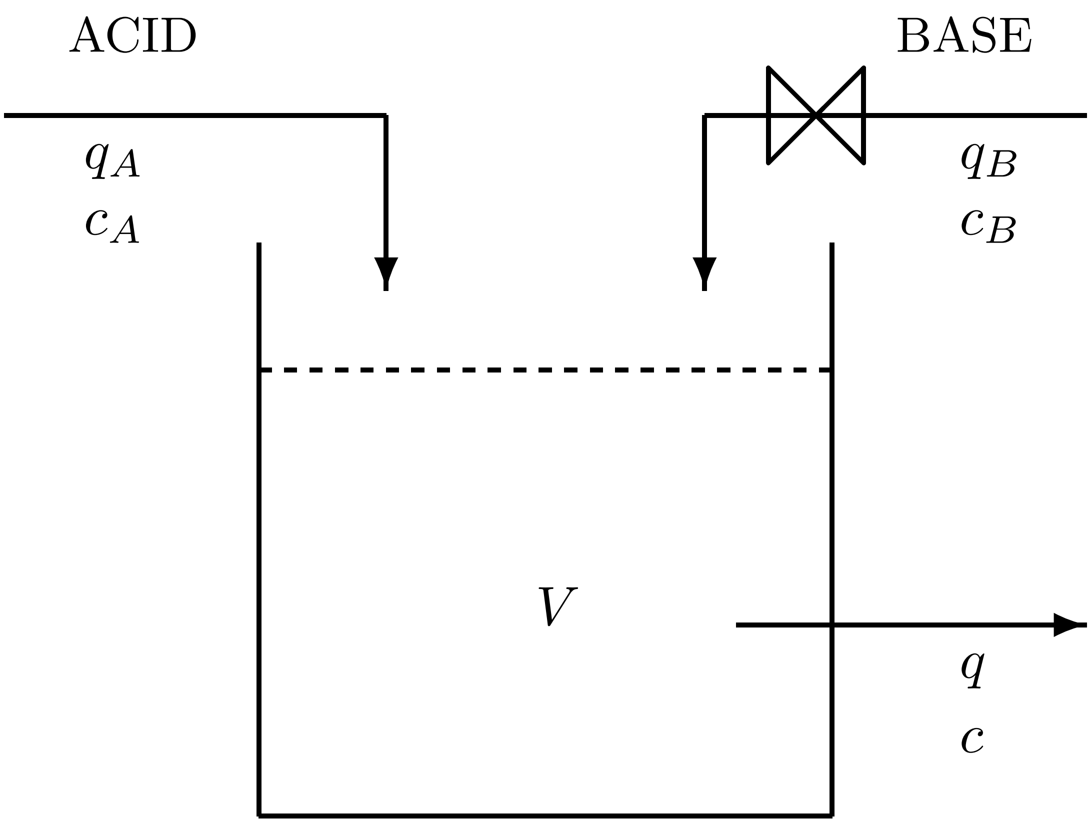


Figure 28: Neutralization process with one mixing tank

Problem statement. Consider process in Figure 28, where a strong acid with pH$= -1$ is neutralized by a strong base (pH=15) in a mixing tank with volume $V = 10\text{m}^3$.

Feedback control to keep the pH in the product stream (output y) in the range 7 ± 1 ("salt water") by manipulating the amount of base, q_B (input u) in spite of variations in the flow of acid, q_A (disturbance d). The delay in the pH-measurement is $\theta_m = 10$ s.

1. Controlled output is the excess of acid, c [mol/l], defined as $c = c_{H^+} - c_{OH^-}$.

2. Objective is to keep $|c| \leq c_{\max} = 10^{-6}$ mol/l, and the plant is

$$\frac{d}{dt}(Vc) = q_A c_A + q_B c_B - qc \tag{3.56}$$

   $q_A^* = q_B^* = 0.005$ [m^3/s] resulting in $q^* = 0.01$ [m^3/s]= 10 [l/s].

3. Scaled variables:

$$y = \frac{c}{10^{-6}}; \quad u = \frac{q_B}{q_B^*}; \quad d = \frac{q_A}{0.5 q_A^*} \tag{3.57}$$

4. Scaled linear model:

$$G_d(s) = \frac{k_d}{1 + \tau_h s}; \quad G(s) = \frac{-2k_d}{1 + \tau_h s}; \quad k_d = 2.5 \cdot 10^6 \tag{3.58}$$

   where $\tau_h = V/q = 1000$ s is the residence time for the liquid in the tank.

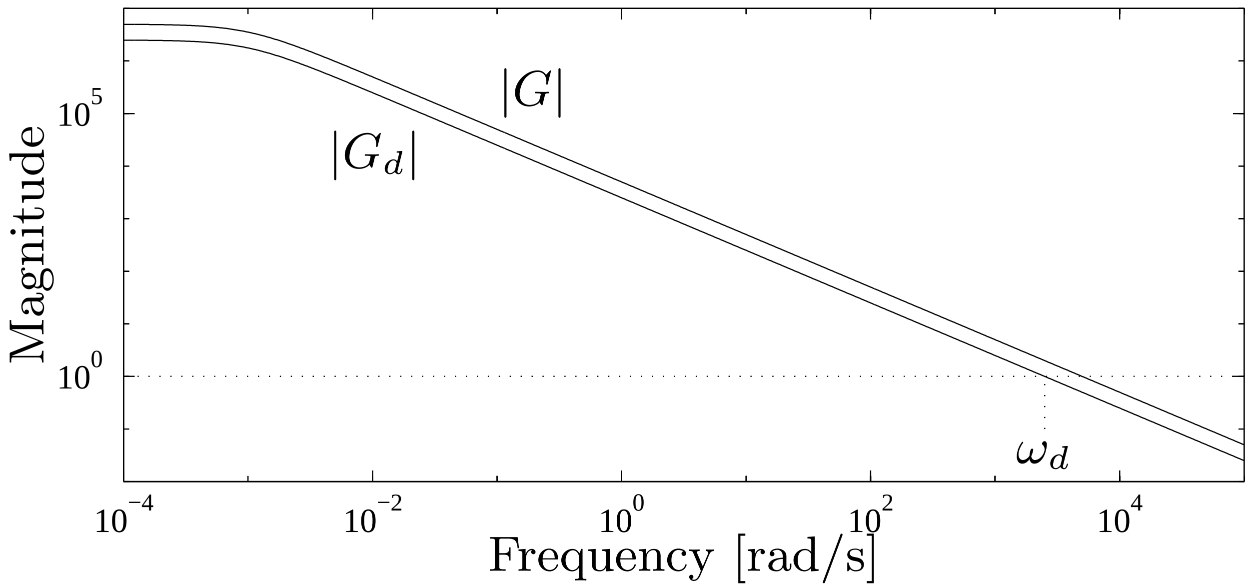


Figure 29: Frequency responses for the neutralization process with one mixing tank

Controllability analysis.

Figure 29: From Rule 2, input constraints do not pose a problem since $|G| = 2|G_d|$ at all frequencies. From Rule 1 we find the frequency up to which feedback is needed

$$\omega_d \approx k_d/\tau = 2500 \text{ rad/s} \qquad (3.59)$$

This requires a response time of $1/2500 = 0.4$ milliseconds which is clearly impossible in a process control application (also: delay of 10 s).

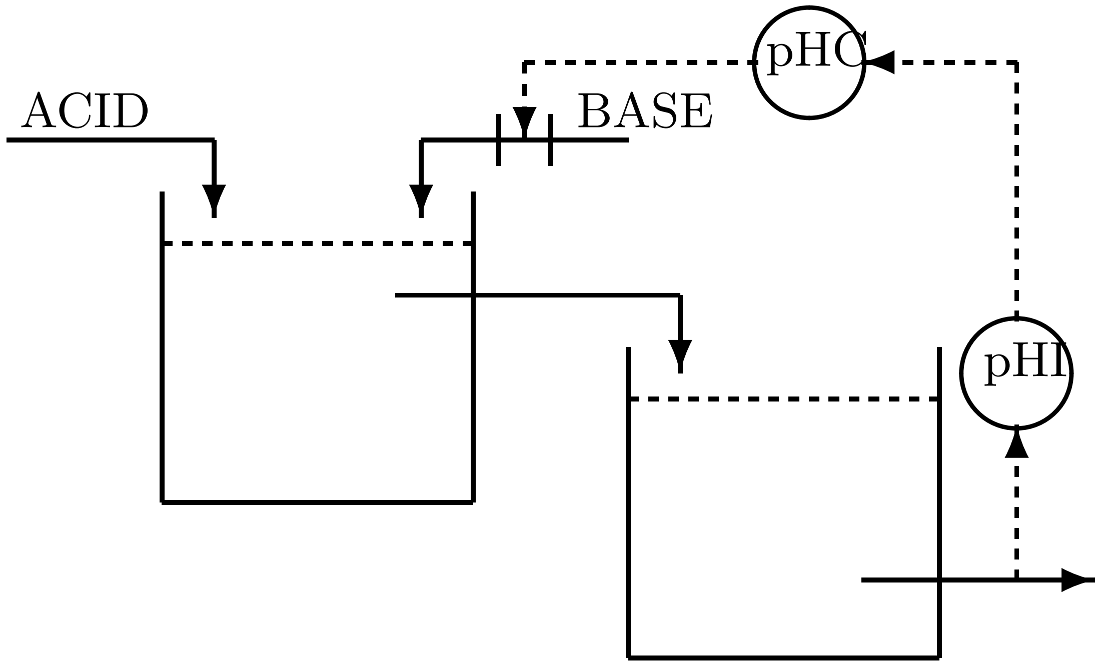


Figure 30: Neutralization process with two tanks and one controller

Design change: Multiple tanks.

To improve controllability modify the process ⇒ Perform the neutralization in several steps as illustrated in Figure 30 for the case of two tanks.

With n equal mixing tanks in series

$$G_d(s) = k_d h_n(s); \quad h_n(s) = \frac{1}{(\frac{\tau_h}{n}s + 1)^n} \qquad (3.60)$$

$h_n(s)$ is transfer function of the mixing tanks, and τ_h is total residence time, V_{tot}/q.

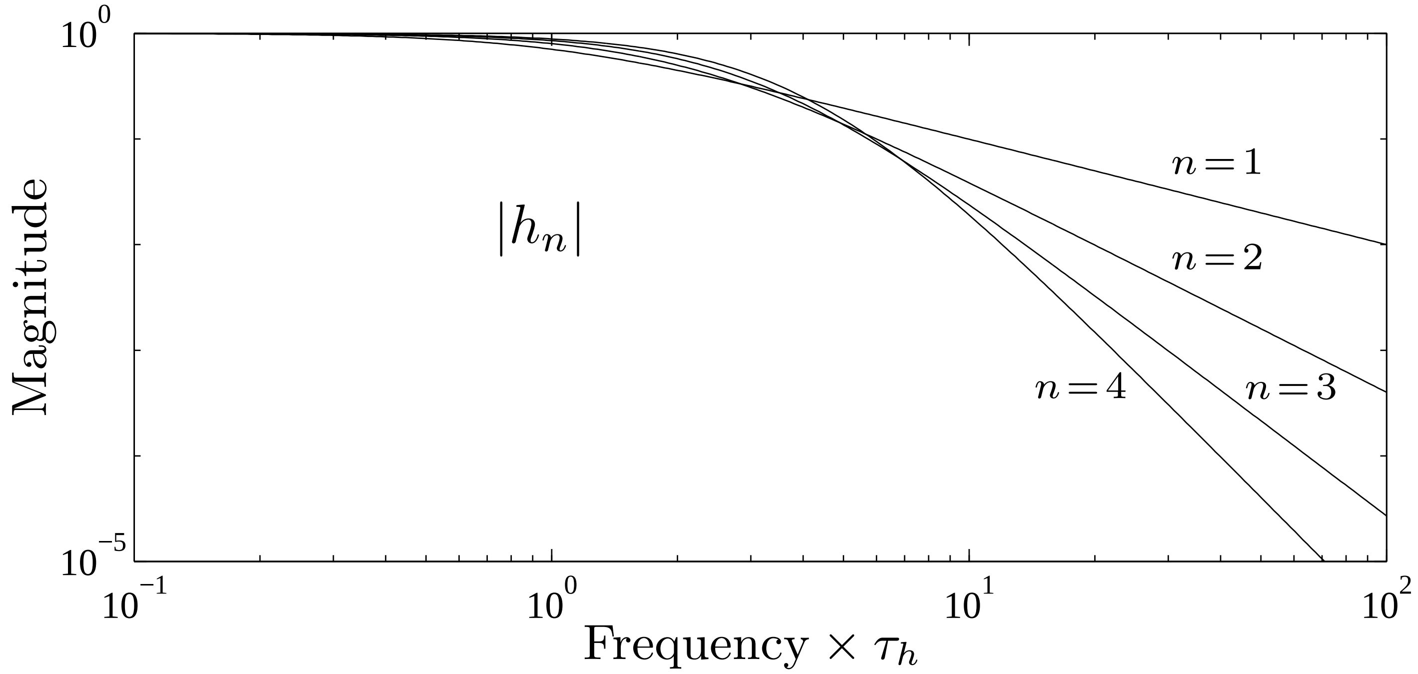


Figure 31: Frequency responses for n tanks in series with the same total residence time τ_h; $h_n(s) = 1/(\frac{\tau_h}{n}s + 1)^n,\ n = 1, 2, 3, 4$

From Rules 1 and 5, we must require

$$\boxed{|G_d(j\omega_\theta)| \leq 1} \quad \omega_\theta \stackrel{\Delta}{=} 1/\theta \tag{3.61}$$

where θ is the delay in the feedback loop. Purpose of mixing tanks $h_n(s)$ is to reduce the effect of the disturbance by a factor $k_d (= 2.5 \cdot 10^6)$ at the frequency $\omega_\theta (= 0.1$ [rad/s]), i.e. $|h_n(j\omega_\theta)| \leq 1/k_d$. Minimum value for the total volume for n equal tanks in series

$$V_{tot} = q\theta n \sqrt{(k_d)^{2/n} - 1} \tag{3.62}$$

where $q = 0.01$ m^3/s.

With $\theta = 10$ s we then find that the following designs have the same controllability

No. of tanks n	Total volume V_{tot} $[m^3]$	Volume each tank $[m^3]$
1	250000	250000
2	316	158
3	40.7	13.6
4	15.9	3.98
5	9.51	1.90
6	6.96	1.16
7	5.70	0.81

$n = 1 \Rightarrow$ Supertanker.

Minimum total volume is 3.662 m^3 with 18 tanks of about 203 liters each

Practical compromise: 3 or 4 tanks.

Control system design. We have $|S| < 1/|G_d|$ at the crossover frequency $\omega_B \approx \omega_c \approx \omega_\theta$. However, from Rule 1 we also require that $|S| < 1/|G_d|$, or approximately $|L| > |G_d|$, at frequencies lower than ω_c, (difficult since $G_d(s) = k_d h(s)$ is of high order). This requires $|L|$ to drop steeply with frequency, which results in a large negative phase for L

Thus, system in Figure 30 with a single feedback controller will *not* work. ⇒ install *local feedback* control system on each tank (Figure 32.).

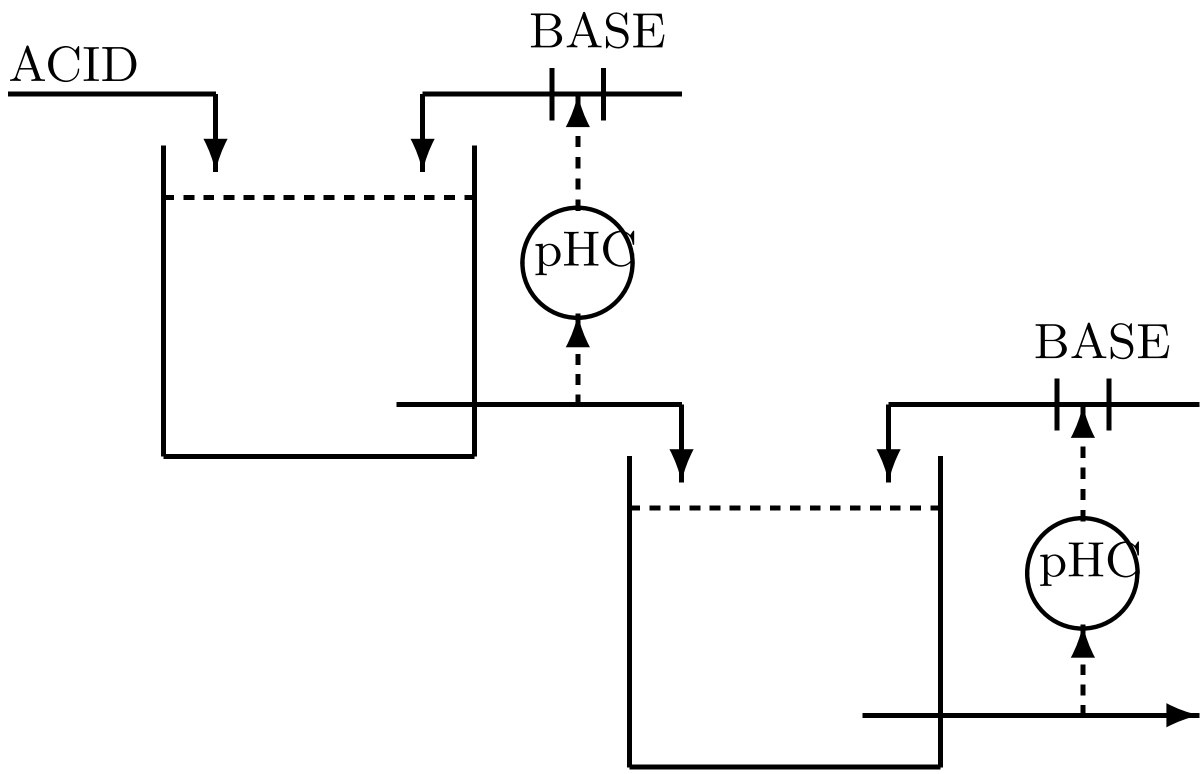


Figure 32: Neutralization process with two tanks and two controllers.

$\Rightarrow$*plant design change*

With n controllers for n tanks the overall closed-loop response from a disturbance into the first tank to the pH in the last tank becomes

$$y = G_d \prod_{i=1}^{n} (\frac{1}{1+L_i}) d \approx \frac{G_d}{L} d, \quad L \triangleq \prod_{i=1}^{n} L_i \qquad (3.63)$$

where $G_d = \prod_{i=1}^{n} G_i$ and $L_i = G_i K_i$, and the approximation applies at low frequencies where feedback is effective.

Design each loop $L_i(s)$ with a slope of -1 and bandwidth $\omega_c \approx \omega_\theta$, such that the overall loop transfer function L has slope $-n$ and achieves $|L| > |G_d|$ at all frequencies lower than ω_d (the size of the tanks are selected as before such that $\omega_d \approx \omega_\theta$).

4 UNCERTAINTY AND ROBUSTNESS FOR SISO SYSTEMS [7]

4.1 Introduction [7.1]

A control system is robust if it is insensitive to differences between the actual system and the model of the system which was used to design the controller. These differences are referred to as *model/plant mismatch* or simply *model uncertainty*.

Our approach is:

1. Determine the uncertainty set: find a mathematical representation of the model uncertainty ("clarify what we know about what we don't know").

2. Check Robust stability (RS): determine whether the system remains stable for all plants in the uncertainty set.

3. Check Robust performance (RP): if RS is satisfied, determine whether the performance specifications are met for all plants in the uncertainty set.

Notation:

Π – a set of possible perturbed plant models ("uncertainty set").

$G(s) \in \Pi$ – nominal plant model (with no uncertainty).

$G_p(s) \in \Pi$ and $G'(s) \in \Pi$ – particular perturbed plant models.

4.2 Classes of uncertainty [7.2]

1. **Parametric uncertainty.** Here the structure of the model (including the order) is known, but some of the parameters are uncertain.
2. **Neglected and unmodelled dynamics uncertainty.** Here the model is in error because of missing dynamics, usually at high frequencies, either through deliberate neglect or because of a lack of understanding of the physical process. Any model of a real system will contain this source of uncertainty.
3. **Lumped uncertainty.** Here the uncertainty description represents one or several sources of parametric and/or unmodelled dynamics uncertainty combined into a single lumped perturbation of a chosen structure.

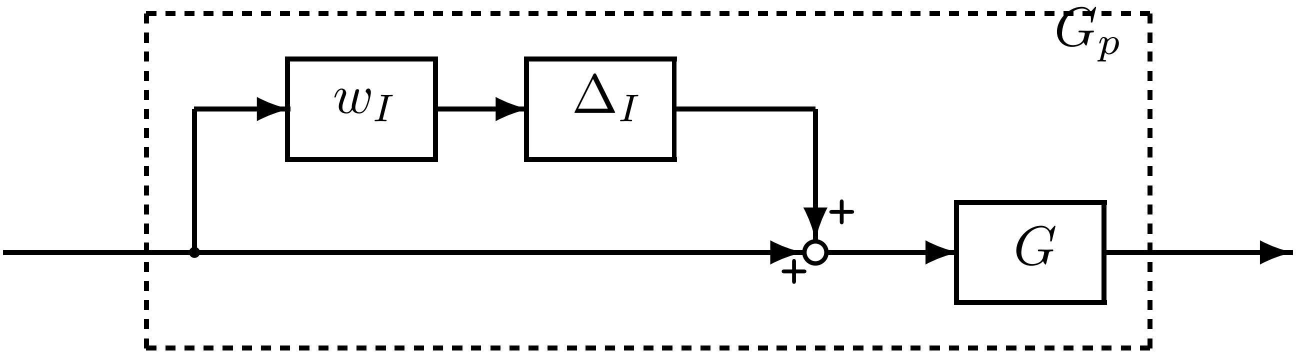


Figure 33: Plant with multiplicative uncertainty

Multiplicative uncertainty of the form

$\Pi_I : \quad G_p(s) = G(s)(1 + w_I(s)\Delta_I(s));$

where

$$\underbrace{|\Delta_I(j\omega)| \leq 1 \ \forall\omega}_{\|\Delta_I\|_\infty \leq 1} \tag{4.1}$$

Here $\Delta_I(s)$ is *any* stable transfer function which at each frequency is less than or equal to one in magnitude. Some allowable $\Delta_I(s)$'s

$$\frac{s-z}{s+z}, \quad \frac{1}{\tau s + 1}, \quad \frac{1}{(5s+1)^3}, \quad \frac{0.1}{s^2 + 0.1s + 1}$$

Inverse multiplicative uncertainty

$$\Pi_{iI} : \quad G_p(s) = G(s)(1 + w_{iI}(s)\Delta_{iI}(s))^{-1};$$
$$|\Delta_{iI}(j\omega)| \leq 1 \ \forall\omega \tag{4.2}$$

4.3 Representing uncertainty in the frequency domain [7.4]

4.3.1 Uncertainty regions [7.4.1]

Example:

$$G_p(s) = \frac{k}{\tau s + 1} e^{-\theta s}, \quad 2 \leq k, \theta, \tau \leq 3 \tag{4.3}$$

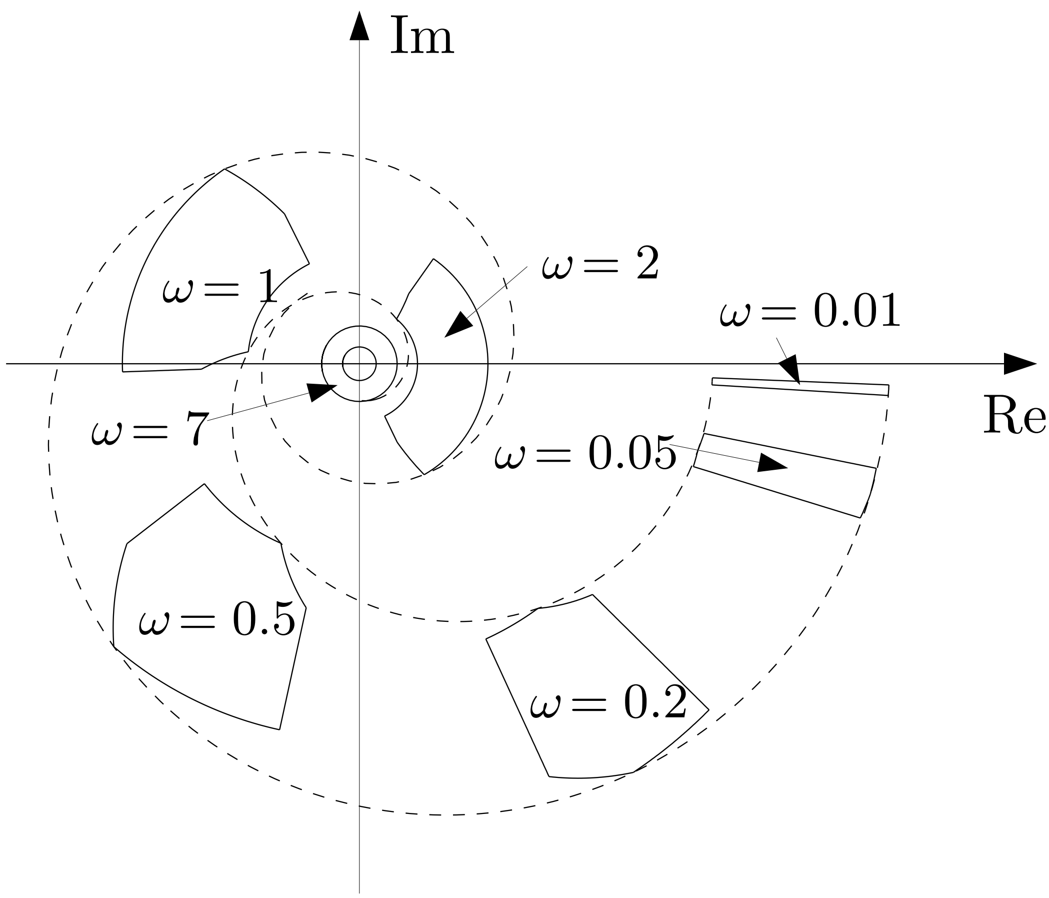


Figure 34: Uncertainty regions of the Nyquist plot at given frequencies. Data from (4.3)

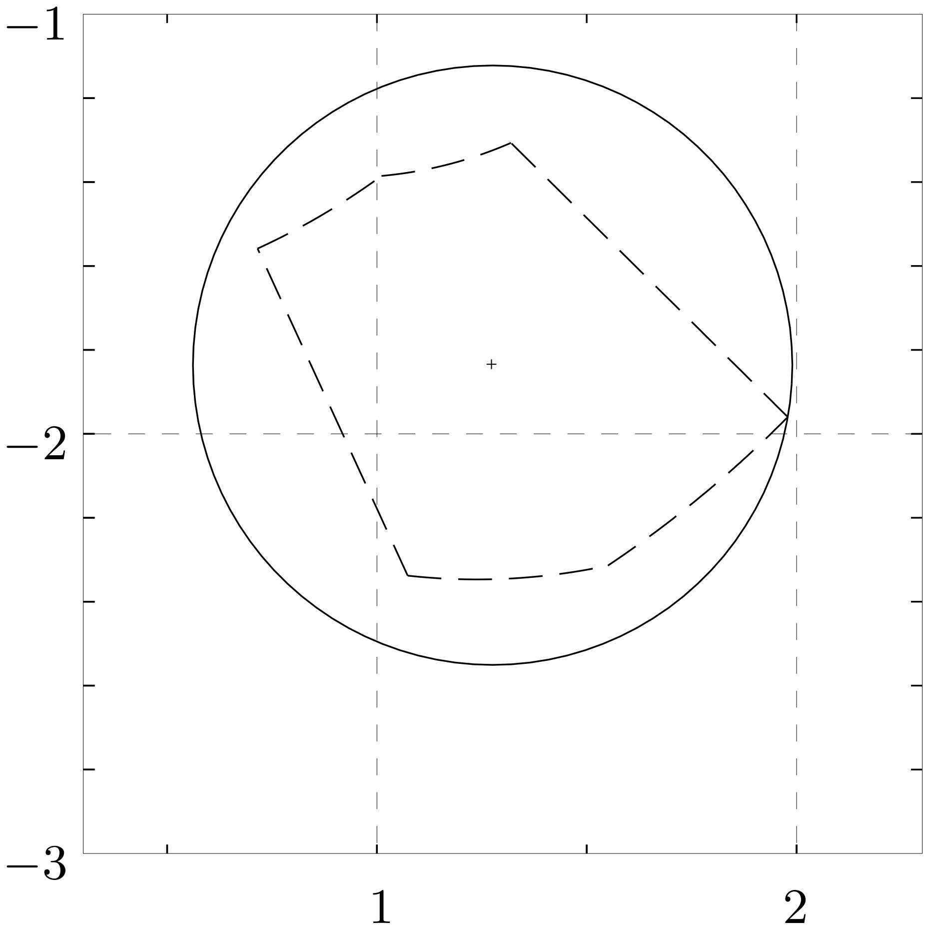


Figure 35: Disc approximation (solid line) of the original uncertainty region (dashed line). Plot corresponds to $\omega = 0.2$ in Figure 34

4.3.2 Approximation by complex perturbations [7.4.2]

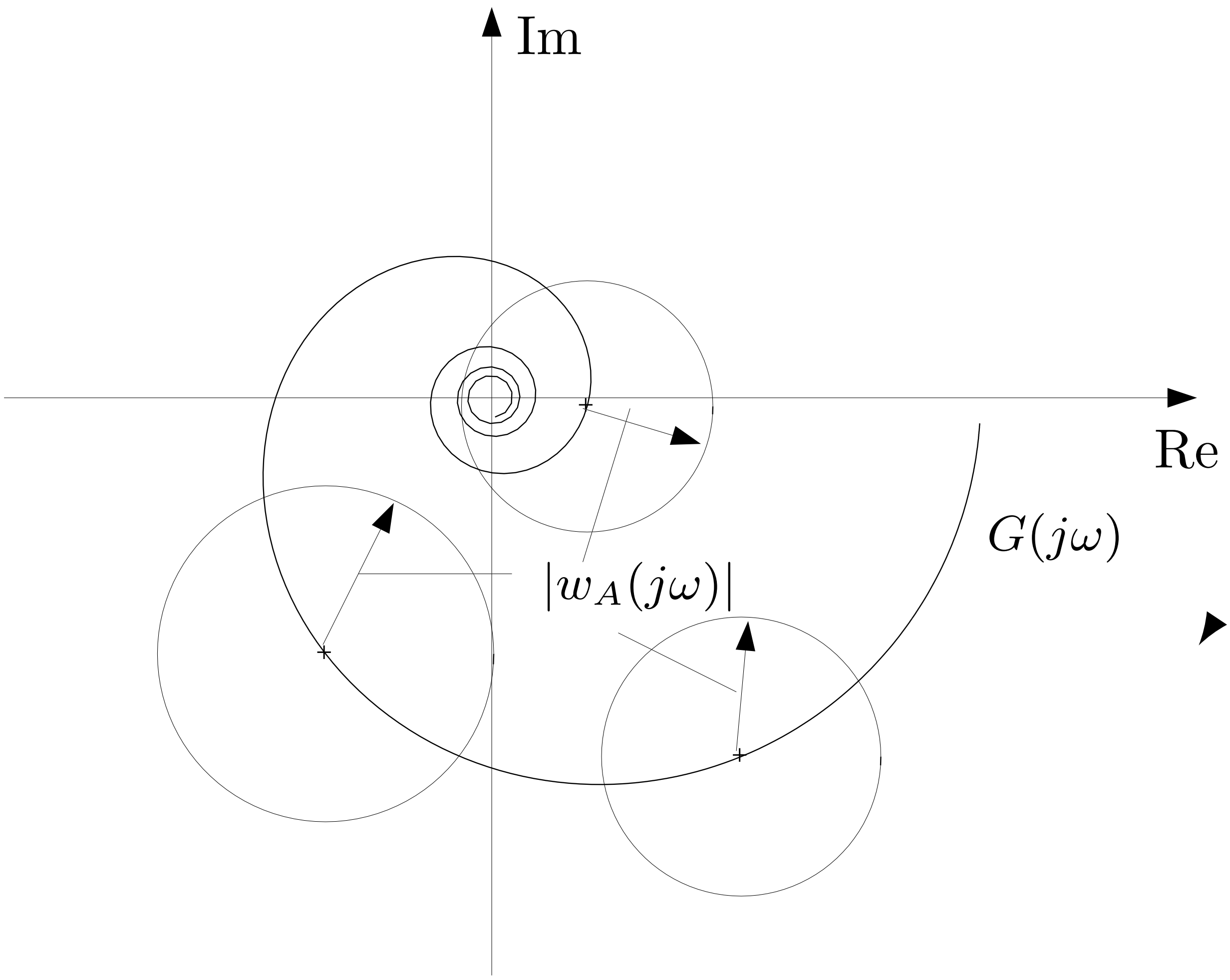


Figure 36: Disc-shaped uncertainty regions generated by complex additive uncertainty, $G_p = G + w_A\Delta$

We use disc-shaped regions to represent uncertainty regions (Figures 35 and 36) generated by

$$\Pi_A : \quad G_p(s) = G(s) + w_A(s)\Delta_A(s); \quad |\Delta_A(j\omega)| \leq 1 \, \forall \omega \tag{4.4}$$

where $\Delta_A(s)$ is *any* stable transfer function which at each frequency is no larger than one in magnitude.

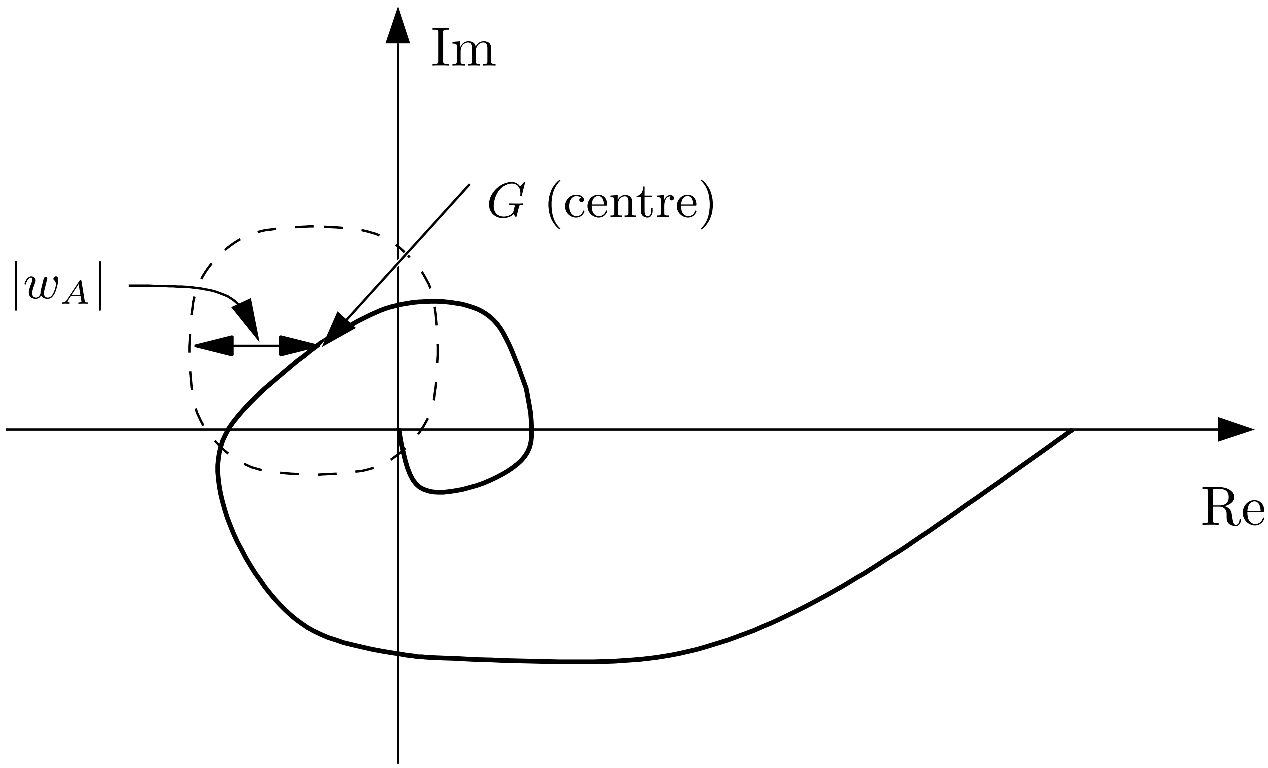


Figure 37: The set of possible plants includes the origin at frequencies where $|w_A(j\omega)| \geq |G(j\omega)|$, or equivalently $|w_I(j\omega)| \geq 1$

Alternative: *multiplicative uncertainty* description as in (4.1),

$$\Pi_I: \quad G_p(s) = G(s)(1+w_I(s)\Delta_I(s)); \quad |\Delta_I(j\omega)| \leq 1, \forall\omega \tag{4.5}$$

(4.4) and (4.5) are equivalent if at each frequency

$$|w_I(j\omega)| = |w_A(j\omega)|/|G(j\omega)| \tag{4.6}$$

4.3.3 Obtaining the weight for complex uncertainty [7.4.3]

1. Select a nominal model $G(s)$.
2. *Additive uncertainty.* At each frequency find the smallest radius $l_A(\omega)$ which includes all the possible plants Π:

$$|w_A(jw)| \geq l_A(\omega) = \max_{G_P \in \Pi} |G_p(j\omega) - G(j\omega)| \tag{4.7}$$

3. *Multiplicative (relative) uncertainty.* (preferred uncertainty form)

$$|w_I(jw)| \geq l_I(\omega) = \max_{G_p \in \Pi} \left| \frac{G_p(j\omega) - G(j\omega)}{G(j\omega)} \right| \tag{4.8}$$

Example 1 Multiplicative weight for parametric uncertainty. *Consider again the set of plants with parametric uncertainty given in (4.3)*

$$\Pi: \quad G_p(s) = \frac{k}{\tau s + 1} e^{-\theta s}, \quad 2 \leq k, \theta, \tau \leq 3 \qquad (4.9)$$

We want to represent this set using multiplicative uncertainty with a rational weight $w_I(s)$. We select a delay-free nominal model

$$G(s) = \frac{\bar{k}}{\bar{\tau} s + 1} = \frac{2.5}{2.5 s + 1} \qquad (4.10)$$

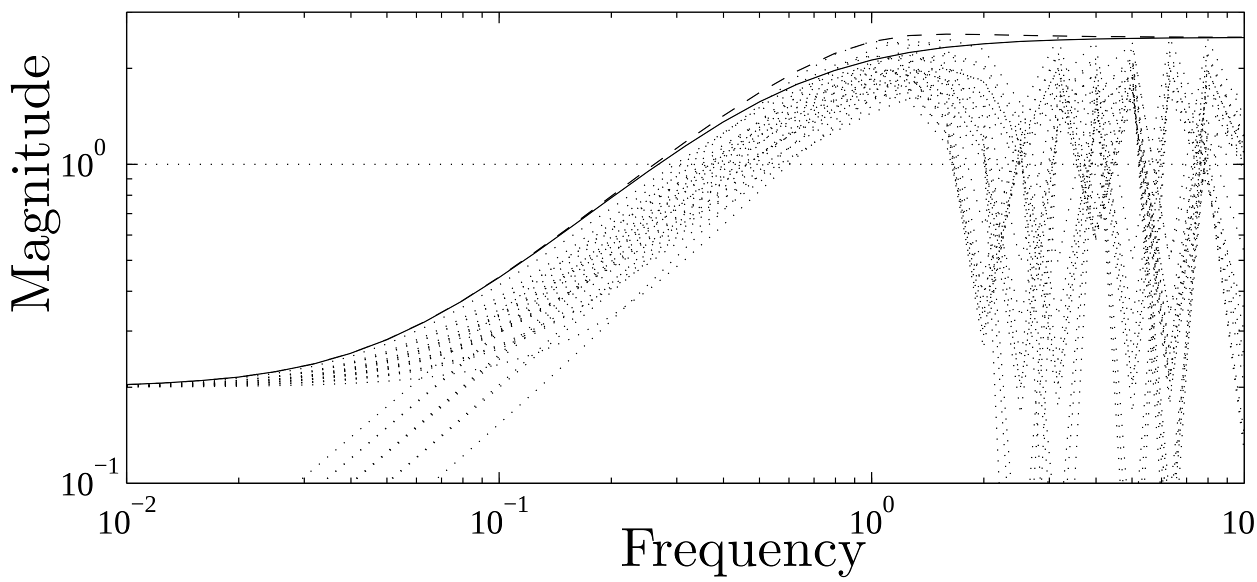


Figure 38: Relative errors for 27 combinations of k, τ and θ with delay-free nominal plant (dotted lines). Solid line: First-order weight $|w_{I1}|$ in (4.11). Dashed line: Third-order weight $|w_I|$ in (4.12)

$$w_{I1}(s) = \frac{Ts + 0.2}{(T/2.5)s + 1}, \quad T = 4 \tag{4.11}$$

$$w_I(s) = \omega_{I1}(s) \; \frac{s^2 + 1.6s + 1}{s^2 + 1.4s + 1} \tag{4.12}$$

4.4 SISO Robust stability [7.5]

4.4.1 RS with multiplicative uncertainty

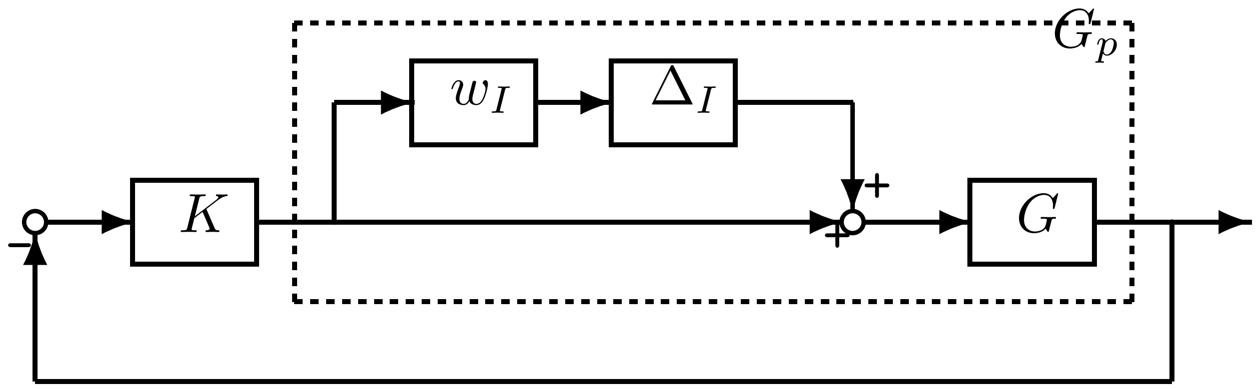


Figure 39: Feedback system with multiplicative uncertainty

Graphical derivation of RS-condition.
In Figure 40 $|-1-L| = |1+L|$ is the distance from the point -1 to the centre of the disc representing L_p, $|w_I L|$ is the radius of the disc. Encirclements are avoided if none of the discs cover -1, and we get from Figure 40

$$\text{RS} \Leftrightarrow |w_I L| < |1+L|, \quad \forall\omega \tag{4.13}$$

$$\Leftrightarrow \left|\frac{w_I L}{1+L}\right| < 1, \forall\omega \Leftrightarrow |w_I T| < 1, \forall\omega \tag{4.14}$$

$$\overset{\text{def}}{\Leftrightarrow} \|w_I T\|_\infty < 1 \tag{4.15}$$

$$\boxed{\text{RS} \Leftrightarrow |T| < 1/|w_I|, \quad \forall\omega} \tag{4.16}$$

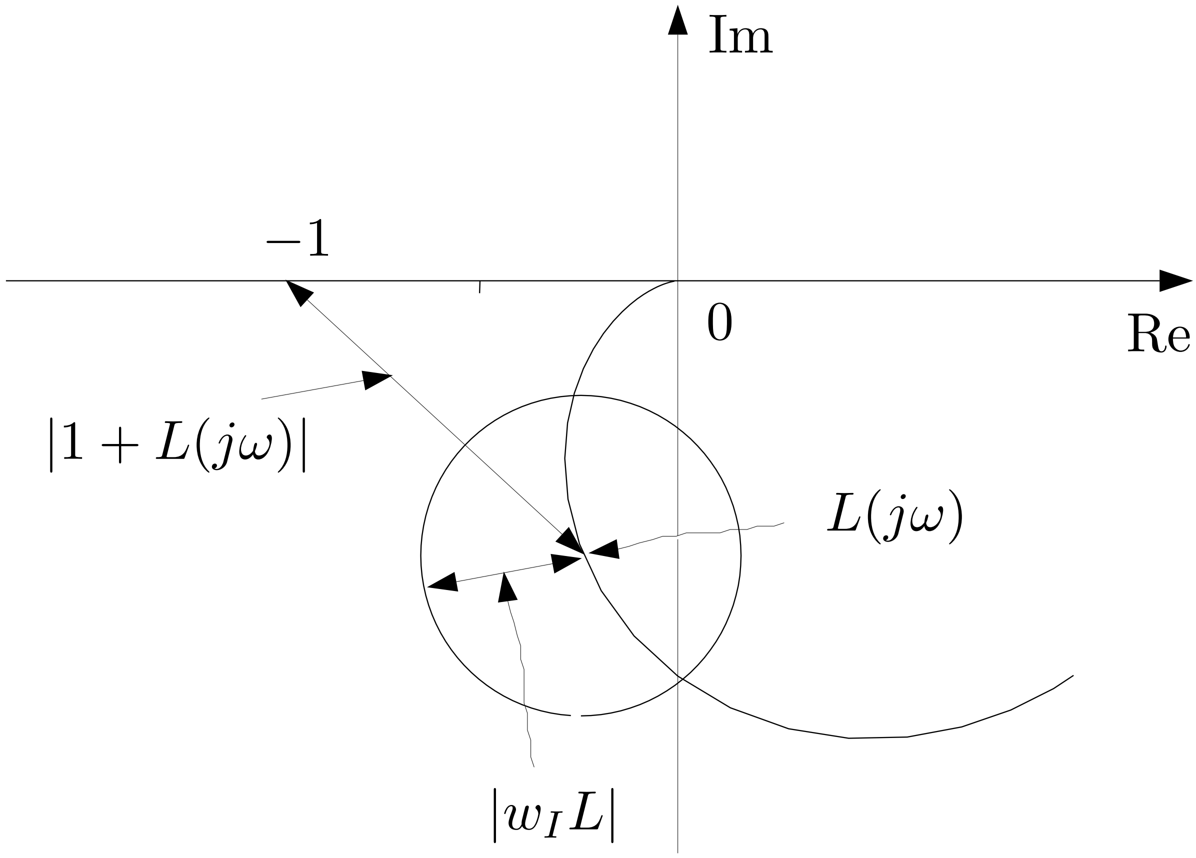


Figure 40: Nyquist plot of L_p for robust stability

Example 2 *Consider the following nominal plant and PI-controller*

$$G(s) = \frac{3(-2s+1)}{(5s+1)(10s+1)} \quad K(s) = K_c \frac{12.7s+1}{12.7s}$$

$K_c = K_{c1} = 1.13$ *(Ziegler-Nichols). One "extreme" uncertain plant is* $G'(s) = 4(-3s+1)/(4s+1)^2$*. For this plant the relative error* $|(G'-G)/G|$ *is* 0.33 *at low frequencies; it is* 1 *at about* 0.1 *rad/s, and it is* 5.25 *at high frequencies* $\Rightarrow$ *uncertainty weight*

$$w_I(s) = \frac{10s+0.33}{(10/5.25)s+1}$$

which closely matches this relative error.

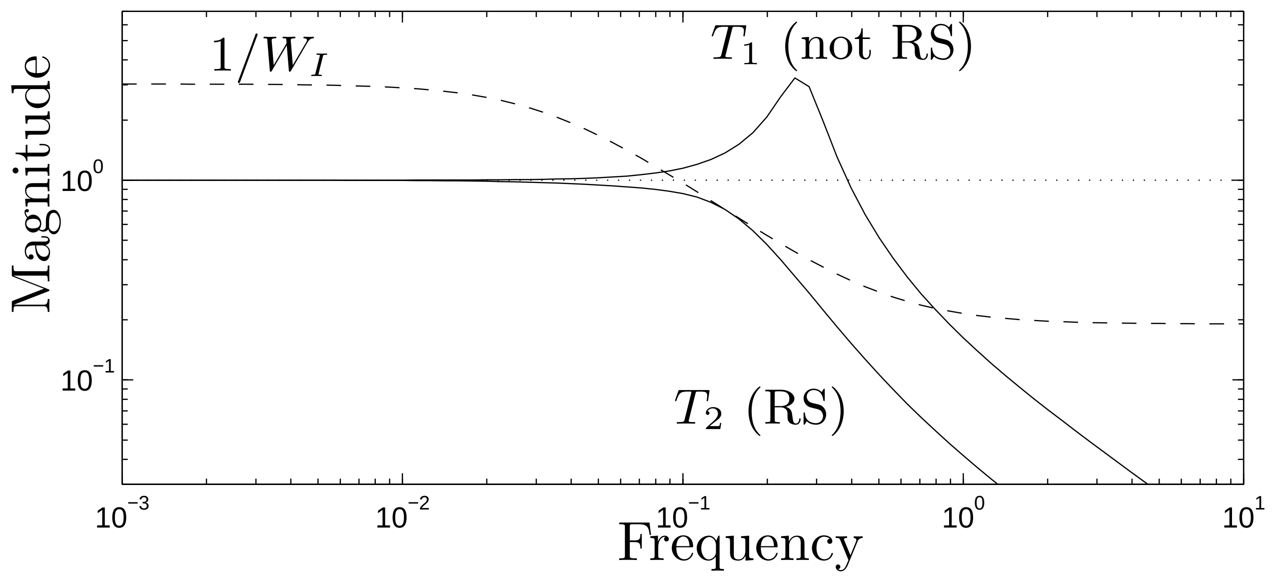


Figure 41: Checking robust stability with multiplicative uncertainty

By trial and error we find that reducing the gain to $K_{c2} = 0.31$ just achieves RS as seen from T_2 in Fig. 41.

Remark:

The procedure is *conservative*. For K_{c2} the system with the "extreme" plant is not at the limit of instability; we can increase the gain to $k_{c2} = 0.58$ before we get instability.

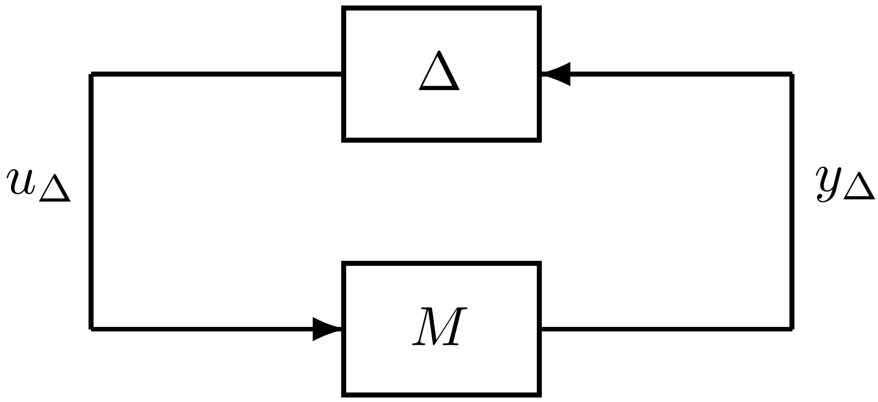


Figure 42: $M\Delta$-structure

$M\Delta$-structure derivation of RS-condition. The stability of the system in Figure 39 is equivalent to stability of the system in Figure 42, where $\Delta = \Delta_I$ and

$$M = w_I K(1 + GK)^{-1} G = w_I T \tag{4.17}$$

The Nyquist stability condition then determines RS if and only if the "loop transfer function" $M\Delta$ does not encircle -1 for all Δ. Thus,

$$\text{RS} \quad \Leftrightarrow \quad |1 + M\Delta| > 0, \quad \forall \omega, \forall |\Delta| \leq 1 \tag{4.18}$$

$$\text{RS} \quad \Leftrightarrow \quad 1 - |M(j\omega)| > 0, \quad \forall \omega \tag{4.19}$$

$$\Leftrightarrow \quad |M(j\omega)| < 1, \quad \forall \omega \tag{4.20}$$

4.4.2 RS with inverse multiplicative uncertainty [7.5.3]

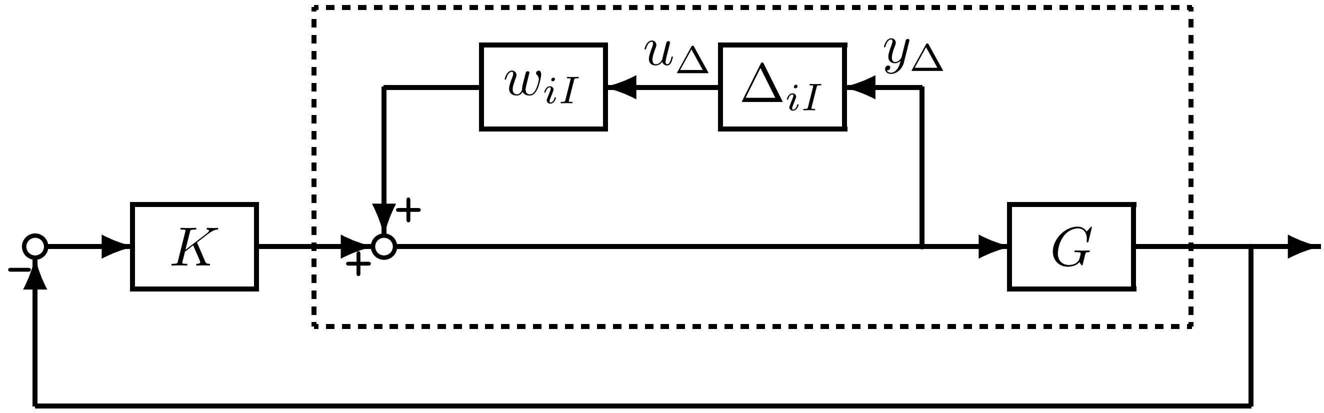


Figure 43: Feedback system with inverse multiplicative uncertainty

$$\boxed{\text{RS} \quad \Leftrightarrow \quad |S| < 1/|w_{iI}|, \quad \forall\omega} \tag{4.21}$$

4.5 SISO Robust performance [7.6]

4.5.1 Nominal performance in the Nyquist plot

$$\text{NP} \quad \Leftrightarrow \quad |w_P S| < 1 \quad \forall\omega \quad \Leftrightarrow \quad |w_P| < |1+L| \quad \forall\omega \tag{4.22}$$

See Figure:

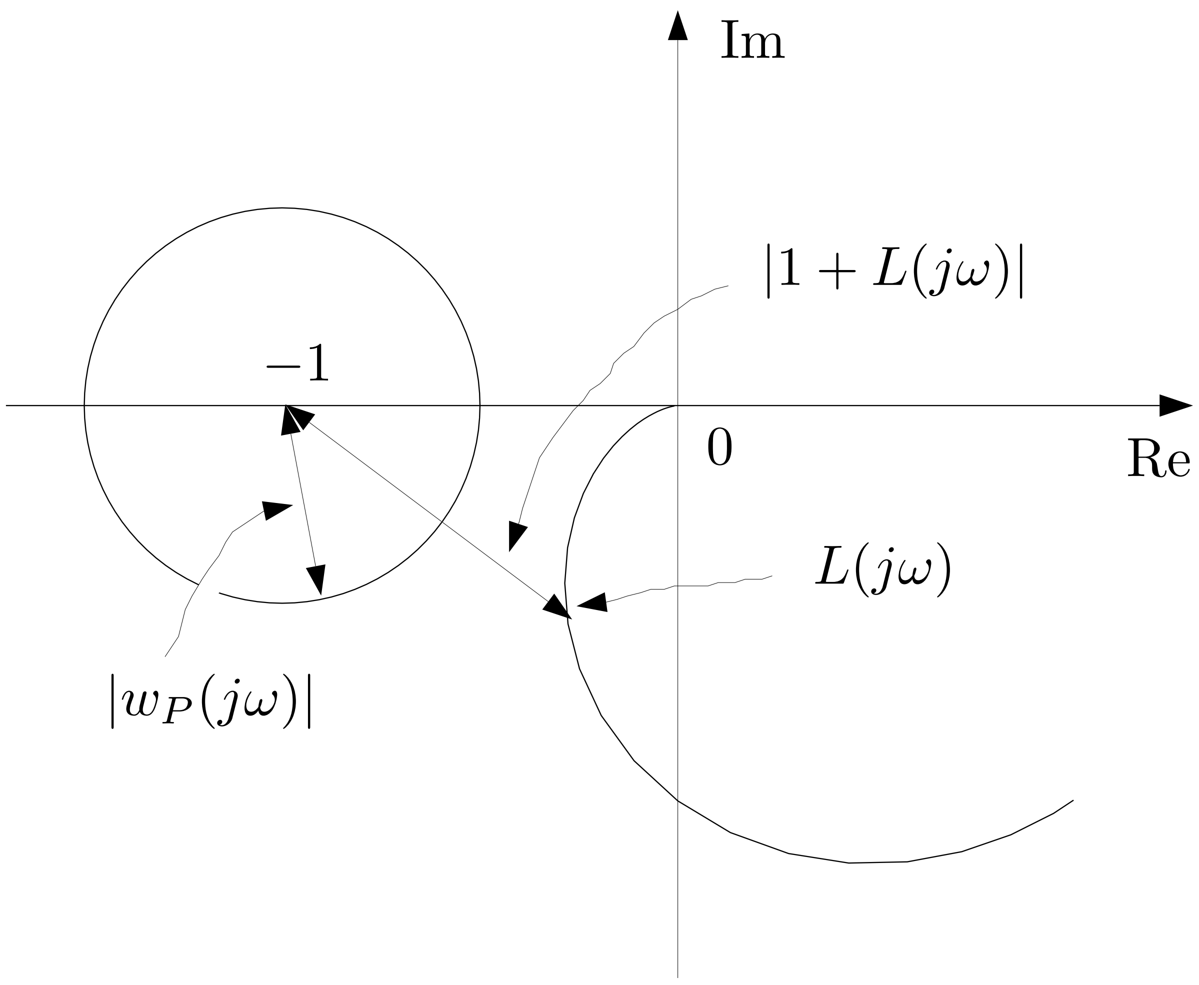


Figure 44: Nyquist plot illustration of nominal performance condition $|w_P| < |1 + L|$

4.5.2 Robust performance [7.6.2]

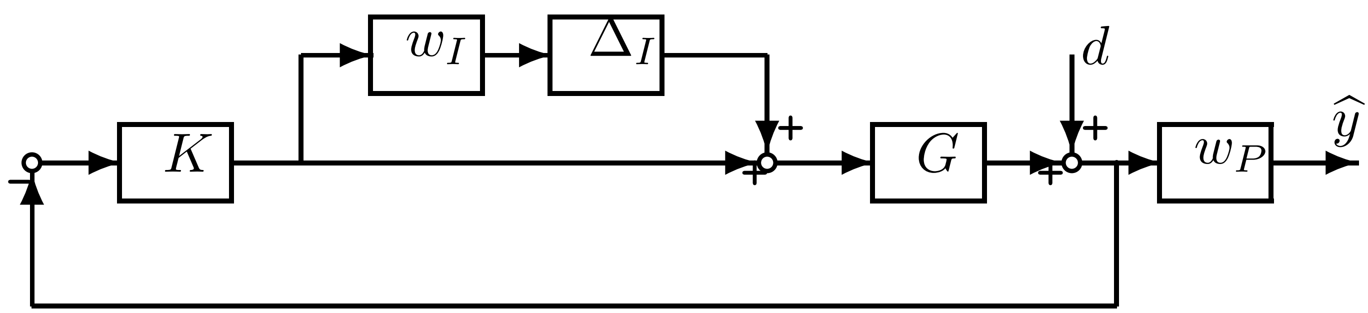


Figure 45: Diagram for robust performance with multiplicative uncertainty

For robust performance we require the performance condition (4.22) to be satisfied for *all* possible plants, that is, including the worst-case uncertainty.

$$\text{RP} \quad \stackrel{\text{def}}{\Leftrightarrow} \quad |w_P S_p| < 1 \quad \forall S_p, \forall \omega \tag{4.23}$$

$$\Leftrightarrow \quad |w_P| < |1 + L_p| \quad \forall L_p, \forall \omega \tag{4.24}$$

This corresponds to requiring $|\widehat{y}/d| < 1 \ \forall \Delta_I$ in Figure 45, where we consider multiplicative uncertainty, and the set of possible loop transfer functions is

$$L_p = G_p K = L(1 + w_I \Delta_I) = L + w_I L \Delta_I \tag{4.25}$$

Graphical derivation of RP-condition.
(Figure 46)

$$\text{RP} \quad \Leftrightarrow \quad |w_P| + |w_I L| < |1+L|, \quad \forall\omega \tag{4.26}$$
$$\Leftrightarrow \quad |w_P(1+L)^{-1}| + |w_I L(1+L)^{-1}| < 1, \forall\omega \tag{4.27}$$

$$\boxed{\text{RP} \;\Leftrightarrow\; \max_\omega \left(|w_P S| + |w_I T|\right) < 1} \tag{4.28}$$

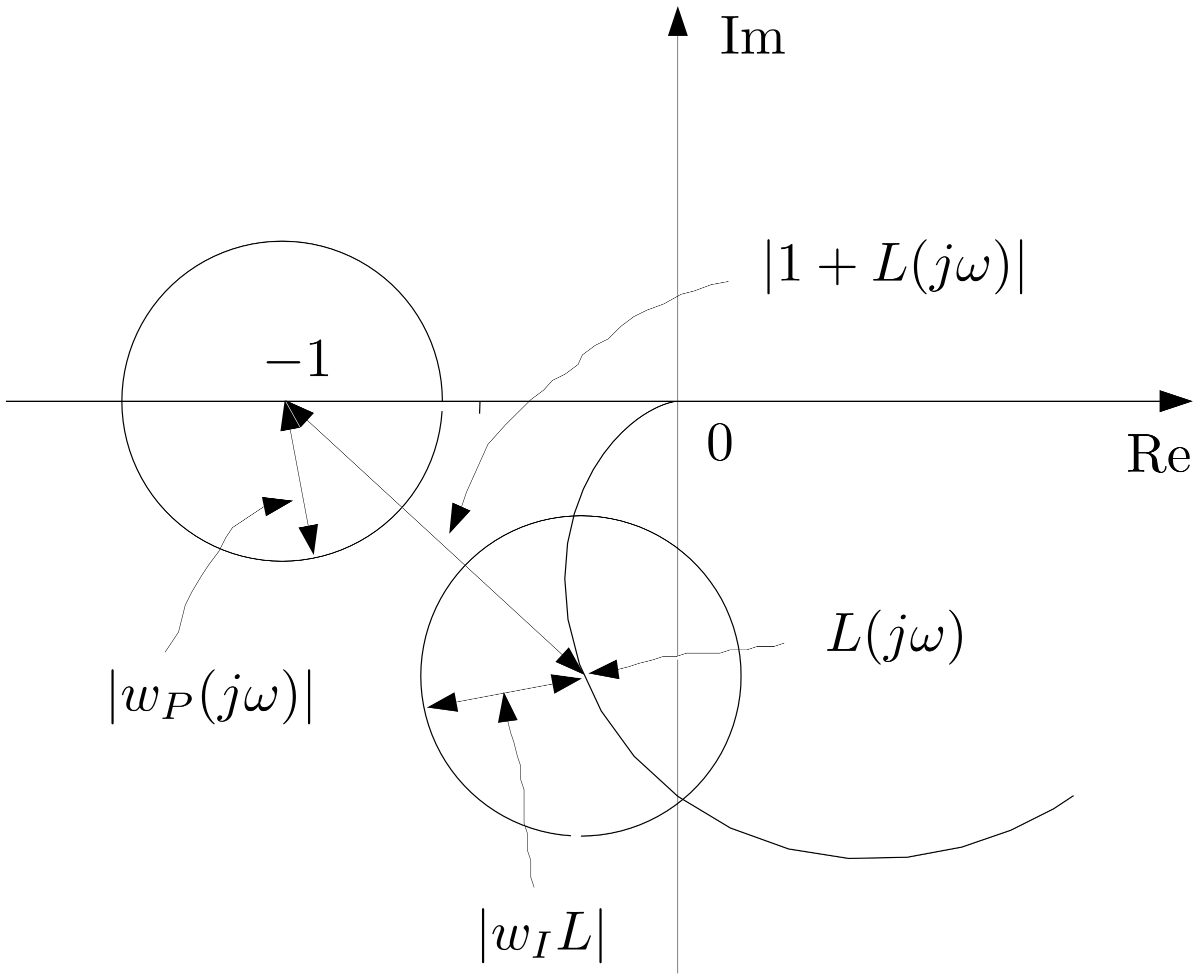


Figure 46: Nyquist plot illustration of robust performance condition $|w_P| < |1 + L_p|$

4.5.3 The relationship between NP, RS and RP [7.6.3]

$$\text{NP} \quad \Leftrightarrow \quad |w_P S| < 1, \forall \omega \tag{4.29}$$

$$\text{RS} \quad \Leftrightarrow \quad |w_I T| < 1, \forall \omega \tag{4.30}$$

$$\text{RP} \quad \Leftrightarrow \quad |w_P S| + |w_I T| < 1, \forall \omega \tag{4.31}$$

- A prerequisite for RP is that we satisfy NP and RS. This applies in general, both for SISO and MIMO systems and for any uncertainty.
- For SISO systems, if we satisfy both RS and NP, then we have at each frequency

$$|w_P S| + |w_I T| \leq 2 \max\{|w_P S|, |w_I T|\} < 2 \tag{4.32}$$

Therefore, within a factor of at most 2, we will automatically get RP when NP and RS are satisfied.

-

$$|w_P S| + |w_I T| \geq \min\{|w_P|, |w_I|\} \tag{4.33}$$

We *cannot have both* $|w_P| > 1$ *(i.e. good performance) and* $|w_I| > 1$ *(i.e. more than* 100% *uncertainty) at the same frequency.*

5 ELEMENTS OF LINEAR SYSTEM THEORY [4]

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{5.1}$$

$$y(t) = Cx(t) + Du(t) \tag{5.2}$$

or:

$$\begin{bmatrix} \dot{x} \\ y \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} \tag{5.3}$$

or:

$$G \stackrel{s}{=} \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right] \tag{5.4}$$

(5.1)–(5.3) is *not* a unique description of the input-output behaviour of a linear system.

Define new states $q = Sx$, i.e. $x = S^{-1}q$.

Equivalent state-space realization (i.e. with same input-output behaviour):

$$A_q = SAS^{-1}, \quad B_q = SB, \quad C_q = CS^{-1}, \quad D_q = D \tag{5.5}$$

Dynamical system response $x(t)$ for $t \geq t_0$

$$x(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)d\tau \tag{5.6}$$

For a system with disturbances d and measurement noise n:

$$\dot{x} = Ax + Bu + Ed \tag{5.7}$$

$$y = Cx + Du + Fd + n \tag{5.8}$$

Let $A_q = SAS^{-1} = \Lambda = \text{diag}\{\lambda_i\}$ be diagonal then

$e^{At} = S^{-1}\{\text{diag}(e^{\lambda_i t})\}S$

where $e^{\lambda_i t}$ is the *mode* associated with eigenvalue $\lambda_i(A)$.

5.1 System descriptions [4.1]

5.1.1 Impulse response [4.1.2]

The impulse response matrix is

$$g(t) = \begin{cases} 0 & t < 0 \\ Ce^{At}B + D\delta(t) & t \geq 0 \end{cases} \tag{5.9}$$

With initial state $x(0) = 0$, the dynamic response to an arbitrary input $u(t)$ is

$$y(t) = g(t) * u(t) = \int_0^t g(t-\tau)u(\tau)d\tau \tag{5.10}$$

where $*$ denotes the convolution operator.

5.1.2 Transfer function representation - Laplace transforms [4.1.3]

Laplace transforms of (5.1) and (5.3) become for $x = 0$

$$sx(s) = Ax(s) + Bu(s) \quad \Rightarrow$$

$$\Rightarrow \quad x(s) = (sI - A)^{-1}Bu(s) \tag{5.11}$$

$$y(s) = Cx(s) + Du(s) \quad \Rightarrow$$

$$\Rightarrow \quad y(s) = \underbrace{\left(C(sI - A)^{-1}B + D\right)}_{G(s)} u(s) \tag{5.12}$$

where $G(s)$ is the transfer function matrix. Equivalently,

$$G(s) = \frac{1}{\det(sI - A)}\left[C\mathrm{adj}(sI - A)B + D\det(sI - A)\right] \tag{5.13}$$

From Appendix A.2.1,

$$\det(sI - A) = \prod_{i=1}^{n} \lambda_i(sI - A) = \prod_{i=1}^{n}(s - \lambda_i(A)) \tag{5.14}$$

5.1.3 *State-space realizations [4.1.6]

Inverse system. For a square $G(s)$ we have

$$G^{-1} \stackrel{s}{=} \left[\begin{array}{c|c} A - BD^{-1}C & BD^{-1} \\ \hline -D^{-1}C & D^{-1} \end{array}\right] \tag{5.15}$$

If $D = 0$, set $D = \varepsilon I$. Be careful not to introduce RHP zeros with this modification.

Improper systems cannot be represented in state space form.

Realization of SISO transfer functions.

$$G(s) = \frac{\beta_{n-1}s^{n-1} + \cdots + \beta_1 s + \beta_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0} \tag{5.16}$$

$y(s) = G(s)u(s)$ corresponds to

$$\begin{gathered} y^n(t) + a_{n-1}y^{n-1}(t) + \cdots + a_1 y'(t) + a_0 y(t) = \\ \beta_{n-1}u^{n-1}(t) + \cdots + \beta_1 u'(t) + \beta_0 u(t) \end{gathered} \tag{5.17}$$

where $y^{n-1}(t)$ and $u^{n-1}(t)$ represent $n-1$'th order derivatives, etc.

Write this as

$$y^n = (-a_{n-1}y^{n-1} + \beta_{n-1}u^{n-1}) + \cdots$$

$$\cdots + \underbrace{(-a_1 y' + \beta_1 u') + \underbrace{(-a_0 y + \beta_0 u)}_{x_n'}}_{x_{n-1}^2}$$

With the notation $\dot{x} \equiv x'(t) = dx/dt$, we get

$$\begin{aligned} \dot{x}_n &= -a_0 x_1 + \beta_0 u \\ \dot{x}_{n-1} &= -a_1 x_1 + x_n + \beta_1 u \\ &\vdots \\ \dot{x}_1 &= -a_{n-1} x_1 + x_2 + \beta_{n-1} u \end{aligned}$$

corresponding to the realization (*observer canonical form*)

$$A = \begin{bmatrix} -a_{n-1} & 1 & 0 & \cdots & 0 & 0 \\ -a_{n-2} & 0 & 1 & & 0 & 0 \\ \vdots & \vdots & & \ddots & & \vdots \\ -a_2 & 0 & 0 & & 1 & 0 \\ -a_1 & 0 & 0 & \cdots & 0 & 1 \\ -a_0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} \beta_{n-1} \\ \beta_{n-2} \\ \vdots \\ \beta_2 \\ \beta_1 \\ \beta_0 \end{bmatrix} \tag{5.18}$$

$$C = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

Example: To obtain the state-space realization of $G(s) = \frac{s-a}{s+a}$, first bring out a constant term by division to get

$$G(s) = \frac{s-a}{s+a} = \frac{-2a}{s+a} + 1$$

Thus $D = 1$. Then (5.18) yields $A = -a, B = -2a$ and $C = 1$.

Example: Ideal PID-controller

$$K(s) = K_c(1 + \frac{1}{\tau_I s} + \tau_D s) = K_c \frac{\tau_I \tau_D s^2 + \tau_I s + 1}{\tau_I s} \quad (5.19)$$

$\Rightarrow$ Improper $\Rightarrow$ no realization

Proper PID controller

$$K(s) = K_c(1 + \frac{1}{\tau_I s} + \frac{\tau_D s}{1 + \epsilon \tau_D s}), \epsilon \leq 0.1 \quad (5.20)$$

Four common realizations

$$D = K_c \frac{1 + \epsilon}{\epsilon} \quad (5.21)$$

1. Diagonalized form (Jordan canonical form)

$$A = \begin{bmatrix} 0 & 0 \\ 0 & -\frac{1}{\epsilon \tau_D} \end{bmatrix}, \quad B = \begin{bmatrix} K_c/\tau_I \\ K_c/(\epsilon^2 \tau_D) \end{bmatrix}, \quad C = \begin{bmatrix} 1 & -1 \end{bmatrix} \quad (5.22)$$

2. Observability canonical form

$$A = \begin{bmatrix} 0 & 1 \\ 0 & -\frac{1}{\epsilon \tau_D} \end{bmatrix}, \quad B = \begin{bmatrix} \gamma_1 \\ \gamma_2 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix} \quad (5.23)$$

$$\text{where} \quad \gamma_1 = K_c(\frac{1}{\tau_I} - \frac{1}{\epsilon^2 \tau_D}), \quad \gamma_2 = \frac{K_c}{\epsilon^3 \tau_D^2} \quad (5.24)$$

3. Controllability canonical form

$$A = \begin{bmatrix} 0 & 0 \\ 1 & -\frac{1}{\epsilon\tau_D} \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} \gamma_1 & \gamma_2 \end{bmatrix} \quad (5.25)$$

4. Observer canonical form in (5.18)

$$A = \begin{bmatrix} -\frac{1}{\epsilon\tau_D} & 1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} \beta_1 \\ \beta_0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix} \quad (5.26)$$

$$\text{where} \quad \beta_0 = \frac{K_c}{\epsilon\tau_I\tau_D}, \quad \beta_1 = K_c\frac{\epsilon^2\tau_D - \tau_I}{\epsilon^2\tau_I\tau_D} \quad (5.27)$$

Note: Transfer function offers more immediate insight.

5.2 State controllability and state observability [4.2]

Definition

State controllability. The dynamical system $\dot{x} = Ax + Bu$, or equivalently the pair (A, B), is said to be state controllable if, for any initial state $x(0) = x_0$, any time $t_1 > 0$ and any final state x_1, there exists an input $u(t)$ such that $x(t_1) = x_1$. Otherwise the system is said to be state uncontrollable.

1. The pair: (A, B) is state controllable if and only if the controllability matrix

$$\mathcal{C} \triangleq [\, B \quad AB \quad A^2B \quad \cdots \quad A^{n-1}B \,] \qquad (5.28)$$

has rank n (full row rank). Here n is the number of states.

2. From (5.6) one can verify that for $x(t_1) = x_1$

$$u(t) = -B^T e^{A^T(t_1-t)} W_c(t_1)^{-1}(e^{At_1}x_0 - x_1) \tag{5.29}$$

where $W_c(t)$ is the Gramian matrix at time t,

$$W_c(t) \triangleq \int_0^t e^{A\tau}BB^T e^{A^T\tau} d\tau \tag{5.30}$$

Thus (A, B) is state controllable if and only if $W_c(t)$ has full rank (and thus is positive definite) for any $t > 0$. For a stable system (A is stable) check only $P \triangleq W_c(\infty)$,

$$P \triangleq \int_0^\infty e^{A\tau}BB^T e^{A^T\tau} d\tau \tag{5.31}$$

P may also be obtained as the solution to the Lyapunov equation

$$AP + PA^T = -BB^T \tag{5.32}$$

3. Let p_i be the i'th eigenvalue of A and q_i the corresponding left eigenvector, $q_i^H A = p_i q_i^H$. Then the system is state controllable if and only if $q_i^H B \neq 0, \forall i$.

Example:

$$A = \begin{bmatrix} -2 & -2 \\ 0 & -4 \end{bmatrix},\ B = \begin{bmatrix} 1 \\ 1 \end{bmatrix},\ C = [\,1 \quad 0\,],\ D = 0$$

The transfer function

$$G(s) = C(sI - A)^{-1}B = \frac{1}{s+4}$$

has only <u>one</u> state.

1. The controllability matrix has two linearly dependent rows:

$$\mathcal{C} = [\,B \quad AB\,] = \begin{bmatrix} 1 & -4 \\ 1 & -4 \end{bmatrix}.$$

2. The controllability Gramian is singular

$$P = \begin{bmatrix} 0.125 & 0.125 \\ 0.125 & 0.125 \end{bmatrix}$$

3. $p_1 = -2$ and $p_2 = -4$, $q_1 = [\,0.707 \quad -0.707\,]^T$ and $q_2 = [\,0 \quad 1\,]^T$.

$$q_1^H B = 0, \quad q_2^H B = 1$$

the first mode (eigenvalue) is not state controllable.

Controllability is a system-theoretic concept important for computation and realizations; but no practical insight:

1. It says nothing about how the states behave, e.g. it does not imply that one can *hold* (as $t \to \infty$) the states at a given value.

2. Required inputs may be very large with sudden changes.

3. Some states may be of no practical importance.

4. Existence result which provides no "degree of controllability".

Definition State observability. The dynamical system $\dot{x} = Ax + Bu$, $y = Cx + Du$ (or the pair (A, C)) is said to be state observable if, for any time $t_1 > 0$, the initial state $x(0) = x_0$ can be determined from the time history of the input $u(t)$ and the output $y(t)$ in the interval $[0, t_1]$. Otherwise the system, or (A, C), is said to be state unobservable.

1. (A, C) is state observable if and only if the observability matrix

$$\mathcal{O} \triangleq \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \tag{5.33}$$

has rank n (full column rank).

2. For a stable system the observability Gramian

$$Q \triangleq \int_0^\infty e^{A^T\tau} C^T C e^{A\tau} d\tau \tag{5.34}$$

must have full rank n (and thus be positive definite). Q can also be found as the solution to the following Lyapunov equation

$$A^T Q + QA = -C^T C \tag{5.35}$$

3. Let p_i be the i'th eigenvalue of A and t_i the corresponding eigenvector, $At_i = p_i t_i$. Then the system is state observable if and only if $Ct_i \neq 0, \forall i$.

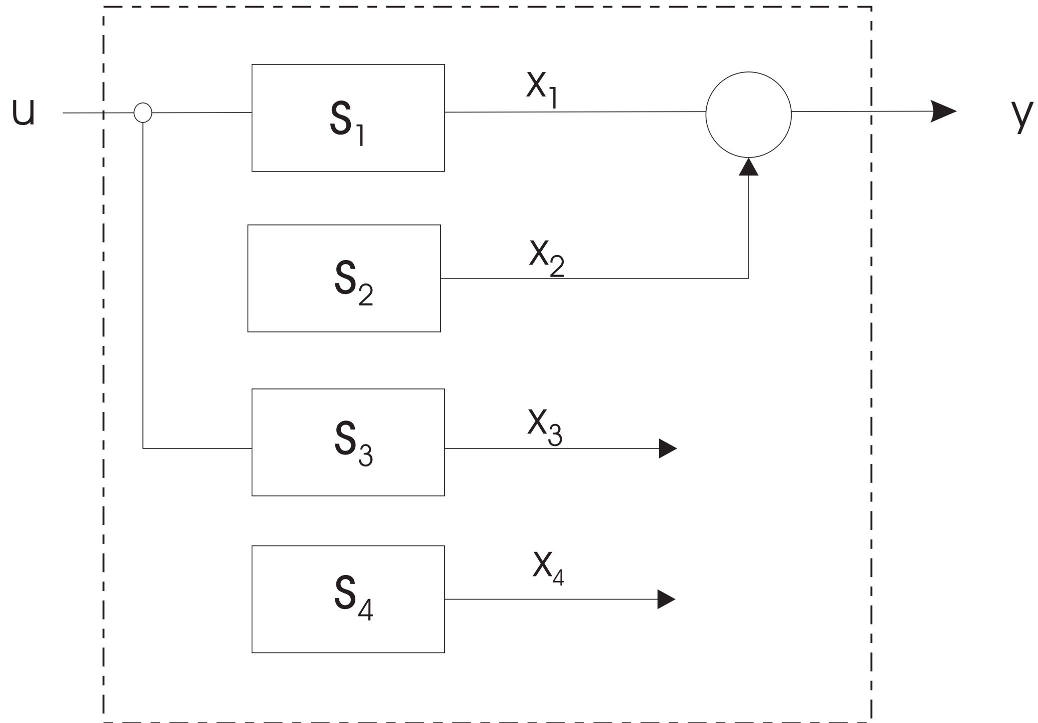
Observability is a system theoretical concept but may not give practical insight.

Kalman's decomposition

By performing an appropriate coordinate transformation, any system can be reduce to a decomposition indicating the state that are or aren't controllable and/or observable.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & 0 & 0 \\ 0 & A_{22} & 0 & 0 \\ A_{31} & A_{32} & A_{33} & A_{34} \\ 0 & A_{42} & 0 & A_{44} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} B_1 \\ 0 \\ B_3 \\ 0 \end{bmatrix} u$$

$$y = [\, C_1 \quad C_2 \quad 0 \quad 0 \,] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$



5.3 Stability [4.3]

Definition

A system is **(internally) stable** if none of its components contains hidden unstable modes and the injection of bounded external signals at any place in the system results in bounded output signals measured anywhere in the system. “internal”, i.e. all the states must be stable not only inputs/outputs.

Definition

State stabilizable, state detectable and hidden unstable modes. A system is state stabilizable if all unstable modes are state controllable. A system is state detectable if all unstable modes are state observable. A system with unstabilizable or undetectable modes is said to contain hidden unstable modes.

5.4 Poles [4.4]

Definition

Poles. The poles p_i of a system with state-space description (5.1)–(5.2) are the eigenvalues $\lambda_i(A), i = 1, \ldots, n$ of the matrix A. The pole or characteristic polynomial $\phi(s)$ is defined as $\phi(s) \triangleq \det(sI - A) = \prod_{i=1}^{n}(s - p_i)$. Thus the poles are the roots of the characteristic equation

$$\phi(s) \triangleq \det(sI - A) = 0 \tag{5.36}$$

5.4.1 Poles and stability

Theorem 6 *A linear dynamic system $\dot{x} = Ax + Bu$ is stable if and only if all the poles are in the open left-half plane (LHP), that is,* $\mathrm{Re}\{\lambda_i(A)\} < 0, \forall i$. *A matrix A with such a property is said to be "stable" or Hurwitz.*

5.4.2 Poles from transfer functions

Theorem 7 *The pole polynomial $\phi(s)$ corresponding to a minimal realization of a system with transfer function $G(s)$, is the least common denominator of all non-identically-zero minors of all orders of $G(s)$.*

Example:

$$G(s) = \frac{1}{1.25(s+1)(s+2)} \begin{bmatrix} s-1 & s \\ -6 & s-2 \end{bmatrix} \tag{5.37}$$

The minors of order 1 are the four elements all have $(s+1)(s+2)$ in the denominator.

Minor of order 2

$$\det G(s) = \frac{(s-1)(s-2)+6s}{1.25^2(s+1)^2(s+2)^2} = \frac{1}{1.25^2(s+1)(s+2)} \tag{5.38}$$

Least common denominator of all the minors:

$$\phi(s) = (s+1)(s+2) \tag{5.39}$$

Minimal realization has two poles: $s=-1$; $s=-2$.

Example: Consider the 2×3 system, with 3 inputs and 2 outputs,

$$G(s) = \frac{1}{(s+1)(s+2)(s-1)} *$$

$$* \begin{bmatrix} (s-1)(s+2) & 0 & (s-1)^2 \\ -(s+1)(s+2) & (s-1)(s+1) & (s-1)(s+1) \end{bmatrix} \tag{5.40}$$

Minors of order 1:

$$\frac{1}{s+1}, \; \frac{s-1}{(s+1)(s+2)}, \; \frac{-1}{s-1}, \; \frac{1}{s+2}, \; \frac{1}{s+2} \tag{5.41}$$

Minor of order 2 corresponding to the deletion of column 2:

$$M_2 = \frac{(s-1)(s+2)(s-1)(s+1) + (s+1)(s+2)(s-1)^2}{((s+1)(s+2)(s-1))^2} =$$

$$= \frac{2}{(s+1)(s+2)} \tag{5.42}$$

The other two minors of order two are

$$M_1 = \frac{-(s-1)}{(s+1)(s+2)^2}, \quad M_3 = \frac{1}{(s+1)(s+2)} \tag{5.43}$$

Least common denominator:

$$\phi(s) = (s+1)(s+2)^2(s-1) \tag{5.44}$$

The system therefore has four poles: $s = -1$, $s = 1$ and two at $s = -2$.

Note MIMO-poles are essentially the poles of the elements. A procedure is needed to determine multiplicity.

5.5 Zeros [4.5]

- SISO system: zeros z_i are the solutions to $G(z_i) = 0$.

In general, zeros are values of s at which $G(s)$ loses rank.

Example

$$\left[Y = \frac{s+2}{s^2+7s+12}U\right]$$

Compute the response when

$$\begin{aligned} u(t) &= e^{-2t},\ y(0) = 0,\ \dot{y}(0) = -1 \\ \mathcal{L}\{u(t)\} &= \frac{1}{s+2} \\ s^2Y &- sy(0) - \dot{y}(0) + 7sY - 7y(0) + 12Y = 1 \\ s^2Y &+ 7sY + 12Y + 1 = 1 \\ &\Rightarrow Y(s) = 0 \end{aligned}$$

Assumption: $g(s)$ has a zero z, $g(z) = 0$. Then for input $u(t) = u_0e^{zt}$ the output is $y(t) \equiv 0$, $t > 0$. (with appropriate initial conditions)

5.5.1 Zeros from state-space realizations [4.5.1]

Setup:

$$\begin{aligned} u &= u_0 e^{zt},\ x(t) = x_0 e^{zt},\ y(t) \equiv 0 \\ \dot{x} &= z e^{zt} x_0 = F e^{zt} x_0 + G u_0 e^{zt} \end{aligned}$$

$$\begin{bmatrix} zI - F & -G \end{bmatrix} \begin{bmatrix} x_0 \\ u_0 \end{bmatrix} = 0$$

and

$$\begin{aligned} y &= Hx + Ju \\ &= H e^{zt} x_0 + J u_0 e^{zt} \equiv 0 \end{aligned}$$

Combined

$$\begin{bmatrix} zI - F & -G \\ H & J \end{bmatrix} \begin{bmatrix} x_0 \\ u_0 \end{bmatrix} = 0$$

The zeros are the solutions of

$$\det \begin{bmatrix} zI - F & -G \\ H & J \end{bmatrix} = 0$$

MATLAB

```
zero = tzero(F,G,H,J)
```

5.5.2 Zeros from transfer functions [4.5.2]

Definition Zeros. z_i is a zero of $G(s)$ if the rank of $G(z_i)$ is less than the normal rank of $G(s)$. The zero polynomial is defined as $z(s) = \prod_{i=1}^{n_z}(s - z_i)$ where n_z is the number of finite zeros of $G(s)$.

Theorem The zero polynomial $z(s)$, corresponding to a minimal realization of the system, is the greatest common divisor of all the numerators of all order-r minors of $G(s)$, where r is the normal rank of $G(s)$, provided that these minors have been adjusted in such a way as to have the pole polynomial $\phi(s)$ as their denominators.

Example

$$G(s) = \frac{1}{s+2}\begin{bmatrix} s-1 & 4 \\ 4.5 & 2(s-1) \end{bmatrix} \tag{5.45}$$

The normal rank of $G(s)$ is 2.

Minor of order 2: $\det G(s) = \frac{2(s-1)^2-18}{(s+2)^2} = 2\frac{s-4}{s+2}$.

Pole polynomial: $\phi(s) = s + 2$.

Zero polynomial: $z(s) = s - 4$.

Note Multivariable zeros have no relationship with the zeros of the transfer function elements.

Example

$$G(s) = \frac{1}{1.25(s+1)(s+2)} \begin{bmatrix} s-1 & s \\ -6 & s-2 \end{bmatrix} \quad (5.46)$$

Minor of order 2 is the determinant

$$\det G(s) = \frac{(s-1)(s-2)+6s}{1.25^2(s+1)^2(s+2)^2} = \frac{1}{1.25^2(s+1)(s+2)} \quad (5.47)$$

$$\phi(s) = 1.25^2(s+1)(s+2)$$

Zero polynomial = numerator of (5.47)

$\Rightarrow$ no multivariable zeros.

Example

$$G(s) = \begin{bmatrix} \frac{s-1}{s+1} & \frac{s-2}{s+2} \end{bmatrix} \quad (5.48)$$

- The normal rank of $G(s)$ is 1
- no value of s for which $G(s) = 0$

  $\Rightarrow$ $G(s)$ has no zeros.

5.6 More on poles and zeros[4.6]

5.6.1 *Directions of poles and zeros

Let $G(s) = C(sI - A)^{-1}B + D$.

Zero directions. Let $G(s)$ have a zero at $s = z$. Then $G(s)$ loses rank at $s = z$, and there exist non-zero vectors u_z and y_z such that

$$G(z)u_z = 0, \quad y_z^H G(z) = 0 \tag{5.49}$$

$u_z =$ input zero direction

$y_z =$ output zero direction

y_z gives information about which output (or combination of outputs) may be difficult to control.

SVD:

$$G(z) = U\Sigma V^H$$

$u_z =$ last column in V

$y_z =$ last column of U

(corresponding to the zero singular value of $G(z)$)

Pole directions. Let $G(s)$ have a pole at $s = p$. Then $G(p)$ is infinite, and we may write

$$G(p)u_p = \infty, \quad y_p^H G(p) = \infty \tag{5.50}$$

$u_p =$ input pole direction

$y_p =$ output pole direction.

Example

Plant in (5.45) has a RHP-zero at $z = 4$ and a LHP-pole at $p = -2$.

$$\begin{aligned} G(z) &= G(4) = \frac{1}{6}\begin{bmatrix} 3 & 4 \\ 4.5 & 6 \end{bmatrix} \\ &= \frac{1}{6}\begin{bmatrix} 0.55 & -0.83 \\ 0.83 & 0.55 \end{bmatrix}\begin{bmatrix} 9.01 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0.6 & -0.8 \\ 0.8 & 0.6 \end{bmatrix}^H \end{aligned}$$

$$u_z = \begin{bmatrix} -0.80 \\ 0.60 \end{bmatrix} \quad y_z = \begin{bmatrix} -0.83 \\ 0.55 \end{bmatrix} \tag{5.51}$$

For pole directions consider

$$G(p+\epsilon) = G(-2+\epsilon) = \frac{1}{\epsilon^2}\begin{bmatrix} -3+\epsilon & 4 \\ 4.5 & 2(-3+\epsilon) \end{bmatrix} \tag{5.52}$$

The SVD as $\epsilon \to 0$ yields

$$G(-2+\epsilon) = \frac{1}{\epsilon^2}\begin{bmatrix} -0.55 & -0.83 \\ 0.83 & -0.55 \end{bmatrix}\begin{bmatrix} 9.01 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0.6 & -0.8 \\ -0.8 & -0.6 \end{bmatrix}^H$$

$$u_p = \begin{bmatrix} 0.60 \\ -0.80 \end{bmatrix} \quad y_p = \begin{bmatrix} -0.55 \\ 0.83 \end{bmatrix} \tag{5.53}$$

Note Locations of poles and zeros are independent of input and output scalings, their directions are *not*.

5.6.2 Remarks on poles and zeros [4.6.2]

1. For square systems the poles and zeros of $G(s)$ are "essentially" the poles and zeros of $\det G(s)$.

   This fails when zero and pole in different parts of the system cancel when forming $\det G(s)$.

   $$G(s) = \begin{bmatrix} (s+2)/(s+1) & 0 \\ 0 & (s+1)/(s+2) \end{bmatrix} \tag{5.54}$$

   $\det G(s) = 1$, although the system obviously has poles at -1 and -2 and (multivariable) zeros at -1 and -2.

2. System (5.54) has poles and zeros at the same locations (at -1 and -2). Their directions are different. They do not cancel or otherwise interact.

3. There are no zeros if the outputs contain direct information about all the states; that is, if from y we can directly obtain x (e.g. $C = I$ and $D = 0$);

4. Zeros usually appear when there are fewer inputs or outputs than states

5. **Moving poles.** (a) feedback control $(G(I+KG)^{-1})$ moves the poles, (b) series compensation (GK, feedforward control) can cancel poles in G by placing zeros in K (but not move them), and (c) parallel compensation $(G+K)$ cannot affect the poles in G.

6. **Moving zeros.** (a) With feedback, the zeros of $G(I+KG)^{-1}$ are the zeros of G plus the poles of K. , i.e. the zeros are unaffected by feedback. (b) Series compensation can counter the effect of zeros in G by placing poles in K to cancel them, but cancellations are not possible for RHP-zeros due to internal stability (see Section 5.7). (c) The only way to move zeros is by parallel compensation, $y=(G+K)u$, which, if y is a physical output, can only be accomplished by adding an extra input (actuator).

Example

Effect of feedback on poles and zeros.

SISO plant $G(s) = z(s)/\phi(s)$ and $K(s) = k$.

$$T(s) = \frac{L(s)}{1+L(s)} = \frac{kG(s)}{1+kG(s)} = \frac{kz(s)}{\phi(s)+kz(s)} = k\frac{z_{cl}(s)}{\phi_{cl}(s)} \quad (5.55)$$

Note the following:

1. Zero polynomial: $z_{cl}(s) = z(s)$
   $\Rightarrow$ zero locations are unchanged.

2. Pole locations are changed by feedback.
   For example,

$$k \to 0 \quad \Rightarrow \quad \phi_{cl}(s) \to \phi(s) \quad (5.56)$$

$$k \to \infty \quad \Rightarrow \quad \phi_{cl}(s) \to z(s).\widetilde{z}(s) \quad (5.57)$$

where roots of $\widetilde{z}(s)$ move with k to infinity (complex pattern)
(cf. root locus)

5.7 Internal stability of feedback systems [4.7]

Note: Checking the pole of S or T is not sufficient to determine internal stability

Example (Figure 47). In forming $L = GK$ we cancel the term $(s-1)$ (a RHP pole-zero cancellation) to obtain

$$L = GK = \frac{k}{s}, \text{ and } S = (I+L)^{-1} = \frac{s}{s+k} \qquad (5.58)$$

$S(s)$ is stable, i.e. transfer function from d_y to y is stable. However, the transfer function from d_y to u is unstable:

$$u = -K(I+GK)^{-1}d_y = -\frac{k(s+1)}{(s-1)(s+k)}d_y \qquad (5.59)$$

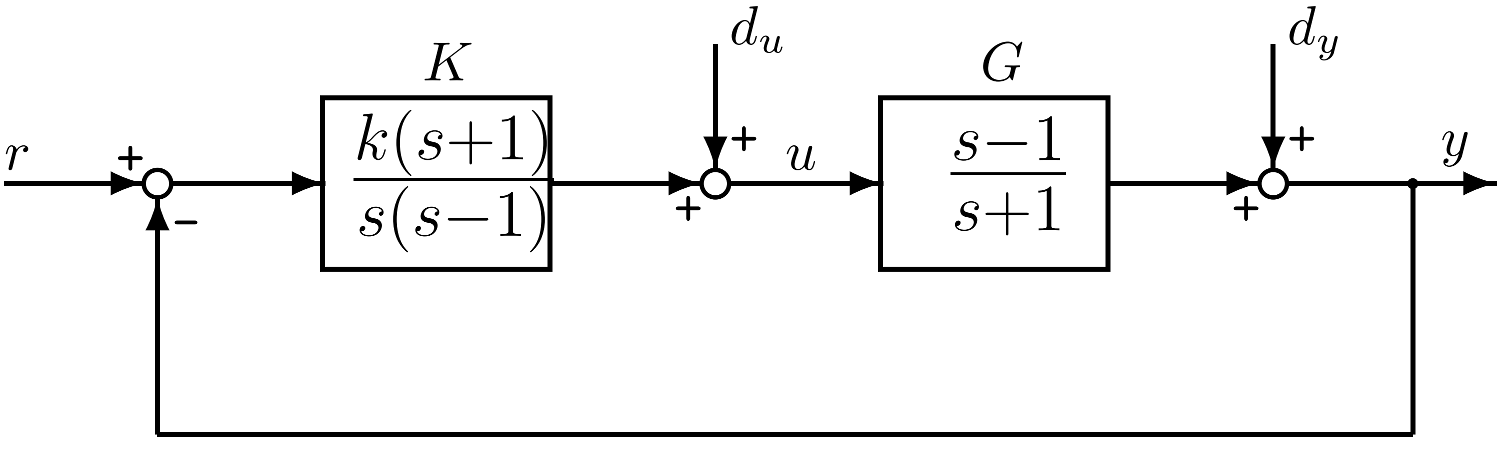


Figure 47: Internally unstable system

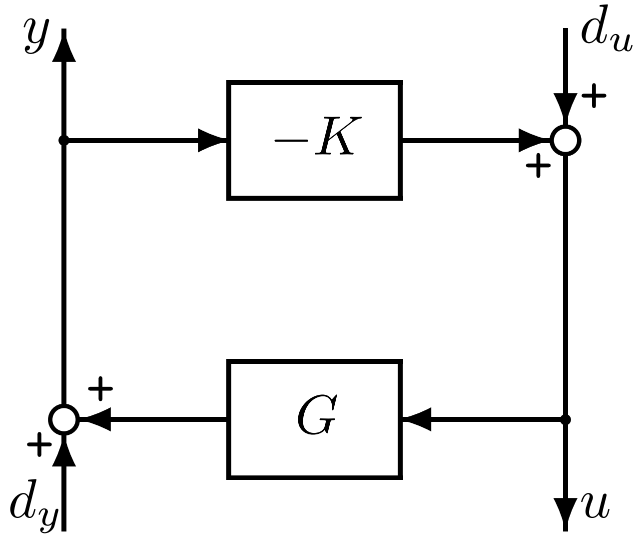


Figure 48: Block diagram used to check internal stability of feedback system

For *internal* stability consider

$$u = (I + KG)^{-1} d_u - K(I + GK)^{-1} d_y \tag{5.60}$$

$$y = G(I + KG)^{-1} d_u + (I + GK)^{-1} d_y \tag{5.61}$$

Theorem 4.4 The feedback system in Figure 48 is **internally stable** if and only if all four closed-loop transfer matrices in (5.60) and (5.61) are stable.

Theorem 4.5 Assume there are no RHP pole-zero cancellations between $G(s)$ and $K(s)$. Then the feedback system in Figure 48 is internally stable if and only if <u>one</u> of the four closed-loop transfer function matrices in (5.60) and (5.61) is stable.

Implications of the internal stability requirement

1. *If $G(s)$ has a RHP-zero at z, then $L = GK$, $T = GK(I+GK)^{-1}$, $SG = (I+GK)^{-1}G$, $L_I = KG$ and $T_I = KG(I+KG)^{-1}$ will each have a RHP-zero at z.*

2. *If $G(s)$ has a RHP-pole at p, then $L = GK$ and $L_I = KG$ also have a RHP-pole at p, while $S = (I+GK)^{-1}, KS = K(I+GK)^{-1}$ and $S_I = (I+KG)^{-1}$ have a RHP-zero at p.*

Exercise: Interpolation constraints. Prove for SISO feedback systems when the plant $G(s)$ has a RHP-zero z or a RHP-pole p:

$$G(z) = 0 \quad \Rightarrow \quad L(z) = 0 \quad \Leftrightarrow \quad T(z) = 0, S(z) = 1 \tag{5.62}$$

$$G^{-1}(p) = 0 \quad \Rightarrow \quad L(p) = \infty \quad \Leftrightarrow \quad T(p) = 1, S(p) = 0 \tag{5.63}$$

Remark "Perfect control" implies $S \approx 0$ and $T \approx 1$.

RHP-zero $\Rightarrow$ perfect control impossible.

RHP-pole $\Rightarrow$ perfect control possible.

RHP-poles cause problems when tight (high gain) control is *not* possible.

5.8 Stabilizing controllers [4.8]

Stable plants

Lemma For a stable plant $G(s)$ the negative feedback system in Figure 48 is internally stable if and only if $Q = K(I + GK)^{-1}$ is stable.

Proof: The four transfer functions in (5.60) and (5.61) are

$$K(I + GK)^{-1} = Q \quad (5.64)$$

$$(I + GK)^{-1} = I - GQ \quad (5.65)$$

$$(I + KG)^{-1} = I - QG \quad (5.66)$$

$$G(I + KG)^{-1} = G(I - QG) \quad (5.67)$$

which are clearly all stable if and only if G and Q are stable.

Consequences: *All stabilizing negative feedback controllers for the stable plant* $G(s)$ are given by

$$K = (I - QG)^{-1}Q = Q(I - GQ)^{-1} \qquad (5.68)$$

where the "parameter" Q is *any stable transfer function matrix.* (Identical to the internal model control (IMC) parameterization of stabilizing controllers.)

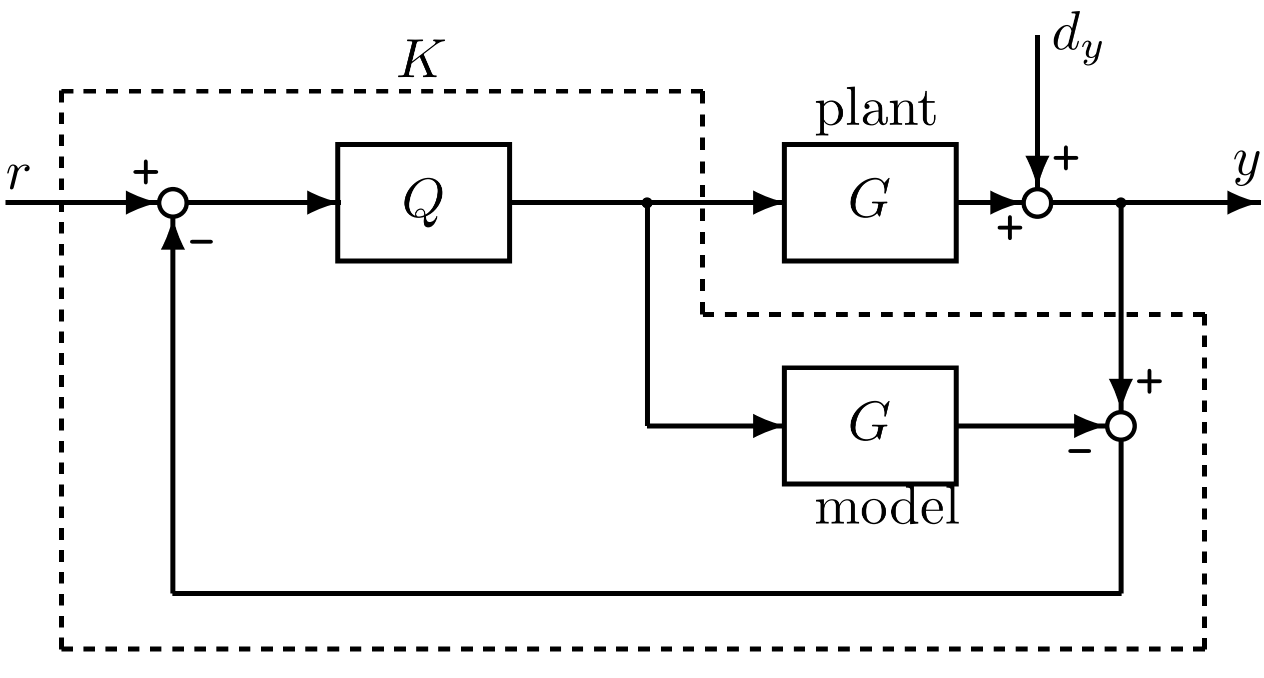


Figure 49: The internal model control (IMC) structure

5.9 Stability analysis in the frequency domain [4.9]

Generalization of Nyquist's stability test for SISO systems.

5.9.1 Open and closed-loop characteristic polynomials [4.9.1]

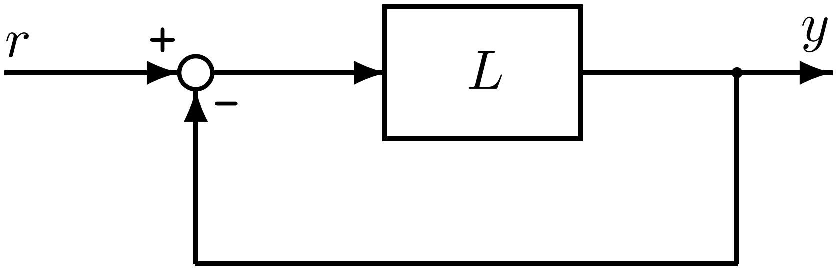


Figure 50: Negative feedback system

Open Loop:

$$L(s) = C_{ol}(sI - A_{ol})^{-1}B_{ol} + D_{ol} \tag{5.69}$$

Poles of $L(s)$ are the roots of the *open-loop* characteristic polynomial

$$\phi_{ol}(s) = \det(sI - A_{ol}) \tag{5.70}$$

Assume no RHP pole-zero cancellations between $G(s)$ and $K(s)$. Then from Theorem 4.5 internal stability of the *closed-loop* system is equivalent to the stability of $S(s) = (I + L(s))^{-1}$.

The realization of $S(s)$ can be derived as follow:

$$\begin{aligned} \dot{x} &= A_{ol}x + B_{ol}(r - y) && (5.71) \\ -e &= r - y = r - C_{ol}x - D_{ol}(r - y) && (5.72) \end{aligned}$$

or

$$r - y = (I + D_{ol})^{-1}(r - C_{ol}x) \tag{5.73}$$

and

$$\dot{x} = (A_{ol} - B_{ol}(I + D_{ol})^{-1}C_{ol})x + B_{ol}(I + D_{ol})^{-1}r \tag{5.74}$$

Therefore the state matrix of $S(s)$ is:

$$A_{cl} = A_{ol} - B_{ol}(I + D_{ol})^{-1}C_{ol} \tag{5.75}$$

And the closed-loop characteristic polynomial is

$$\phi_{cl}(s) \triangleq \det(sI - A_{cl}) = \det(sI - A_{ol} + B_{ol}(I + D_{ol})^{-1}C_{ol}) \tag{5.76}$$

Relationship between characteristic polynomials

From (5.69) we get

$$\det(I + L(s)) = \det(I + C_{ol}(sI - A_{ol})^{-1}B_{ol} + D_{ol}) \quad (5.77)$$

Schur's formula yields (with $A_{11} = I + D_{ol}, A_{12} = -C_{ol}, A_{22} = sI - A_{ol}, A_{21} = B_{ol}$)

$$\det(I + L(s)) = \frac{\phi_{cl}(s)}{\phi_{ol}(s)} \cdot c \quad (5.78)$$

where $c = \det(I + D_{ol})$ is a constant (cf. SISO result from RSI).

Side calculation:

$$\det \begin{bmatrix} I + D_{ol} & -C_{ol} \\ B_{ol} & sI - A_{ol} \end{bmatrix}$$

$$= \det [I + D_{ol}] \det \left[sI - A_{ol} + B_{ol} (I + D_{ol})^{-1} C_{ol}\right]$$

$$= \det [sI - A_{ol}] \det \left[I + D_{ol} + C_{ol} (sI - A_{ol})^{-1} B_{ol}\right]$$

5.9.2 MIMO Nyquist stability criteria [4.9.2]

Theorem: Generalized (MIMO) Nyquist theorem. Let P_{ol} denote the number of open-loop unstable poles in $L(s)$. The closed-loop system with loop transfer function $L(s)$ and negative feedback is stable if and only if the Nyquist plot of $\det(I + L(s))$

i) makes P_{ol} anti-clockwise encirclements of the origin, and

ii) does not pass through the origin.

Note

By "Nyquist plot of $\det(I + L(s))$" we mean "the image of $\det(I + L(s))$ as s goes clockwise around the Nyquist D-contour".

5.9.4 Small gain theorem [4.9.4]

$$\rho(L(j\omega)) \stackrel{\Delta}{=} \max_i |\lambda_i(L(j\omega))| \tag{5.79}$$

Theorem: Spectral radius stability condition. Consider a system with a stable loop transfer function $L(s)$. Then the closed-loop system is stable if

$$\rho(L(j\omega)) \stackrel{\Delta}{=} \max_i |\lambda_i(L(j\omega))| < 1 \quad \forall\omega \tag{5.80}$$

Proof: Assume the system is unstable. Therefore $\det(I + L(s))$ encircles the origin, and there is an eigenvalue, $\lambda_i(L(j\omega))$ which is larger than 1 at some frequency. If $\det(I + L(s))$ does encircle the origin, then there must exists a gain $\epsilon \in (0, 1]$ and a frequency ω' such that

$$\det(I + \epsilon L(j\omega')) = 0 \tag{5.81}$$

or

$$\prod_i \lambda_i(I + \epsilon L(j\omega')) = 0 \tag{5.82}$$

$$\Leftrightarrow \qquad 1 + \epsilon\lambda_i(L(j\omega')) = 0 \quad \text{for some } i \tag{5.83}$$

$$\Leftrightarrow \qquad \lambda_i(L(j\omega')) = -\frac{1}{\epsilon} \quad \text{for some } i \tag{5.84}$$

$$\Rightarrow \qquad |\lambda_i(L(j\omega'))| \geq 1 \quad \text{for some } i \tag{5.85}$$

$$\Leftrightarrow \qquad \rho(L(j\omega')) \geq 1 \tag{5.86}$$

Interpretation: If the system gain is less than 1 in all directions (all eigenvalues) and for all frequencies ($\forall\omega$), then all signal deviations will eventually die out, and the system is stable.

Spectral radius theorem is conservative because phase information is not considered.

Small Gain Theorem. Consider a system with a stable loop transfer function $L(s)$. Then the closed-loop system is stable if

$$\|L(j\omega)\| < 1 \quad \forall\omega \tag{5.87}$$

where $\|L\|$ denotes any matrix norm satisfying $\|AB\| \leq \|A\| \cdot \|B\|$, for example the singular value $\bar{\sigma}(L)$.

Note The small gain theorem is generally more conservative than the spectral radius condition in (5.80).

5.10 System norms [4.10]

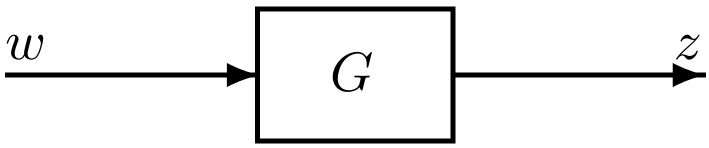


Figure 51: System G

Figure 51: System with stable transfer function matrix $G(s)$ and impulse response matrix $g(t)$.

Question: given information about the allowed input signals $w(t)$, how large can the outputs $z(t)$ become?

We use the 2-norm,

$$\|z(t)\|_2 = \sqrt{\sum_i \int_{-\infty}^{\infty} |z_i(\tau)|^2 d\tau} \tag{5.88}$$

and consider three inputs:

1. $w(t)$ is a series of unit impulses.
2. $w(t)$ is any signal satisfying $\|w(t)\|_2 = 1$.
3. $w(t)$ is any signal satisfying $\|w(t)\|_2 = 1$, but $w(t) = 0$ for $t \geq 0$, and we only measure $z(t)$ for $t \geq 0$.

The relevant system norms in the three cases are the $\mathcal{H}_2$, $\mathcal{H}_\infty$, and Hankel norms, respectively.

5.10.1 $\mathcal{H}_2$ norm [4.10.1]

$G(s)$ strictly proper.

For the $\mathcal{H}_2$ norm we use the Frobenius norm spatially (for the matrix) and integrate over frequency, i.e.

$$\|G(s)\|_2 \triangleq \sqrt{\frac{1}{2\pi}\int_{-\infty}^{\infty} \underbrace{\operatorname{tr}(G(j\omega)^H G(j\omega))}_{\|G(j\omega)\|_F^2=\sum_{ij}|G_{ij}(j\omega)|^2} \; d\omega} \tag{5.89}$$

$G(s)$ must be strictly proper, otherwise the $\mathcal{H}_2$ norm is infinite. By Parseval's theorem, (5.89) is equal to the $\mathcal{H}_2$ norm of the impulse response

$$\|G(s)\|_2 = \|g(t)\|_2 \triangleq \sqrt{\int_0^{\infty} \underbrace{\operatorname{tr}(g^T(\tau)g(\tau))}_{\|g(\tau)\|_F^2=\sum_{ij}|g_{ij}(\tau)|^2} \; d\tau} \tag{5.90}$$

- Note that $G(s)$ and $g(t)$ are dynamic *systems* while $G(j\omega)$ and $g(\tau)$ are constant *matrices* (for a given value of ω or τ).

- We can change the order of integration and summation in (5.90) to get

$$\|G(s)\|_2 = \|g(t)\|_2 = \sqrt{\sum_{ij} \int_0^\infty |g_{ij}(\tau)|^2 d\tau} \tag{5.91}$$

where $g_{ij}(t)$ is the ij'th element of the impulse response matrix, $g(t)$. Thus $\mathcal{H}_2$ norm can be interpreted as the 2-norm output resulting from applying unit impulses $\delta_j(t)$ to each input, one after another (allowing the output to settle to zero before applying an impulse to the next input). Thus $\|G(s)\|^2 = \sqrt{\sum_{i=1}^m \|z_i(t)\|_2^2}$ where $z_i(t)$ is the output vector resulting from applying a unit impulse $\delta_i(t)$ to the i'th input.

Numerical computations of the $\mathcal{H}_2$ norm.

Consider $G(s) = C(sI - A)^{-1}B$. Then

$$\|G(s)\|_2 = \sqrt{\mathrm{tr}(B^T Q B)} \quad \text{or} \quad \|G(s)\|_2 = \sqrt{\mathrm{tr}(C P C^T)} \tag{5.92}$$

where $Q =$ observability Gramian

and $P =$ controllability Gramian

5.10.2 $\mathcal{H}_\infty$ norm [4.10.2]

$G(s)$ proper.

For the $\mathcal{H}_\infty$ norm we use the singular value (induced 2-norm) spatially (for the matrix) and pick out the peak value as a function of frequency

$$\|G(s)\|_\infty \triangleq \max_\omega \bar{\sigma}(G(j\omega)) \tag{5.93}$$

The $\mathcal{H}_\infty$ norm is the peak of the transfer function "magnitude".

Time domain performance interpretations of the $\mathcal{H}_\infty$ norm.

- Worst-case steady-state gain for sinusoidal inputs at any frequency.
- Induced (worst-case) 2-norm in the time domain:

  $$\|G(s)\|_\infty = \max_{w(t)\neq 0} \frac{\|z(t)\|_2}{\|w(t)\|_2} = \max_{\|w(t)\|_2=1} \|z(t)\|_2 \tag{5.94}$$

  (In essence, (5.94) arises because the worst input signal $w(t)$ is a sinusoid with frequency ω^* and a direction which gives $\overline{\sigma}(G(j\omega^*))$ as the maximum gain.)

Numerical computation of the $\mathcal{H}_\infty$ norm.
Consider

$$G(s) = C(sI - A)^{-1}B + D$$

$\mathcal{H}_\infty$ norm is the smallest value of γ such that the Hamiltonian matrix H has no eigenvalues on the imaginary axis, where

$$H = \begin{bmatrix} A + BR^{-1}D^TC & BR^{-1}B^T \\ -C^T(I + DR^{-1}D^T)C & -(A + BR^{-1}D^TC)^T \end{bmatrix} \tag{5.95}$$

and $R = \gamma^2 I - D^T D$

5.10.3 Difference between the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms

Frobenius norm in terms of singular values

$$\|G(s)\|_2 = \sqrt{\frac{1}{2\pi}\int_{-\infty}^{\infty}\sum_i \sigma_i^2(G(j\omega))d\omega} \tag{5.96}$$

Thus when optimizing performance in terms of the different norms:

- $\mathcal{H}_\infty$: “push down peak of largest singular value”.
- $\mathcal{H}_2$: “push down whole thing” (all singular values over all frequencies).

Example

$$G(s) = \frac{1}{s+a} \tag{5.97}$$

$\mathcal{H}_2$ norm:

$$\begin{aligned}\|G(s)\|_2 &= \left(\frac{1}{2\pi}\int_{-\infty}^{\infty} \underbrace{|G(j\omega)|^2}_{\frac{1}{\omega^2+a^2}} d\omega\right)^{\frac{1}{2}} \\ &= \left(\frac{1}{2\pi a}\left[\tan^{-1}(\frac{\omega}{a})\right]_{-\infty}^{\infty}\right)^{\frac{1}{2}} = \sqrt{\frac{1}{2a}}\end{aligned}$$

Alternatively: Consider the impulse response

$$g(t) = \mathcal{L}^{-1}\left(\frac{1}{s+a}\right) = e^{-at}, t \geq 0 \tag{5.98}$$

to get

$$\|g(t)\|_2 = \sqrt{\int_0^{\infty}(e^{-at})^2 dt} = \sqrt{\frac{1}{2a}} \tag{5.99}$$

as expected from Parseval's theorem.

$\mathcal{H}_\infty$ norm:

$$||G(s)||_\infty = \max_\omega |G(j\omega)| = \max_\omega \frac{1}{(\omega^2+a^2)^{\frac{1}{2}}} = \frac{1}{a} \tag{5.100}$$

Example

There is no general relationship between the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms.

$$f_1(s) = \frac{1}{\epsilon s + 1}, \quad f_2(s) = \frac{\epsilon s}{s^2 + \epsilon s + 1} \tag{5.101}$$

$$\begin{aligned} ||f_1||_\infty = 1 \quad ||f_1||_2 = \infty \\ ||f_2||_\infty = 1 \quad ||f_2||_2 = 0 \end{aligned} \tag{5.102}$$

Why is the $\mathcal{H}_\infty$ norm so popular? In robust control convenient for representing unstructured model uncertainty, and because it satisfies the multiplicative property:

$$\|A(s)B(s)\|_\infty \leq \|A(s)\|_\infty \cdot \|B(s)\|_\infty \tag{5.103}$$

What is wrong with the $\mathcal{H}_2$ norm? It is *not* an induced norm and does *not* satisfy the multiplicative property.

Example

Consider again $G(s) = 1/(s+a)$ in (5.97), for which $\|G(s)\|_2 = \sqrt{1/2a}$.

$$\begin{aligned} \|G(s)G(s)\|_2 &= \sqrt{\int_0^\infty |\underbrace{\mathcal{L}^{-1}[(\frac{1}{s+a})^2]}_{te^{-at}}|^2} \\ &= \sqrt{\frac{1}{a}\frac{1}{2a}} = \sqrt{\frac{1}{a}}\ \|G(s)\|_2^2 \end{aligned} \tag{5.104}$$

for $a < 1$,

$$\|G(s)G(s)\|_2 > \|G(s)\|_2 \cdot \|G(s)\|_2 \tag{5.105}$$

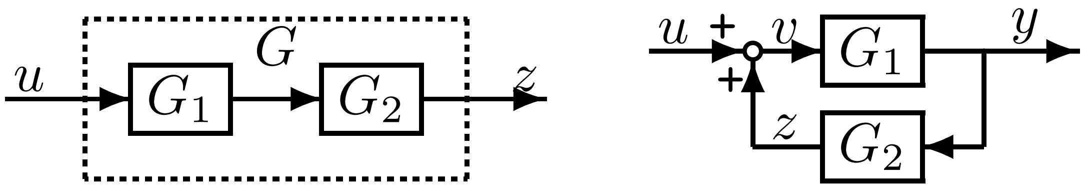
which does not satisfy the multiplicative property.

$\mathcal{H}_\infty$ norm does satisfy the multiplicative property

$$\|G(s)G(s)\|_\infty = \frac{1}{a^2} = \|G(s)\|_\infty \cdot \|G(s)\|_\infty$$

6 INTRODUCTION TO MULTIVARIABLE CONTROL [3]

6.1 Transfer functions for MIMO systems [3.2]



(a) Cascade system

(b) Positive feedback system

Figure 52: Block diagrams for the cascade rule and the feedback rule

1. **Cascade rule.** (Figure 52(a)) $G = G_2 G_1$
2. **Feedback rule.** (Figure 52(b)) $v = (I - L)^{-1} u$ where $L = G_2 G_1$
3. **Push-through rule.**

$$G_1(I - G_2 G_1)^{-1} \quad = \quad (I - G_1 G_2)^{-1} G_1$$

MIMO Rule: *Start from the output, move backwards. If you exit from a feedback loop then include a term* $(I - L)^{-1}$ *where* L *is the transfer function around that loop (evaluated against the signal flow starting at the point of exit from the loop).*

Example

$$z = (P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21})w \qquad (6.1)$$

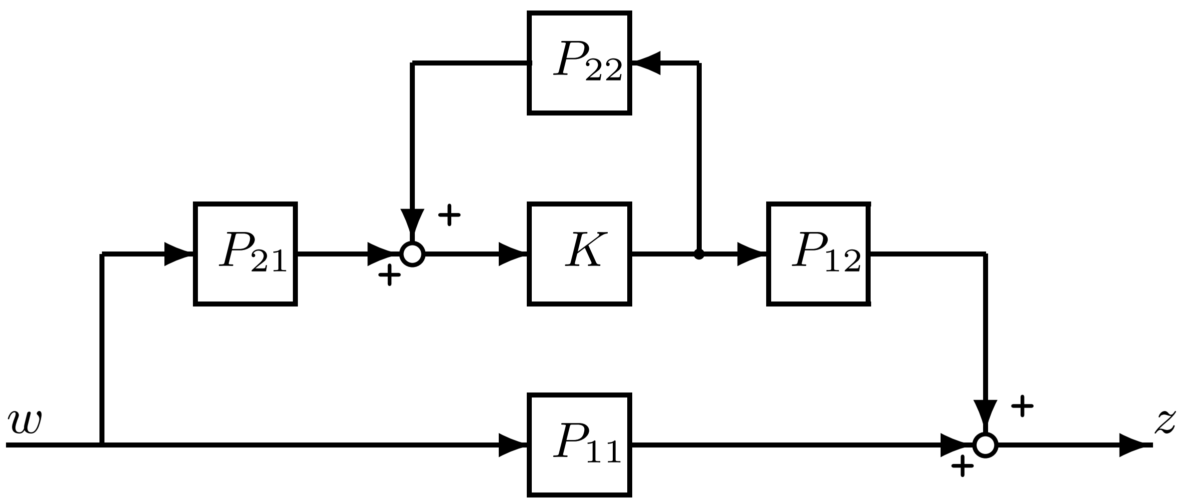


Figure 53: Block diagram corresponding to (6.1)

Negative feedback control systems

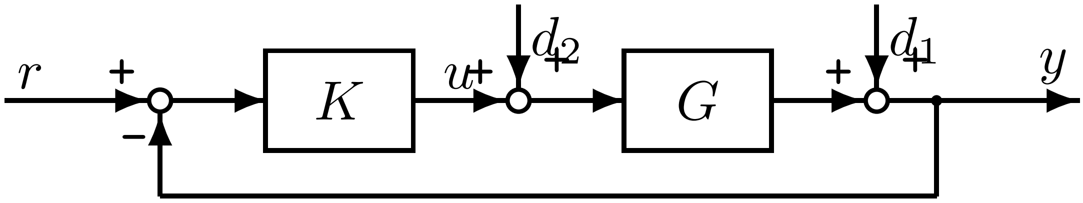


Figure 54: Conventional negative feedback control system

- L is the loop transfer function when breaking the loop at the *output* of the plant.

$$L = GK \tag{6.2}$$

Accordingly

$$S \triangleq (I + L)^{-1}$$

output sensitivity (6.3)

$$T \triangleq I - S = (I + L)^{-1} L = L(I + L)^{-1}$$

output complementary sensitivity (6.4)

$L_O \equiv L$, $S_O \equiv S$ and $T_O \equiv T$.

- L_I is the loop transfer function at the *input* to the plant

$$L_I = KG \tag{6.5}$$

Input sensitivity:

$$S_I \triangleq (I + L_I)^{-1}$$

Input complementary sensitivity:

$$T_I \triangleq I - S_I = L_I(I + L_I)^{-1}$$

- Some relationships:

$$(I + L)^{-1} + (I + L)^{-1}L = S + T = I \tag{6.6}$$

$$G(I + KG)^{-1} = (I + GK)^{-1}G \tag{6.7}$$

$$GK(I+GK)^{-1} = G(I+KG)^{-1}K = (I+GK)^{-1}GK \tag{6.8}$$

$$T = L(I + L)^{-1} = (I + L^{-1})^{-1} = (I + L)^{-1}L \tag{6.9}$$

Rule to remember: "G comes first and then G and K alternate in sequence".

6.2 Multivariable frequency response analysis [3.3]

$$\begin{aligned} G(s) &= \text{transfer (function) matrix} \\ G(j\omega) &= \text{complex matrix representing response} \\ &\quad \text{to sinusoidal signal of frequency } \omega \end{aligned}$$

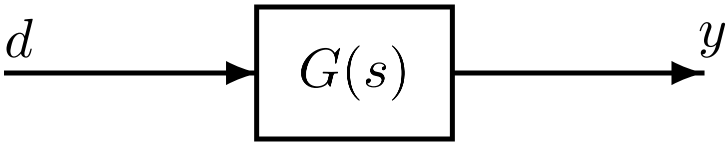


Figure 55: System $G(s)$ with input d and output y

$$y(s) = G(s)d(s) \tag{6.10}$$

Sinusoidal input to channel j

$$d_j(t) = d_{j0} \sin(\omega t + \alpha_j) \tag{6.11}$$

starting at $t = -\infty$. Output in channel i is a sinusoid with the same frequency

$$y_i(t) = y_{i0} \sin(\omega t + \beta_i) \tag{6.12}$$

Amplification (gain):

$$\frac{y_{io}}{d_{jo}} = |g_{ij}(j\omega)| \tag{6.13}$$

Phase shift:

$$\beta_i - \alpha_j = \angle g_{ij}(j\omega) \tag{6.14}$$

$g_{ij}(j\omega)$ represents the sinusoidal response from input j to output i.

Example 2×2 multivariable system, sinusoidal signals of the same frequency ω to the two input channels:

$$d(t) = \begin{bmatrix} d_1(t) \\ d_2(t) \end{bmatrix} = \begin{bmatrix} d_{10}\sin(\omega t + \alpha_1) \\ d_{20}\sin(\omega t + \alpha_2) \end{bmatrix} \quad (6.15)$$

The output signal

$$y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} y_{10}\sin(\omega t + \beta_1) \\ y_{20}\sin(\omega t + \beta_2) \end{bmatrix} \quad (6.16)$$

can be computed by multiplying the complex matrix $G(j\omega)$ by the complex vector $d(\omega)$:

$$\begin{aligned} y(\omega) &= G(j\omega)d(\omega) \\ y(\omega) &= \begin{bmatrix} y_{10}e^{j\beta_1} \\ y_{20}e^{j\beta_2} \end{bmatrix}, \; d(\omega) = \begin{bmatrix} d_{10}e^{j\alpha_1} \\ d_{20}e^{j\alpha_2} \end{bmatrix} \quad (6.17) \end{aligned}$$

6.2.1 Directions in multivariable systems [3.3.2]

SISO system ($y = Gd$): gain

$$\frac{|y(\omega)|}{|d(\omega)|} = \frac{|G(j\omega)d(\omega)|}{|d(\omega)|} = |G(j\omega)|$$

The gain depends on ω, but is independent of $|d(\omega)|$.

MIMO system: input and output are vectors.

$\Rightarrow$ need to "sum up" magnitudes of elements in each vector by use of some norm

$$\|d(\omega)\|_2 = \sqrt{\sum_j |d_j(\omega)|^2} = \sqrt{d_{10}^2 + d_{20}^2 + \cdots} \quad (6.18)$$

$$\|y(\omega)\|_2 = \sqrt{\sum_i |y_i(\omega)|^2} = \sqrt{y_{10}^2 + y_{20}^2 + \cdots} \quad (6.19)$$

The *gain* of the system $G(s)$ is

$$\frac{\|y(\omega)\|_2}{\|d(\omega)\|_2} = \frac{\|G(j\omega)d(\omega)\|_2}{\|d(\omega)\|_2} = \frac{\sqrt{y_{10}^2 + y_{20}^2 + \cdots}}{\sqrt{d_{10}^2 + d_{20}^2 + \cdots}} \quad (6.20)$$

The gain depends on ω, and is independent of $\|d(\omega)\|_2$. However, for a MIMO system the gain depends on the *direction* of the input d.

Example Consider the five inputs (all $\|d\|_2 = 1$)

$$\begin{aligned} d_1 &= \begin{bmatrix} 1 \\ 0 \end{bmatrix},\ d_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\ d_3 = \begin{bmatrix} 0.707 \\ 0.707 \end{bmatrix}, \\ d_4 &= \begin{bmatrix} 0.707 \\ -0.707 \end{bmatrix},\ d_5 = \begin{bmatrix} 0.6 \\ -0.8 \end{bmatrix} \end{aligned}$$

For the 2×2 system

$$G_1 = \begin{bmatrix} 5 & 4 \\ 3 & 2 \end{bmatrix} \tag{6.21}$$

The five inputs d_j lead to the following output vectors

$$y_1 = \begin{bmatrix} 5 \\ 3 \end{bmatrix},\ y_2 = \begin{bmatrix} 4 \\ 2 \end{bmatrix},\ y_3 = \begin{bmatrix} 6.36 \\ 3.54 \end{bmatrix},\ y_4 = \begin{bmatrix} 0.707 \\ 0.707 \end{bmatrix}, y_5 = \begin{bmatrix} -0.2 \\ 0.2 \end{bmatrix}$$

with the 2-norms (i.e. the gains for the five inputs)

$$\begin{aligned} \|y_1\|_2 &= 5.83,\ \|y_2\|_2 = 4.47,\ \|y_3\|_2 = 7.30, \\ \|y_4\|_2 &= 1.00,\ \|y_5\|_2 = 0.28 \end{aligned}$$

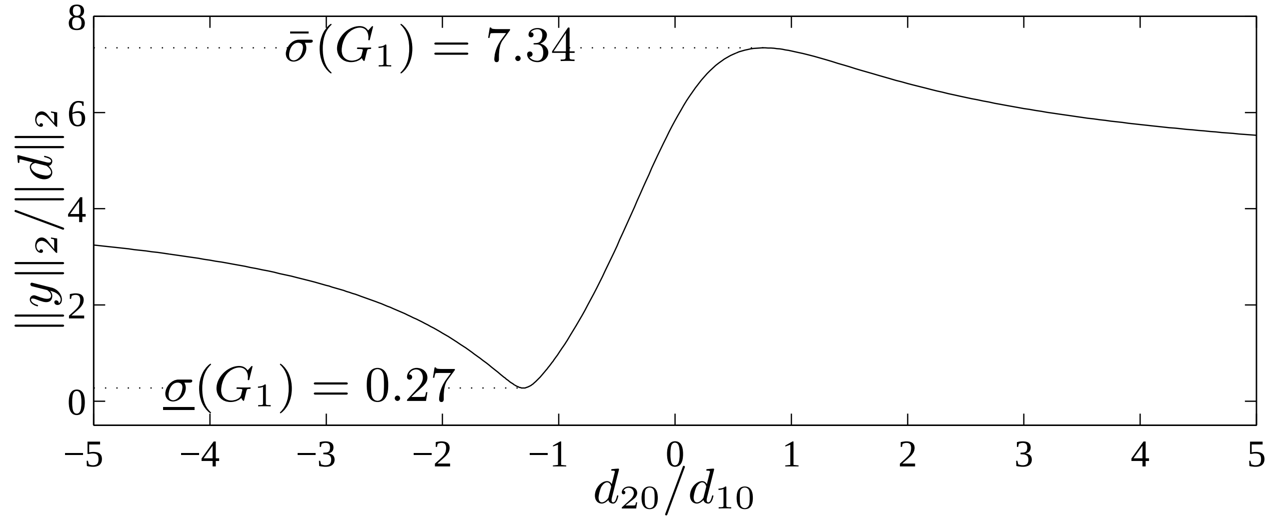


Figure 56: Gain $\|G_1 d\|_2/\|d\|_2$ as a function of d_{20}/d_{10} for G_1 in (6.21)

The maximum value of the gain in (6.20) as the direction of the input is varied, is the maximum singular value of G,

$$\max_{d\neq 0} \frac{\|Gd\|_2}{\|d\|_2} = \max_{\|d\|_2=1} \|Gd\|_2 = \bar{\sigma}(G) \qquad (6.22)$$

whereas the minimum gain is the minimum singular value of G,

$$\min_{d\neq 0} \frac{\|Gd\|_2}{\|d\|_2} = \min_{\|d\|_2=1} \|Gd\|_2 = \underline{\sigma}(G) \qquad (6.23)$$

6.2.2 Eigenvalues are a poor measure of gain [3.3.3]

Example

$$G = \begin{bmatrix} 0 & 100 \\ 0 & 0 \end{bmatrix}; \quad G \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 100 \\ 0 \end{bmatrix} \tag{6.24}$$

Both eigenvalues are equal to zero, but gain is equal to 100.

Problem: eigenvalues measure the gain for the special case when the inputs and the outputs are in the same direction (in the direction of the eigenvectors).

For generalizations of $|G|$ when G is a matrix, we need the concept of a *matrix norm*, denoted $\|G\|$. Two important properties: *triangle inequality*

$$\|G_1 + G_2\| \leq \|G_1\| + \|G_2\| \tag{6.25}$$

and the multiplicative property

$$\|G_1 G_2\| \leq \|G_1\| \cdot \|G_2\| \tag{6.26}$$

$\rho(G) \triangleq |\lambda_{max}(G)|$ (the spectral radius), does *not* satisfy the properties of a matrix norm

6.2.3 Singular value decomposition [3.3.4]

Any matrix G may be decomposed into its singular value decomposition,

$$G = U\Sigma V^H \tag{6.27}$$

where

Σ is an $l \times m$ matrix with $k = \min\{l, m\}$ non-negative singular values, σ_i, arranged in descending order along its main diagonal;

U is an $l \times l$ unitary matrix of output singular vectors, u_i,

V is an $m \times m$ unitary matrix of input singular vectors, v_i,

Example SVD of a real 2×2 matrix can always be written as

$$G = \underbrace{\begin{bmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix}}_{\Sigma} \underbrace{\begin{bmatrix} \cos\theta_2 & \pm\sin\theta_2 \\ -\sin\theta_2 & \pm\cos\theta_2 \end{bmatrix}^T}_{V^T} \tag{6.28}$$

U and V involve rotations and their columns are orthonormal.

Input and output directions.

The column vectors of U, denoted u_i, represent the *output directions* of the plant. They are orthogonal and of unit length (orthonormal), that is

$$\|u_i\|_2 = \sqrt{|u_{i1}|^2 + |u_{i2}|^2 + \ldots + |u_{il}|^2} = 1 \qquad (6.29)$$

$$u_i^H u_i = 1, \quad u_i^H u_j = 0, \quad i \neq j \qquad (6.30)$$

The column vectors of V, denoted v_i, are orthogonal and of unit length, and represent the *input directions*.

$$G v_i = \sigma_i u_i \qquad (6.31)$$

If we consider an *input* in the direction v_i, then the *output* is in the direction u_i. Since $\|v_i\|_2 = 1$ and $\|u_i\|_2 = 1$ σ_i gives the gain of the matrix G in this direction.

$$\sigma_i(G) = \|G v_i\|_2 = \frac{\|G v_i\|_2}{\|v_i\|_2} \qquad (6.32)$$

Maximum and minimum singular values.

The largest gain for *any* input direction is

$$\bar{\sigma}(G) \equiv \sigma_1(G) = \max_{d \neq 0} \frac{\|Gd\|_2}{\|d\|_2} = \frac{\|Gv_1\|_2}{\|v_1\|_2} \tag{6.33}$$

The smallest gain for any input direction is

$$\underline{\sigma}(G) \equiv \sigma_k(G) = \min_{d \neq 0} \frac{\|Gd\|_2}{\|d\|_2} = \frac{\|Gv_k\|_2}{\|v_k\|_2} \tag{6.34}$$

where $k = \min\{l, m\}$. For any vector d we have

$$\underline{\sigma}(G) \leq \frac{\|Gd\|_2}{\|d\|_2} \leq \bar{\sigma}(G) \tag{6.35}$$

Define $u_1 = \bar{u}, v_1 = \bar{v}, u_k = \underline{u}$ and $v_k = \underline{v}$. Then

$$G\bar{v} = \bar{\sigma}\bar{u}, \qquad G\underline{v} = \underline{\sigma}\ \underline{u} \tag{6.36}$$

$\bar{v}$ corresponds to the input direction with largest amplification, and $\bar{u}$ is the corresponding output direction in which the inputs are most effective. The directions involving $\bar{v}$ and $\bar{u}$ are sometimes referred to as the “strongest”, “high-gain” or “most important” directions.

Example

$$G_1 = \begin{bmatrix} 5 & 4 \\ 3 & 2 \end{bmatrix} \tag{6.37}$$

The singular value decomposition of G_1 is

$$G_1 = \underbrace{\begin{bmatrix} 0.872 & 0.490 \\ 0.490 & -0.872 \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} 7.343 & 0 \\ 0 & 0.272 \end{bmatrix}}_{\Sigma} \underbrace{\begin{bmatrix} 0.794 & -0.608 \\ 0.608 & 0.794 \end{bmatrix}^H}_{V^H}$$

The largest gain of 7.343 is for an input in the direction $\bar{v} = \begin{bmatrix} 0.794 \\ 0.608 \end{bmatrix}$, the smallest gain of 0.272 is for an input in the direction $\underline{v} = \begin{bmatrix} -0.608 \\ 0.794 \end{bmatrix}$. Since in (6.37) both inputs affect both outputs, we say that the system is *interactive*. The system is *ill-conditioned*, that is, some combinations of the inputs have a strong effect on the outputs, whereas other combinations have a weak effect on the outputs. Quantified by the *condition number*; $\bar{\sigma}/\underline{\sigma} = 7.343/0.272 = 27.0$.

Example

Shopping cart. Consider a shopping cart (supermarket trolley) with fixed wheels which we may want to move in three directions; forwards, sideways and upwards. For the shopping cart the gain depends strongly on the input direction, i.e. the plant is ill-conditioned.

Example: Distillation process.

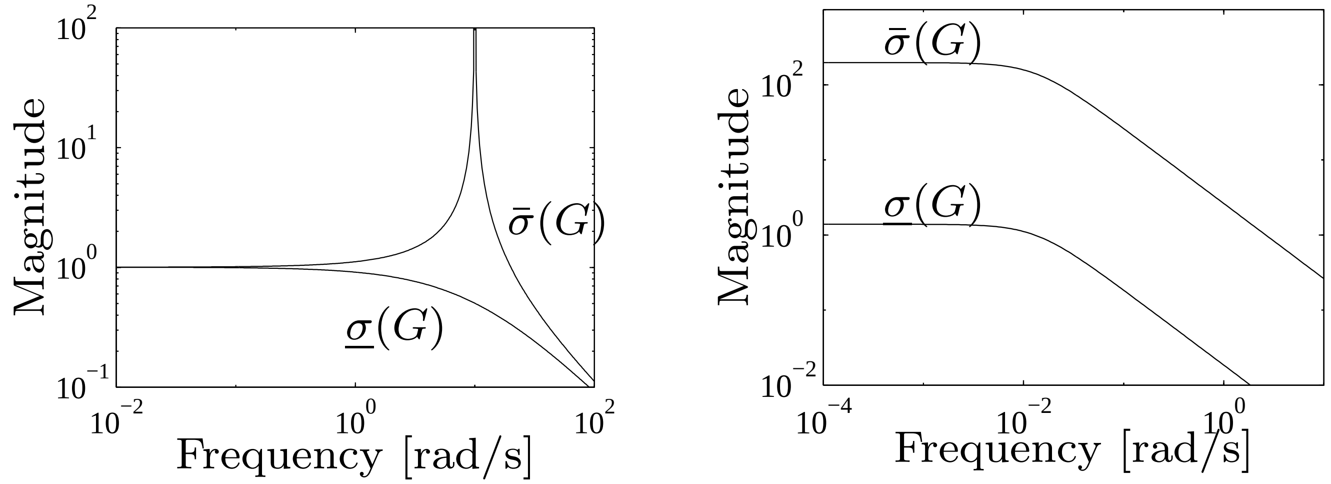
Steady-state model of a distillation column

$$G = \begin{bmatrix} 87.8 & -86.4 \\ 108.2 & -109.6 \end{bmatrix} \tag{6.38}$$

Since the elements are much larger than 1 in magnitude there should be no problems with input constraints. However, the gain in the low-gain direction is only just above 1.

$$G = \underbrace{\begin{bmatrix} 0.625 & -0.781 \\ 0.781 & 0.625 \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} 197.2 & 0 \\ 0 & 1.39 \end{bmatrix}}_{\Sigma} \underbrace{\begin{bmatrix} 0.707 & -0.708 \\ -0.708 & -0.707 \end{bmatrix}^H}_{V^H} \tag{6.39}$$

The distillation process is *ill-conditioned*, and the condition number is $197.2/1.39 = 141.7$. For dynamic systems the singular values and their associated directions vary with frequency (Figure 57).



(a) Spinning satellite in (6.44)

(b) Distillation process in (6.49)

Figure 57: Typical plots of singular values

6.2.4 Singular values for performance [3.3.5]

Maximum singular value is very useful in terms of frequency-domain performance and robustness.

Performance measure for SISO systems:

$$|e(\omega)|/|r(\omega)| = |S(j\omega)|$$

.

Generalization for MIMO systems $\|e(\omega)\|_2/\|r(\omega)\|_2$

$$\underline{\sigma}(S(j\omega)) \le \frac{\|e(\omega)\|_2}{\|r(\omega)\|_2} \le \bar{\sigma}(S(j\omega)) \tag{6.40}$$

For *performance* we want the gain $\|e(\omega)\|_2/\|r(\omega)\|_2$ small for any direction of $r(\omega)$

$$\begin{aligned} \bar{\sigma}(S(j\omega)) < 1/|w_P(j\omega)|, \ \forall\omega \quad &\Leftrightarrow \quad \bar{\sigma}(w_P S) < 1, \ \forall\omega \\ &\Leftrightarrow \quad \|w_P S\|_\infty < 1 \end{aligned} \tag{6.41}$$

where the $\mathcal{H}_\infty$ norm is defined as the peak of the maximum singular value of the frequency response

$$\|M(s)\|_\infty \triangleq \max_\omega \bar{\sigma}(M(j\omega)) \tag{6.42}$$

Typical singular values of $S(j\omega)$ in Figure 58.

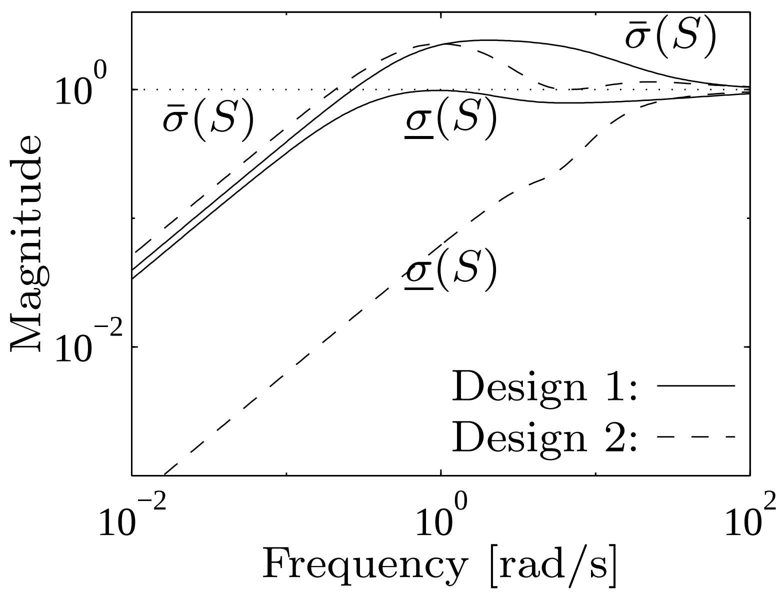


Figure 58: Singular values of S for a 2×2 plant with RHP-zero

- *Bandwidth*, ω_B: frequency where $\bar{\sigma}(S)$ crosses $\frac{1}{\sqrt{2}} = 0.7$ from below.

Since $S = (I + L)^{-1}$, the singular values inequality $\underline{\sigma}(A) - 1 \leq \frac{1}{\bar{\sigma}(I+A)^{-1}} \leq \underline{\sigma}(A) + 1$ yields

$$\underline{\sigma}(L) - 1 \leq \frac{1}{\bar{\sigma}(S)} \leq \underline{\sigma}(L) + 1 \qquad (6.43)$$

- low ω : $\underline{\sigma}(L) \gg 1 \Rightarrow \bar{\sigma}(S) \approx \frac{1}{\underline{\sigma}(L)}$
- high ω: $\bar{\sigma}(L) \ll 1 \Rightarrow \bar{\sigma}(S) \approx 1$

6.3 Introduction to MIMO robustness [3.7]

6.3.1 Motivating robustness example no. 1: Spinning Satellite [3.7.1]

Angular velocity control of a satellite spinning about one of its principal axes:

$$G(s) = \frac{1}{s^2+a^2}\begin{bmatrix} s-a^2 & a(s+1) \\ -a(s+1) & s-a^2 \end{bmatrix}; \quad a = 10 \tag{6.44}$$

A minimal, state-space realization, $G = C(sI-A)^{-1}B + D$, is

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{cc|cc} 0 & a & 1 & 0 \\ -a & 0 & 0 & 1 \\ \hline 1 & a & 0 & 0 \\ -a & 1 & 0 & 0 \end{array}\right] \tag{6.45}$$

Poles at $s = \pm ja$ For stabilization:

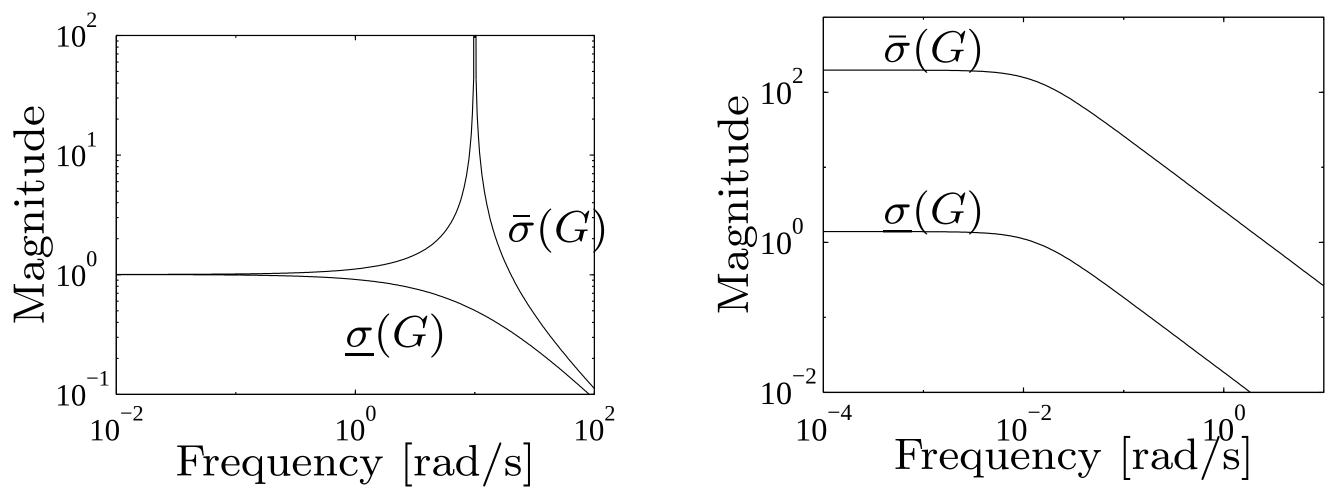
$$K = I$$

$$T(s) = GK(I+GK)^{-1} = \frac{1}{s+1}\begin{bmatrix} 1 & a \\ -a & 1 \end{bmatrix} \tag{6.46}$$

Nominal stability (NS). Two closed loop poles at $s = -1$ and

$$A_{cl} = A - BKC = \begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix} - \begin{bmatrix} 1 & a \\ -a & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

Nominal performance (NP). Figure 59(a)



(a) Spinning satellite in (6.44)

(b) Distillation process in (6.49)

Figure 59: Typical plots of singular values

- $\underline{\sigma}(L) \leq 1 \;\; \forall \omega$ poor performance in low gain direction
- g_{12}, g_{21} large $\Rightarrow$ strong interaction

Robust stability (RS).

Check stability: one loop at a time.

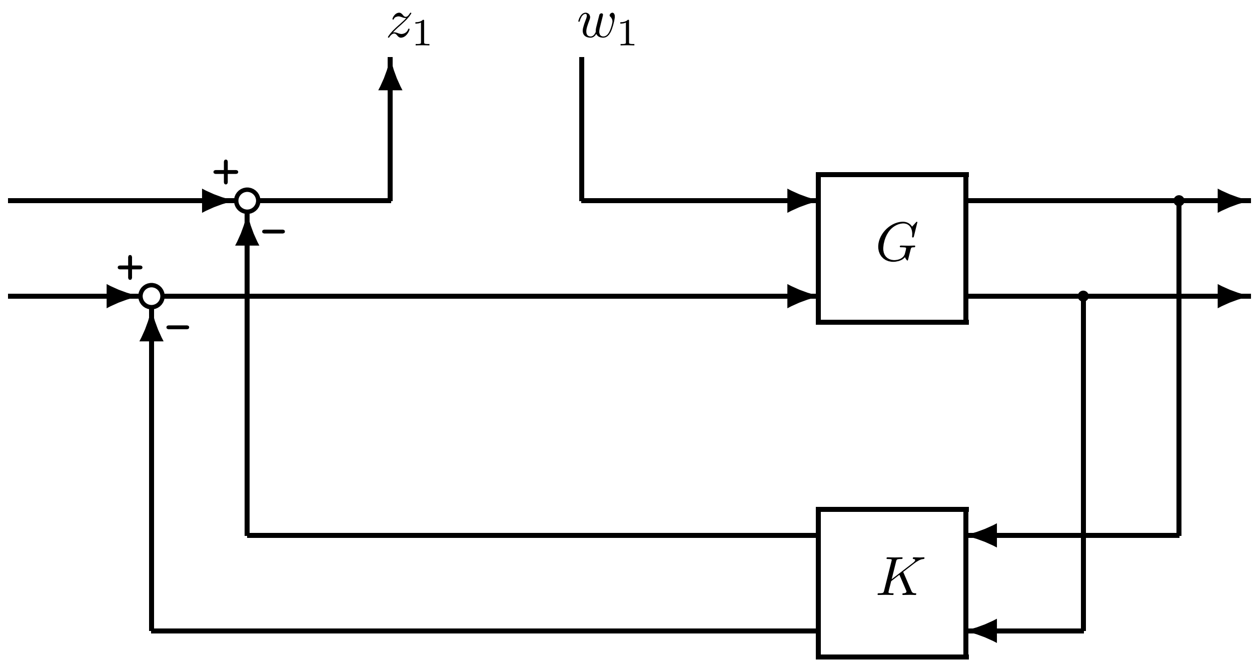


Figure 60: Checking stability margins "one-loop-at-a-time"

$$\frac{z_1}{w_1} \triangleq L_1(s) = \frac{1}{s} \Rightarrow GM = \infty, PM = 90^\circ \qquad (6.47)$$

- Good Robustness? NO
- Consider perturbation in each feedback channel

$$u_1' = (1+\epsilon_1)u_1, \quad u_2' = (1+\epsilon_2)u_2 \qquad (6.48)$$

$$B' = \begin{bmatrix} 1+\epsilon_1 & 0 \\ 0 & 1+\epsilon_2 \end{bmatrix}$$

Closed-loop state matrix:

$$A'_{cl} = A - B'KC = \begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix} - \begin{bmatrix} 1+\epsilon_1 & 0 \\ 0 & 1+\epsilon_2 \end{bmatrix} \begin{bmatrix} 1 & a \\ -a & 1 \end{bmatrix}$$

Characteristic polynomial:

$$\begin{aligned} \det(sI - A'_{cl}) &= s^2 + \underbrace{(2+\epsilon_1+\epsilon_2)}_{a_1} s + \\ &\quad + \underbrace{1+\epsilon_1+\epsilon_2+(a^2+1)\epsilon_1\epsilon_2}_{a_0} \end{aligned}$$

Stability for $(-1 < \epsilon_1 < \infty, \epsilon_2 = 0)$ and $(\epsilon_1 = 0, -1 < \epsilon_2 < \infty)$ (GM=∞)
But only *small simultaneous changes* in the two channels: for example, let $\epsilon_1 = -\epsilon_2$, then the system is unstable ($a_0 < 0$) for

$$|\epsilon_1| > \frac{1}{\sqrt{a^2+1}} \approx 0.1$$

Summary.
Checking single-loop margins is inadequate for MIMO problems.

6.3.2 Motivating robustness example no. 2: Distillation Process [3.7.2]

Idealized dynamic model of a distillation column,

$$G(s) = \frac{1}{75s+1}\begin{bmatrix} 87.8 & -86.4 \\ 108.2 & -109.6 \end{bmatrix} \tag{6.49}$$

(time is in minutes).

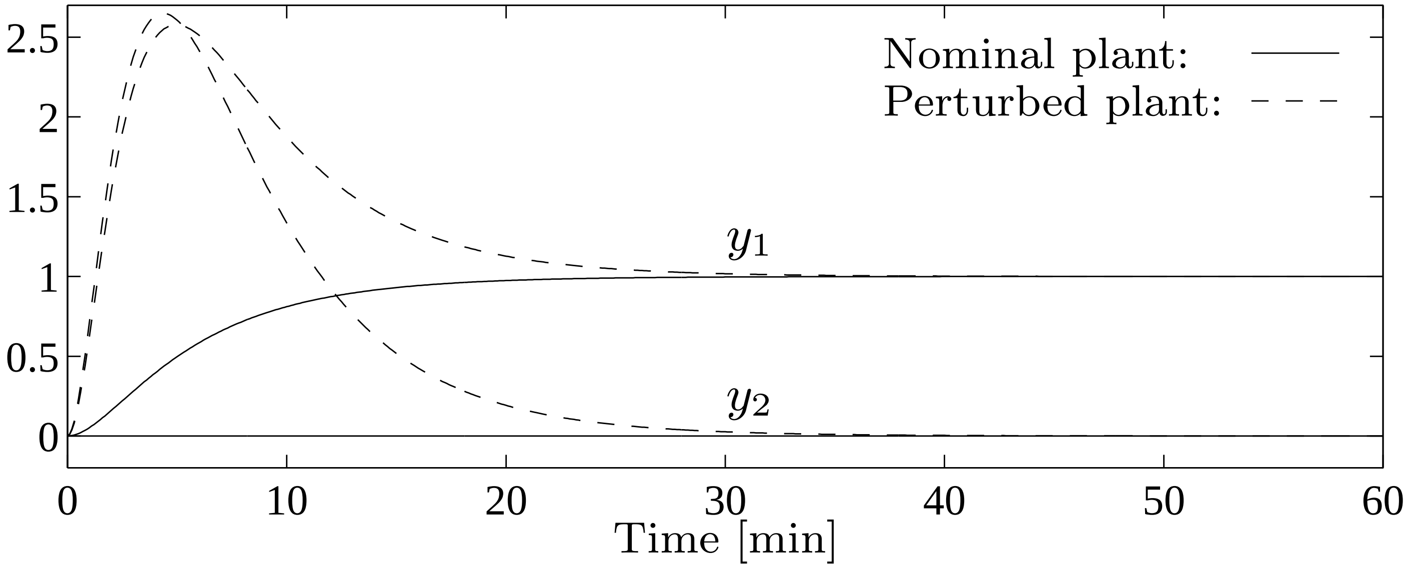


Figure 61: Response with decoupling controller to filtered reference input $r_1 = 1/(5s+1)$. The perturbed plant has 20% gain uncertainty as given by (6.52).

Inverse-based controller or equivalently steady-state decoupler with a PI controller ($k_1 = 0.7$)

$$K_{\text{inv}}(s) = \frac{k_1}{s}G^{-1}(s) = \frac{k_1(1+75s)}{s}\begin{bmatrix} 0.3994 & -0.3149 \\ 0.3943 & -0.3200 \end{bmatrix} \tag{6.50}$$

Nominal performance (NP).

$$GK_{\text{inv}} = K_{\text{inv}}G = \frac{0.7}{s}I$$

first order response with time constant 1.43 (Fig. 61).

Nominal performance (NP) achieved with decoupling controller.

Robust stability (RS).

$$S = S_I = \frac{s}{s+0.7}I; \quad T = T_I = \frac{1}{1.43s+1}I \qquad (6.51)$$

In each channel: GM=∞, PM=90°.

Input gain uncertainty (6.48) with $\epsilon_1 = 0.2$ and $\epsilon_2 = -0.2$:

$$u_1' = 1.2u_1, \quad u_2' = 0.8u_2 \qquad (6.52)$$

$$\begin{aligned} L_I'(s) &= K_{\text{inv}}G' = K_{\text{inv}}G \begin{bmatrix} 1+\epsilon_1 & 0 \\ 0 & 1+\epsilon_2 \end{bmatrix} = \\ & \frac{0.7}{s}\begin{bmatrix} 1+\epsilon_1 & 0 \\ 0 & 1+\epsilon_2 \end{bmatrix} \end{aligned} \qquad (6.53)$$

Perturbed closed-loop poles are

$$s_1 = -0.7(1+\epsilon_1), \quad s_2 = -0.7(1+\epsilon_2) \qquad (6.54)$$

Closed-loop stability as long as the input gains $1 + \epsilon_1$ and $1 + \epsilon_2$ remain positive

$\Rightarrow$ *Robust stability (RS) achieved with respect to input gain errors for the decoupling controller.*

Robust performance (RP).

Performance with model error poor (Fig. 61)

- SISO: NP+RS $\Rightarrow$ RP
- MIMO: NP+RS $\not\Rightarrow$ RP

RP is not achieved by the decoupling controller.

6.3.3 Robustness conclusions [3.7.3]

Multivariable plants can display a sensitivity to uncertainty (in this case input uncertainty) which is fundamentally different from what is possible in SISO systems.

6.4 General control problem formulation [3.8]

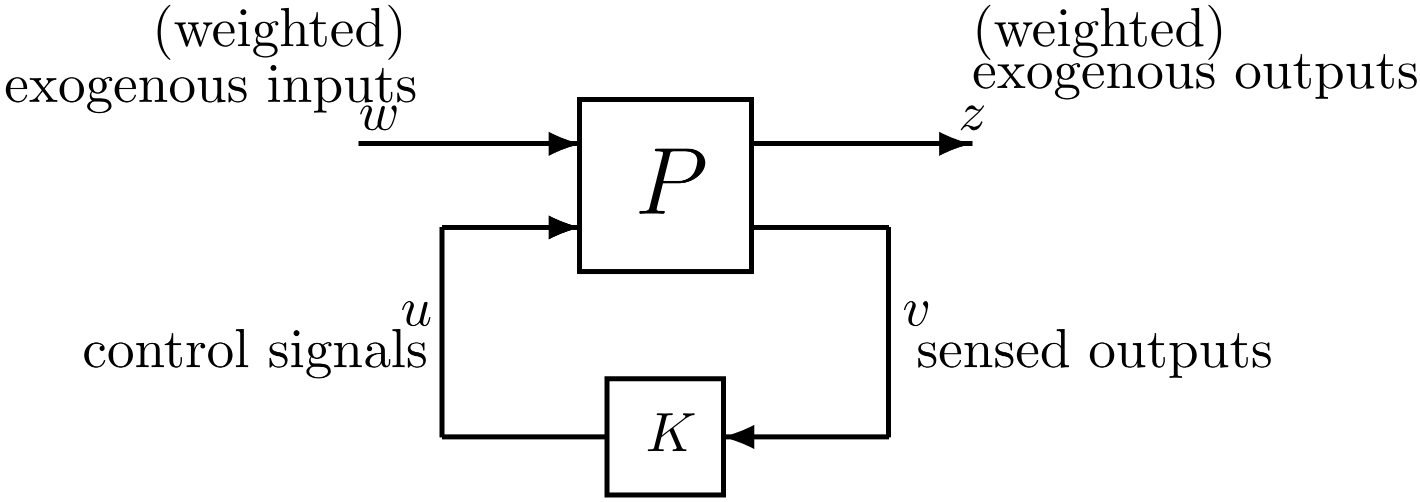


Figure 62: General control configuration for the case with no model uncertainty

The overall control objective is to minimize some norm of the transfer function from w to z, for example, the $\mathcal{H}_\infty$ norm. The controller design problem is then:

Find a controller K which based on the information in v, generates a control signal u which counteracts the influence of w on z, thereby minimizing the closed-loop norm from w to z.

6.4.1 Obtaining the generalized plant P [3.8.1]

The routines in MATLAB for synthesizing $\mathcal{H}_\infty$ and $\mathcal{H}_2$ optimal controllers assume that the problem is in the general form of Figure 62

Example: One degree-of-freedom feedback control configuration.

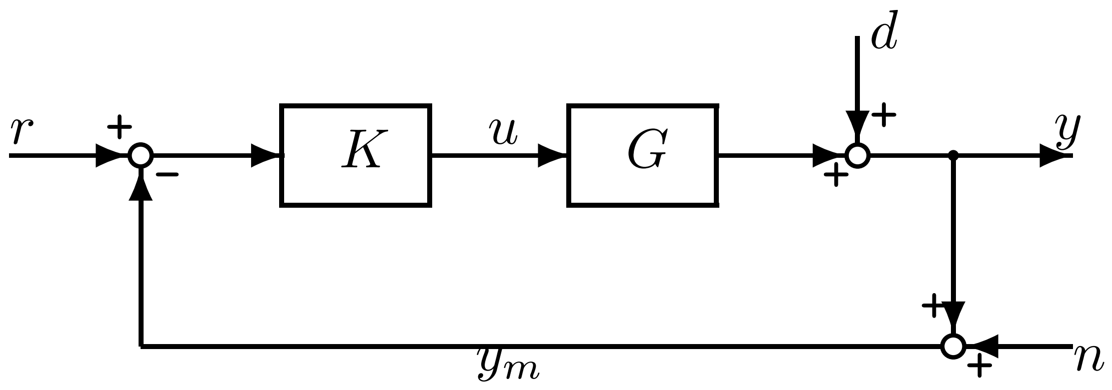


Figure 63: One degree-of-freedom control configuration

Equivalent representation of Figure 63 where the error signal to be minimized is $z = y - r$ and the input to the controller is $v = r - y_m$

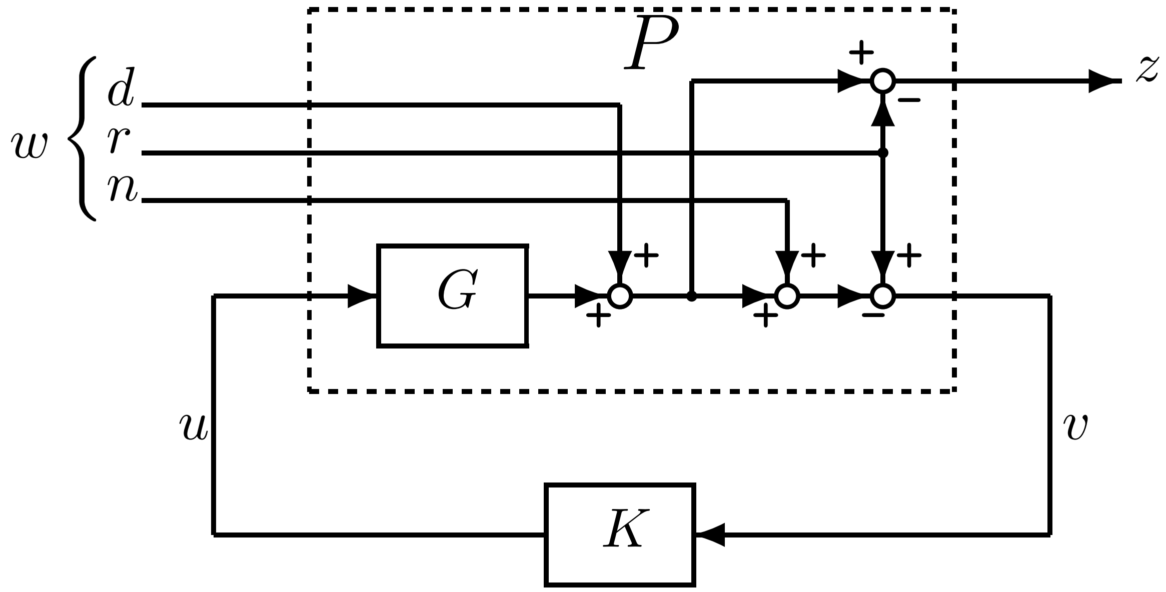


Figure 64: General control configuration equivalent to Figure 63

$$w = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} d \\ r \\ n \end{bmatrix}; z = e = y - r; v = r - y_m = r - y - n \tag{6.55}$$

$$\begin{aligned} z &= y - r = Gu + d - r = Iw_1 - Iw_2 + 0w_3 + Gu \\ v &= r - y_m = r - Gu - d - n = \\ &= -Iw_1 + Iw_2 - Iw_3 - Gu \end{aligned}$$

and P which represents the transfer function matrix from $[\,w \quad u\,]^T$ to $[\,z \quad v\,]^T$ is

$$P = \begin{bmatrix} I & -I & 0 & G \\ -I & I & -I & -G \end{bmatrix} \tag{6.56}$$

Note that P does *not* contain the controller. Alternatively, P can be obtained from Figure 64.

Remark. In MATLAB we may obtain P via `simulink`, or we may use the `sysic` program in the μ-toolbox. The code in Table 2 generates the generalized plant P in (6.56) for Figure 63.

Table 2: **MATLAB program to generate** P

```
% Uses the Mu-toolbox
systemnames = 'G';                     % G is the SISO plant.
inputvar = '[d(1);r(1);n(1);u(1)]'; % Consists of vectors w and u.
input_to_G = '[u]';
outputvar = '[G+d-r; r-G-d-n]';     % Consists of vectors z and v.
sysoutname = 'P';
sysic;
```

6.4.2 Including weights in P [3.8.2]

To get a meaningful controller synthesis problem, for example, in terms of the $\mathcal{H}_\infty$ or $\mathcal{H}_2$ norms, we generally have to include weights W_z and W_w in the generalized plant P, see Figure 65.

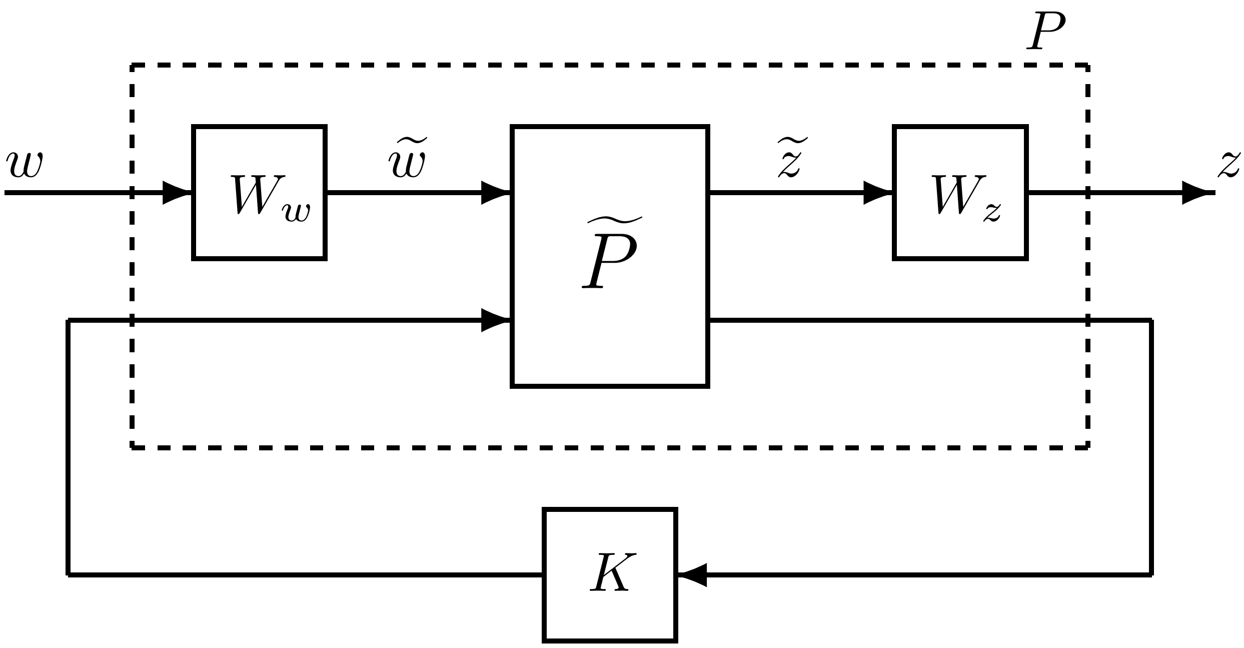


Figure 65: General control configuration for the case with no model uncertainty

That is, we consider the weighted or normalized exogenous inputs w, and the weighted or normalized controlled outputs $z = W_z\widetilde{z}$. The weighting matrices are usually frequency dependent and typically selected such that weighted signals w and z are of magnitude 1, that is, the norm from w to z should be less than 1.

Example: Stacked $S/T/KS$ problem.

Consider an $\mathcal{H}_\infty$ problem where we want to bound $\bar{\sigma}(S)$ (for performance), $\bar{\sigma}(T)$ (for robustness and to avoid sensitivity to noise) and $\bar{\sigma}(KS)$ (to penalize large inputs). These requirements may be combined into a stacked $\mathcal{H}_\infty$ problem

$$\min_K \|N(K)\|_\infty, \quad N = \begin{bmatrix} W_u KS \\ W_T T \\ W_P S \end{bmatrix} \tag{6.57}$$

where K is a stabilizing controller. In other words, we have $z = Nw$ and the objective is to minimize the $\mathcal{H}_\infty$ norm from w to z.

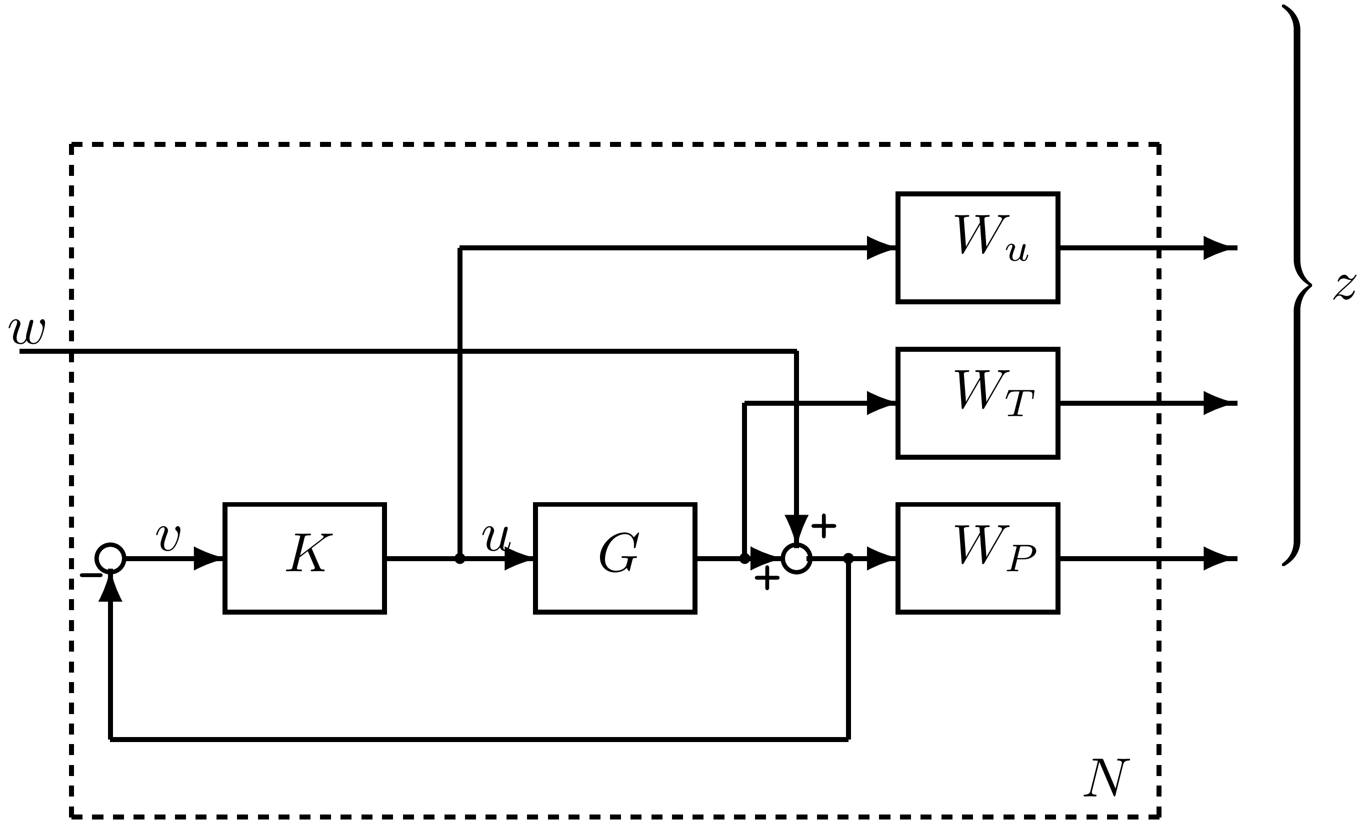


Figure 66: Block diagram corresponding to generalized plant in (6.57)

$$
\begin{aligned}
z_1 &= W_u u \\
z_2 &= W_T G u \\
z_3 &= W_P w + W_P G u \\
v &= -w - Gu
\end{aligned}
$$

so the generalized plant P from $[\,w \quad u\,]^T$ to $[\,z \quad v\,]^T$ is

$$
P = \left[\begin{array}{c|c} 0 & W_u I \\ 0 & W_T G \\ W_P I & W_P G \\ \hline -I & -G \end{array}\right] \qquad (6.58)
$$

6.4.3 Partitioning the generalized plant P [3.8.3]

We often partition P as

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \tag{6.59}$$

so that

$$z = P_{11}w + P_{12}u \tag{6.60}$$

$$v = P_{21}w + P_{22}u \tag{6.61}$$

In Example "Stacked $S/T/KS$ problem" we get from (6.58)

$$P_{11} = \begin{bmatrix} 0 \\ 0 \\ W_P I \end{bmatrix}, \quad P_{12} = \begin{bmatrix} W_u I \\ W_T G \\ W_P G \end{bmatrix} \tag{6.62}$$

$$P_{21} = -I, \quad P_{22} = -G \tag{6.63}$$

Note that P_{22} has dimensions compatible with the controller K in Figure 65

6.4.4 Analysis: Closing the loop to get N [3.8.4]

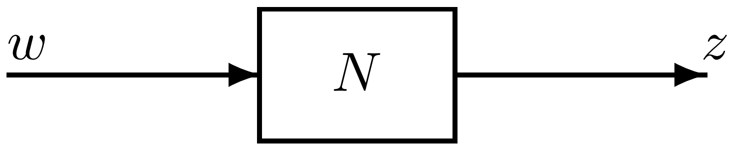


Figure 67: General block diagram for analysis with no uncertainty

For *analysis* of closed-loop performance we may absorb K into the interconnection structure and obtain the system N as shown in Figure 67 where

$$z = Nw \tag{6.64}$$

where N is a function of K. To find N, first partition the generalized plant P as given in (6.59)-(6.61), combine this with the controller equation

$$u = Kv \tag{6.65}$$

and eliminate u and v from equations (6.60), (6.61) and (6.65) to yield $z = Nw$ where N is given by

$$N = P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21} \triangleq F_l(P, K) \tag{6.66}$$

Here $F_l(P,K)$ denotes a lower *linear fractional transformation (LFT)* of P with K as the parameter. In words, N is obtained from Figure 62 by using K to close a lower feedback loop around P. Since positive feedback is used in the general configuration in Figure 62 the term $(I - P_{22}K)^{-1}$ has a negative sign.

Example: We want to derive N for the partitioned P in (6.62) and (6.63) using the LFT-formula in (6.66). We get

$$N = \begin{bmatrix} 0 \\ 0 \\ W_P I \end{bmatrix} + \begin{bmatrix} W_u I \\ W_T G \\ W_P G \end{bmatrix} K(I+GK)^{-1}(-I) = \begin{bmatrix} -W_u KS \\ -W_T T \\ W_P S \end{bmatrix}$$

where we have made use of the identities $S = (I + GK)^{-1}$, $T = GKS$ and $I - T = S$.

In the MATLAB μ-Toolbox we can evaluate $N = F_l(P,K)$ using the command `N=starp(P,K)`. Here `starp` denotes the matrix star product which generalizes the use of LFTs.

6.4.5 Further examples [3.8.5]

Example: Consider the control system in Figure 68, where y_1 is the output we want to control, y_2 is a secondary output (extra measurement), and we also measure the disturbance d. The control configuration includes a two degrees-of-freedom controller, a feedforward controller and a local feedback controller based on the extra measurement y_2.

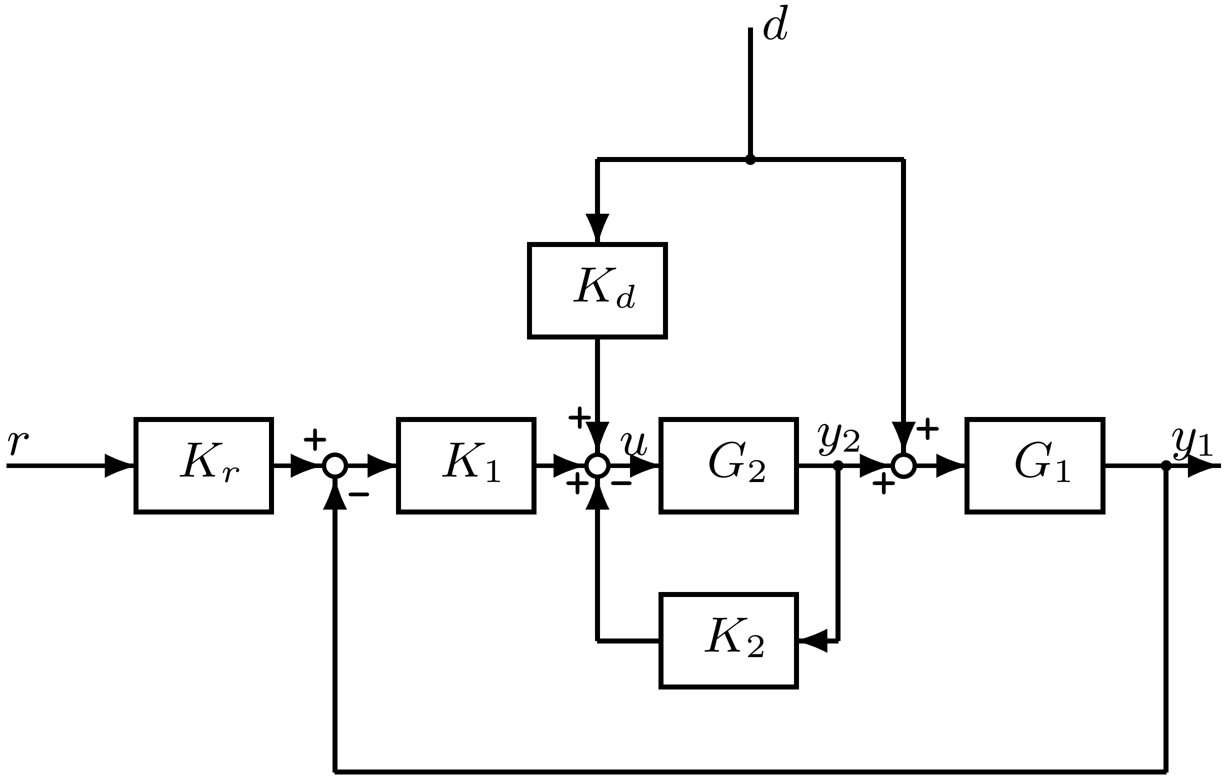


Figure 68: System with feedforward, local feedback and two degrees-of-freedom control

To recast this into our standard configuration of Figure 62 we define

$$w = \begin{bmatrix} d \\ r \end{bmatrix}; \quad z = y_1 - r; \quad v = \begin{bmatrix} r \\ y_1 \\ y_2 \\ d \end{bmatrix} \tag{6.67}$$

$$K = [\, K_1 K_r \quad -K_1 \quad -K_2 \quad K_d \,] \tag{6.68}$$

We get

$$P = \left[\begin{array}{cc|c} G_1 & -I & G_1 G_2 \\ \hline 0 & I & 0 \\ G_1 & 0 & G_1 G_2 \\ 0 & 0 & G_2 \\ I & 0 & 0 \end{array}\right] \tag{6.69}$$

Then partitioning P as in (6.60) and (6.61) yields $P_{22} = [\, 0^T \quad (G_1 G_2)^T \quad G_2^T \quad 0^T \,]^T$.

6.4.6 * Deriving P from N [3.8.6]

For cases where N is given and we wish to find a P such that

$$N = F_l(P, K) = P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21}$$

it is usually best to work from a block diagram representation. This was illustrated above for the stacked N in (6.57). Alternatively, the following procedure may be useful:

1. Set $K = 0$ in N to obtain P_{11}.

2. Define $Q = N - P_{11}$ and rewrite Q such that each term has a common factor $R = K(I - P_{22}K)^{-1}$ (this gives P_{22}).

3. Since $Q = P_{12}RP_{21}$, we can now usually obtain P_{12} and P_{21} by inspection.

Example 3 Weighted sensitivity. *We will use the above procedure to derive P when*

$N = w_P S = w_P(I + GK)^{-1}$,

where w_P is a scalar weight.

1. $P_{11} = N(K = 0) = w_P I$.

2. $Q = N - w_P I = w_P(S - I) = -w_P T = -w_P GK(I + GK)^{-1}$, *and we have* $R = K(I + GK)^{-1}$ *so* $P_{22} = -G$.

3. $Q = -w_P GR$ *so we have* $P_{12} = -w_P G$ *and* $P_{21} = I$, *and we get*

$$P = \begin{bmatrix} w_P I & -w_P G \\ I & -G \end{bmatrix} \tag{6.70}$$

6.4.8 A general control configuration including model uncertainty [3.8.8]

The general control configuration in Figure 62 may be extended to include model uncertainty. Here the matrix Δ is a *block-diagonal* matrix that includes all possible perturbations (representing uncertainty) to the system. It is normalized such that $\|\Delta\|_\infty \leq 1$.

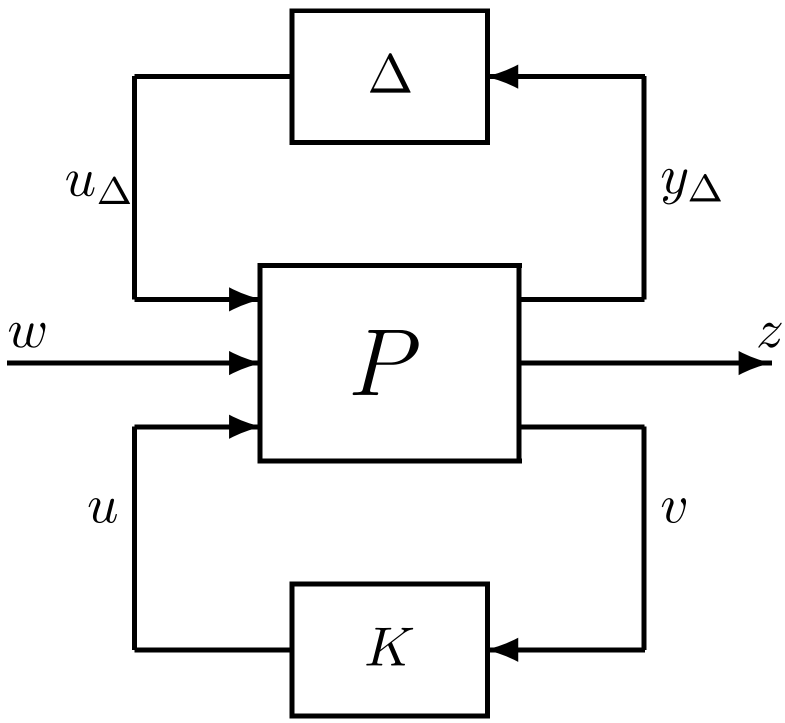


Figure 69: General control configuration for the case with model uncertainty

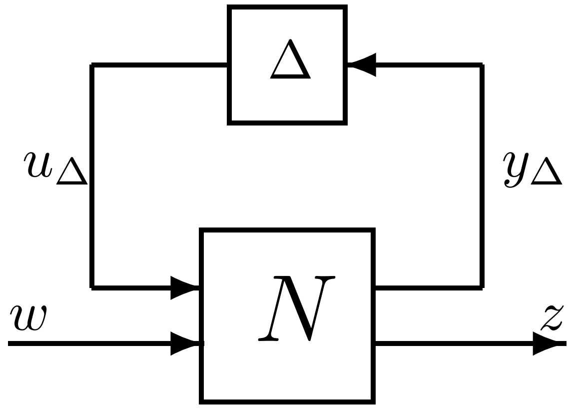


Figure 70: General block diagram for analysis with uncertainty included

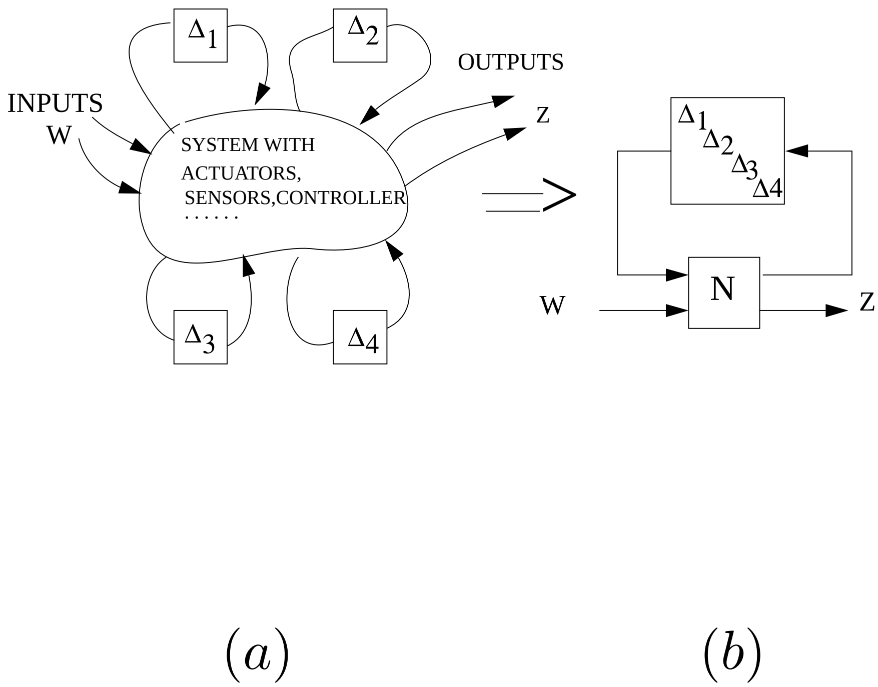


Figure 71: Rearranging a system with multiple perturbations into the $N\Delta$-structure

The block diagram in Figure 69 in terms of P (for synthesis) may be transformed into the block diagram in Figure 70 in terms of N (for analysis) by using K to close a lower loop around P. The same *lower LFT* as found in (6.66) applies, and

$$N = F_l(P, K) = P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21} \quad (6.71)$$

To evaluate the perturbed (uncertain) transfer function from external inputs w to external outputs z, we use Δ to close the upper loop around N (see Figure 70), resulting in an *upper LFT*:

$$z = F_u(N, \Delta)w; \; F_u(N, \Delta) \triangleq N_{22}+N_{21}\Delta(I-N_{11}\Delta)^{-1}N_{12} \quad (6.72)$$

Remark 1 Almost any control problem with uncertainty can be represented by Figure 69. First represent each source of uncertainty by a perturbation block, Δ_i, which is normalized such that $\|\Delta_i\| \leq 1$. Then "pull out" each of these blocks from the system so that an input and an output can be associated with each Δ_i as shown in Figure 71(a). Finally, collect these perturbation blocks into a large block-diagonal matrix having perturbation inputs and outputs as shown in Figure 71(b).

7 LIMITATIONS ON PERFORMANCE IN MIMO SYSTEMS [6]

Differently: In a MIMO system, disturbances, the plant, RHP-zeros, RHP-poles and delays each have directions associated with them. A multivariable plant may have a RHP-zero and a RHP-pole at the same location, but their effects may not interact.

7.1 Constraints on S and T [6.2]

From the identity $S + T = I$ we get

$$|1 - \bar{\sigma}(S)| \leq \bar{\sigma}(T) \leq 1 + \bar{\sigma}(S) \tag{7.1}$$

$$|1 - \bar{\sigma}(T)| \leq \bar{\sigma}(S) \leq 1 + \bar{\sigma}(T) \tag{7.2}$$

$\Rightarrow$ S and T cannot be small simultaneously; $\bar{\sigma}(S)$ is large if and only if $\bar{\sigma}(T)$ is large.

7.2 Sensitivity peaks [6.2.4]

Theorem 8 Weighted sensitivity. *Suppose the plant $G(s)$ has a RHP-zero at $s = z$. Let $w_P(s)$ be any stable scalar weight. Then for closed-loop stability the weighted sensitivity function must satisfy*

$$\|w_P(s)S(s)\|_\infty = \max_\omega \bar{\sigma}(w_P(j\omega)S(j\omega)) \geq |w_P(z)| \tag{7.3}$$

In MIMO systems we generally have the freedom to move the effect of RHP zeros to different outputs by appropriate control.

Theorem 9 Weighted complementary sensitivity. *Suppose the plant $G(s)$ has a RHP-pole at $s = p$. Let $w_T(s)$ be any stable scalar weight. Then for closed-loop stability the weighted complementary sensitivity function must satisfy*

$$\|w_T(s)T(s)\|_\infty = \max_\omega \bar{\sigma}(w_T(j\omega)T(j\omega)) \geq |w_T(p)| \tag{7.4}$$

7.3 Limitations imposed by uncertainty [6.10]

7.3.1 Input and output uncertainty

In a multiplicative (relative) form, the output and input uncertainties (as in Figure 72) are given by

$$\text{Output uncertainty:}\quad G' = (I + E_O)G \quad \text{or}$$
$$E_O = (G' - G)G^{-1} \tag{7.5}$$
$$\text{Input uncertainty:}\quad G' = G(I + E_I) \quad \text{or}$$
$$E_I = G^{-1}(G' - G) \tag{7.6}$$

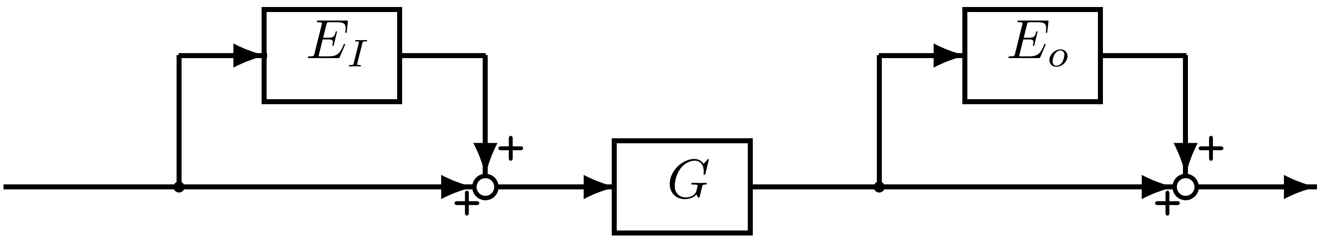


Figure 72: Plant with multiplicative input and output uncertainty

7.3.3 Uncertainty and the benefits of feedback [6.10.3]

Feedback control. With one degree-of-freedom feedback control the nominal transfer function is $y = Tr$ where $T = L(I + L)^{-1}$ is the complementary sensitivity function. Ideally, $T = I$. The change in response with model error is $y' - y = (T' - T)r$ where

$$T' - T = S'E_O T \tag{7.7}$$

Thus, $y' - y = S'E_O Tr = S'E_O y$, and we see that

- with feedback control the effect of the uncertainty is reduced by a factor S' relative to that with feedforward control.

7.3.4 Uncertainty and the sensitivity peak

We will derive *upper bounds* on $\bar{\sigma}(S')$ which involve the plant and controller condition numbers

$$\gamma(G) = \frac{\bar{\sigma}(G)}{\underline{\sigma}(G)}, \qquad \gamma(K) = \frac{\bar{\sigma}(K)}{\underline{\sigma}(K)} \tag{7.8}$$

Factorizations of S' in terms of the nominal sensitivity S

Output uncertainty: $$S' = S(I + E_O T)^{-1} \tag{7.9}$$

Input uncertainty: $$S' = S(I + GE_I G^{-1} T)^{-1} = \\ = SG(I + E_I T_I)^{-1} G^{-1} \tag{7.10}$$

$$S' = (I + TK^{-1} E_I K)^{-1} S = \\ = K^{-1}(I + T_I E_I)^{-1} KS \tag{7.11}$$

We assume: G and G' are stable; closed-loop stability, i.e. S and S' are stable; therefore $(I + E_O T)^{-1}$ and $(I + E_I T_I)^{-1}$ are stable; the magnitude of the multiplicative (relative) uncertainty at each frequency can be bounded in terms of its singular value

$$\bar{\sigma}(E_I) \leq |w_I|, \qquad \bar{\sigma}(E_O) \leq |w_O| \tag{7.12}$$

where $w_I(s)$ and $w_O(s)$ are scalar weights. Typically the uncertainty bound, $|w_I|$ or $|w_O|$, is 0.2 at low frequencies and exceeds 1 at higher frequencies.

Upper bound on $\bar{\sigma}(S')$ for output uncertainty

From (7.9) we derive

$$\bar{\sigma}(S') \leq \bar{\sigma}(S)\bar{\sigma}((I + E_O T)^{-1}) \leq \frac{\bar{\sigma}(S)}{1 - |w_O|\bar{\sigma}(T)} \tag{7.13}$$

Upper bounds on $\bar{\sigma}(S')$ for input uncertainty

The sensitivity function can be much more sensitive to input uncertainty than output uncertainty.

From (7.10) and (7.11) we derive:

$$\bar{\sigma}(S') \leq \gamma(G)\bar{\sigma}(S)\bar{\sigma}((I+E_I T_I)^{-1}) \leq \leq \gamma(G)\frac{\bar{\sigma}(S)}{1-|w_I|\bar{\sigma}(T_I)} \tag{7.14}$$

$$\bar{\sigma}(S') \leq \gamma(K)\bar{\sigma}(S)\bar{\sigma}((I+T_I E_I)^{-1}) \leq \leq \gamma(K)\frac{\bar{\sigma}(S)}{1-|w_I|\bar{\sigma}(T_I)} \tag{7.15}$$

$\Rightarrow$ If we use a "round" controller ($\gamma(K) \approx 1$) then the sensitivity function is *not* sensitive to input uncertainty.

8 ROBUST STABILITY AND PERFORMANCE ANALYSIS [8]

8.1 General control configuration with uncertainty [8.1]

For our robustness analysis we use a system representation in which the uncertain perturbations are "pulled out" into a block-diagonal matrix,

$$\Delta = \text{diag}\{\Delta_i\} = \begin{bmatrix} \Delta_1 & & & \\ & \ddots & & \\ & & \Delta_i & \\ & & & \ddots \end{bmatrix} \tag{8.1}$$

where each Δ_i represents a specific source of uncertainty.

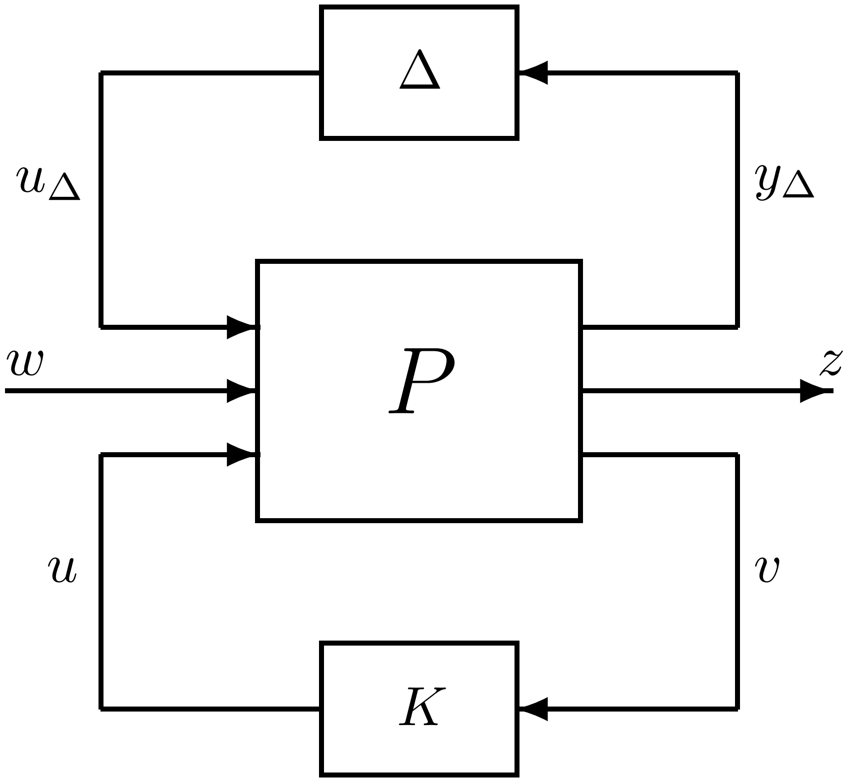


Figure 73: General control configuration (for controller synthesis)

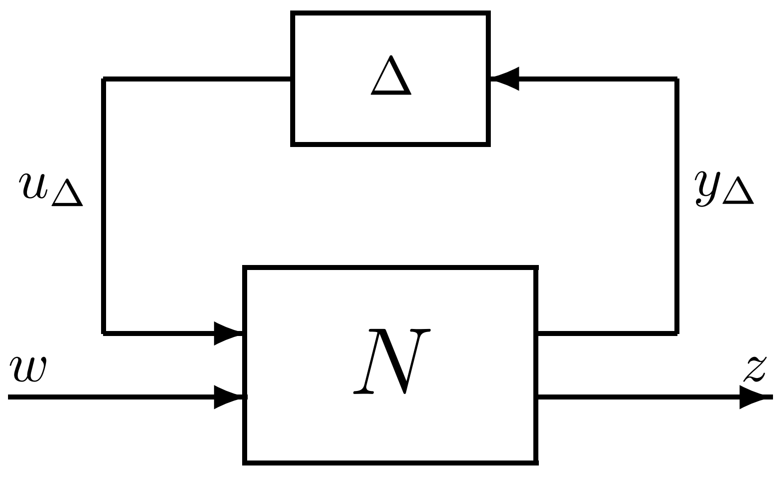


Figure 74: $N\Delta$-structure for robust performance analysis

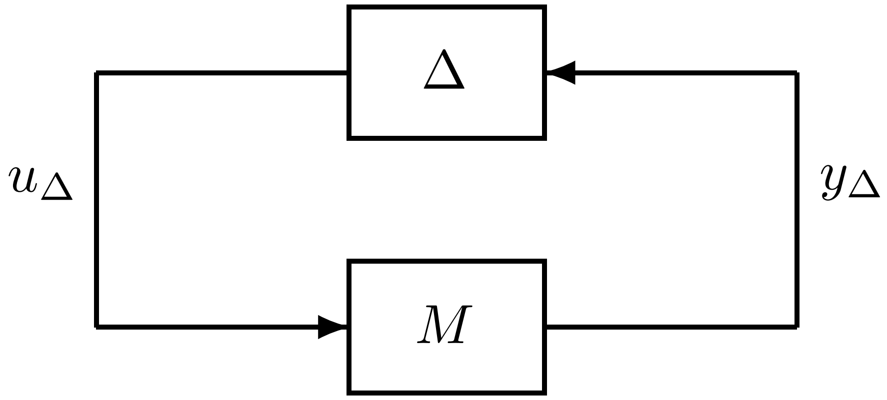
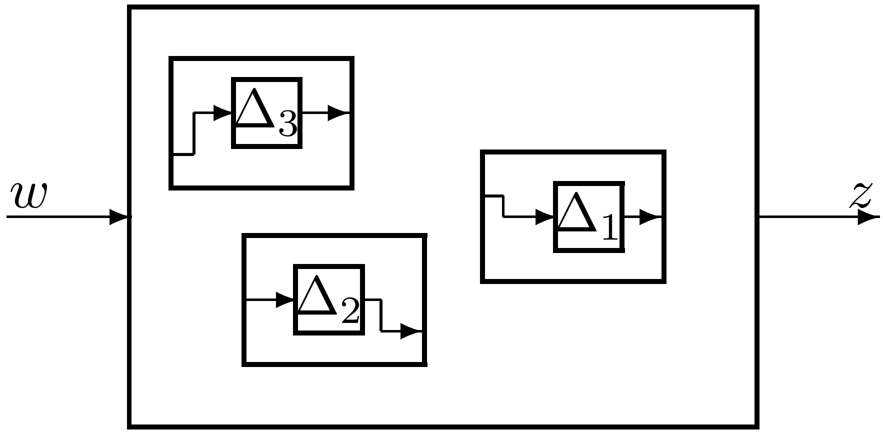
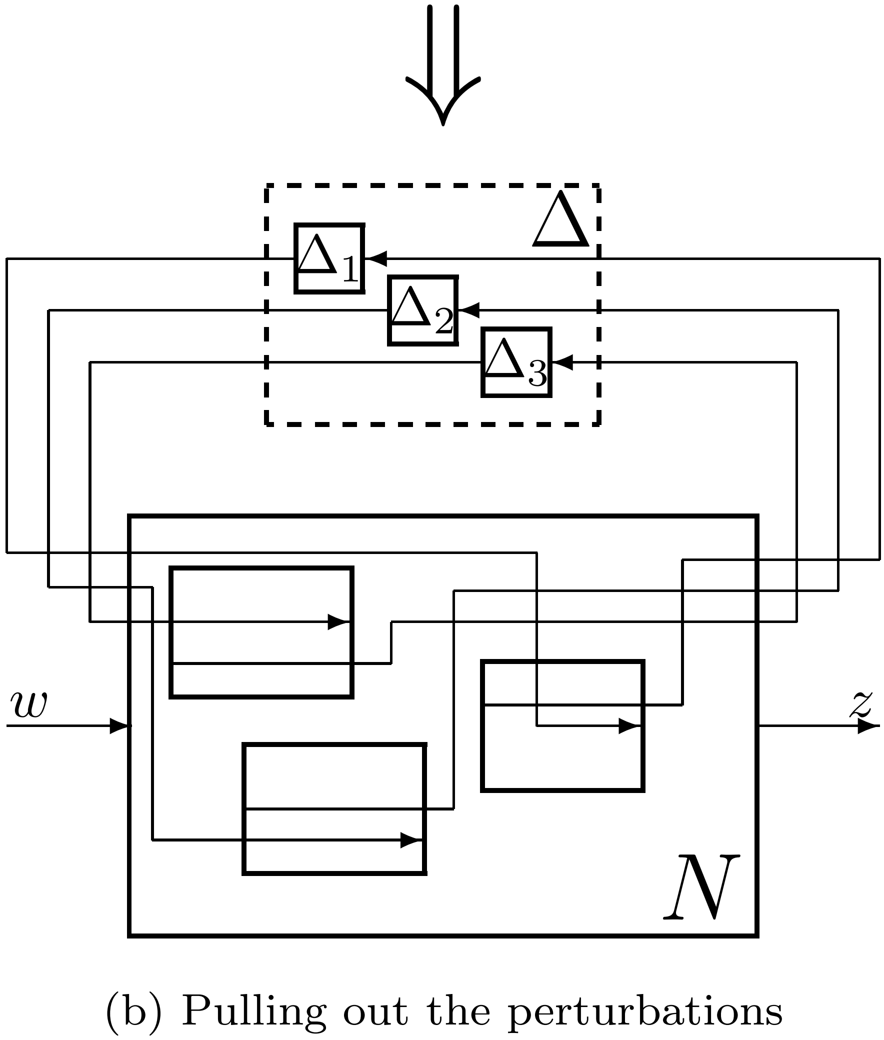


Figure 75: $M\Delta$-structure for robust stability analysis

If we also pull out the controller K, we get the generalized plant P, as shown in Figure 73. *For analysis* of the uncertain system, we use the $N\Delta$-structure in Figure 74.



(a) Original system with multiple perturbations



(b) Pulling out the perturbations

Figure 76: Rearranging an uncertain system into the $N\Delta$-structure

Consider Figure 76 where it is shown how to pull out the perturbation blocks to form Δ and the nominal system N. As shown in (6.71), N is related to P and K by a lower LFT

$$N = F_l(P,K) \triangleq P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21} \quad (8.2)$$

Similarly, the uncertain closed-loop transfer function from w to z, $z = Fw$, is related to N and Δ by an upper LFT (see (6.72)),

$$F = F_u(N,\Delta) \triangleq N_{22} + N_{21}\Delta(I - N_{11}\Delta)^{-1}N_{12} \quad (8.3)$$

To analyze robust stability of F, we can then rearrange the system into the $M\Delta$-structure of Figure 75 where $M = N_{11}$ is the transfer function from the output to the input of the perturbations.

8.2 Representing uncertainty [8.2]

As usual, each individual perturbation is assumed to be stable and is normalized,

$$\bar{\sigma}(\Delta_i(j\omega)) \leq 1 \ \forall\omega \tag{8.4}$$

For a complex scalar perturbation we have $|\delta_i(j\omega)| \leq 1, \ \forall\omega$, and for a real scalar perturbation $-1 \leq \delta_i \leq 1$. Since the maximum singular value of a block diagonal matrix is equal to the largest of the maximum singular values of the individual blocks, it then follows for $\Delta = \text{diag}\{\Delta_i\}$ that

$$\bar{\sigma}(\Delta_i(j\omega)) \leq 1 \ \forall\omega, \ \forall i \quad \Leftrightarrow \quad \boxed{\|\Delta\|_\infty \leq 1} \tag{8.5}$$

Note that Δ has *structure*, and therefore in the robustness analysis we do *not* want to allow all Δ such that (8.5) is satisfied.

8.2.1 Unstructured uncertainty

We define *unstructured* uncertainty as the use of a "full" complex perturbation matrix Δ, usually with dimensions compatible with those of the plant, where at each frequency any $\Delta(j\omega)$ satisfying $\bar{\sigma}(\Delta(j\omega)) \leq 1$ is allowed.

Six common forms of unstructured uncertainty are shown in Figure 77. In Figure 77(a), (b) and (c) are shown three *feedforward* forms; additive uncertainty, multiplicative input uncertainty and multiplicative output uncertainty:

$$\Pi_A: \quad G_p = G + E_A; \quad E_a = w_A \Delta_a \tag{8.6}$$

$$\Pi_I: \quad G_p = G(I + E_I); \quad E_I = w_I \Delta_I \tag{8.7}$$

$$\Pi_O: \quad G_p = (I + E_O)G; \quad E_O = w_O \Delta_O \tag{8.8}$$

In Figure 77(d), (e) and (f) are shown three *feedback* or *inverse* forms; inverse additive uncertainty, inverse multiplicative input uncertainty and inverse multiplicative output uncertainty:

$$\Pi_{iA}: \qquad G_p = G(I - E_{iA}G)^{-1}; \qquad E_{iA} = w_{iA}\Delta_{iA} \tag{8.9}$$

$$\Pi_{iI}: \qquad G_p = G(I - E_{iI})^{-1}; \qquad E_{iI} = w_{iI}\Delta_{iI} \tag{8.10}$$

$$\Pi_{iO}: \qquad G_p = (I - E_{iO})^{-1}G; \qquad E_{iO} = w_{iO}\Delta_{iO} \tag{8.11}$$

The negative sign in front of the E's does not really matter here since we assume that Δ can have any sign. Δ denotes the normalized perturbation and E the "actual" perturbation. We have here used scalar weights w, so $E = w\Delta = \Delta w$, but sometimes one may want to use matrix weights, $E = W_2\Delta W_1$ where W_1 and W_2 are given transfer function matrices.

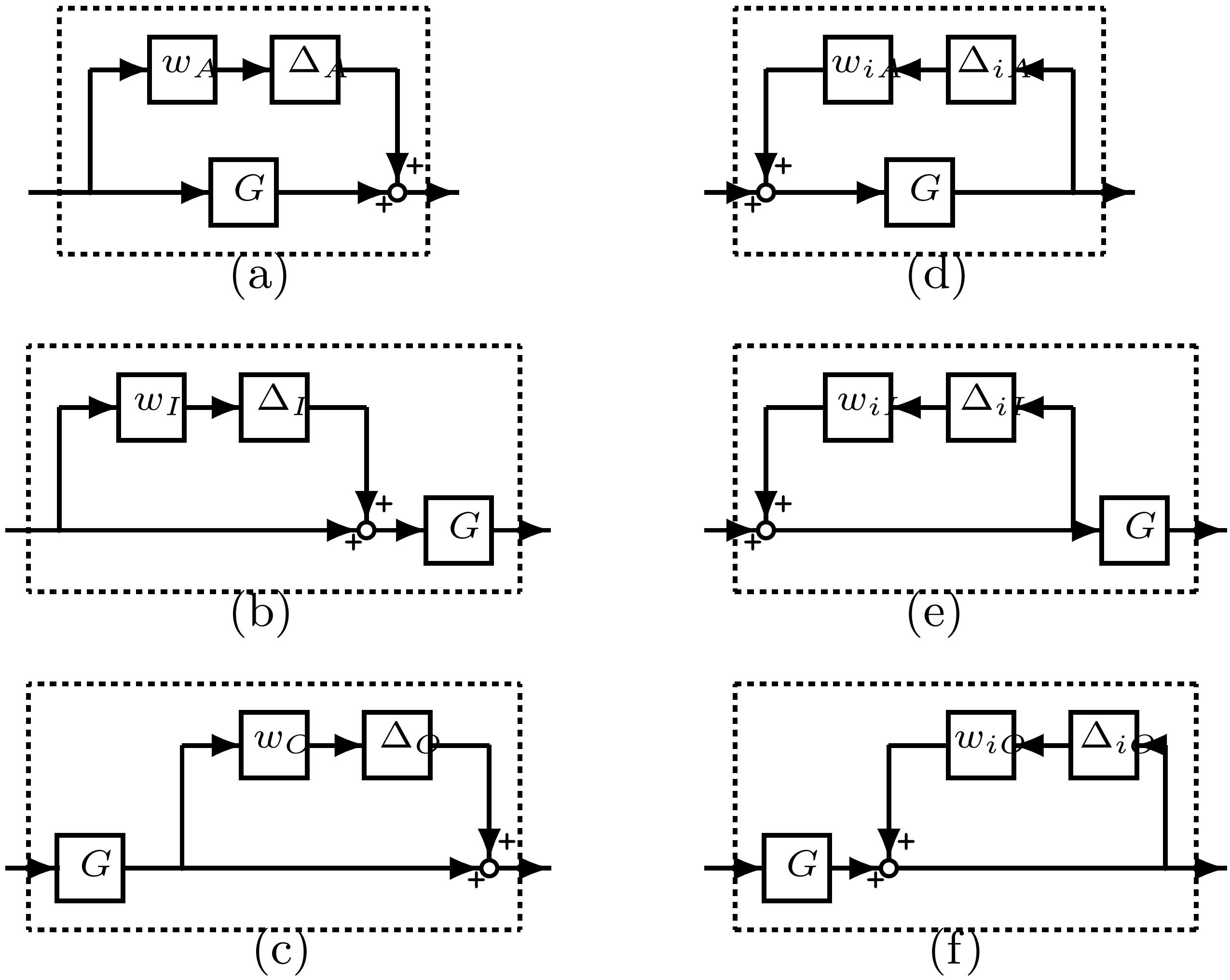


Figure 77: (a) Additive uncertainty, (b) Multiplicative input uncertainty, (c) Multiplicative output uncertainty, (d) Inverse additive uncertainty, (e) Inverse multiplicative input uncertainty, (f) Inverse multiplicative output uncertainty

8.3 Obtaining P, N and M [8.3]

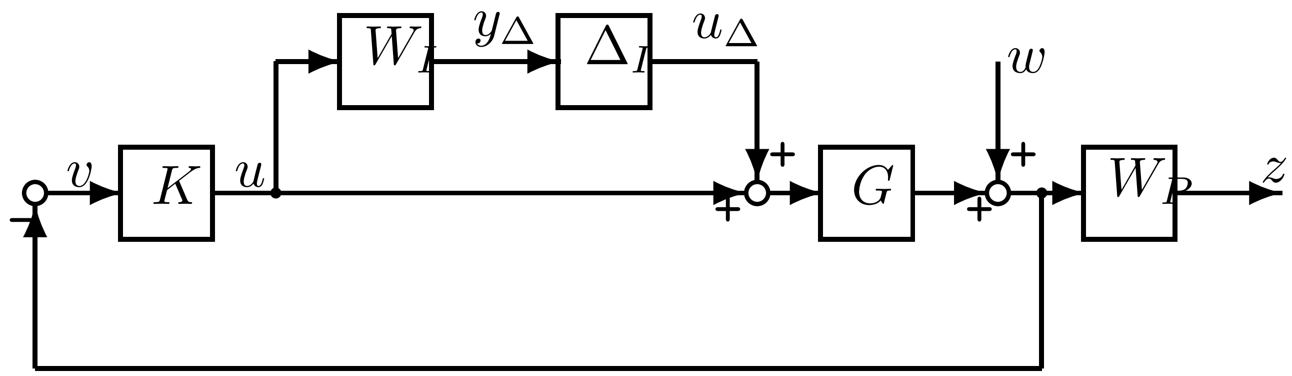


Figure 78: System with multiplicative input uncertainty and performance measured at the output

Example 4 System with input uncertainty (Figure 78). *We want to derive the generalized plant P in Figure 73 which has inputs $[\, u_\Delta \quad w \quad u \,]^T$ and outputs $[\, y_\Delta \quad z \quad v \,]^T$. By writing down the equations or simply by inspecting Figure 78 (remember to break the loop before and after K) we get*

$$P = \begin{bmatrix} 0 & 0 & W_I \\ W_P G & W_P & W_P G \\ -G & -I & -G \end{bmatrix} \tag{8.12}$$

Next, we want to derive the matrix N corresponding to Figure 74. First, partition P to be compatible with K, i.e.

$$P_{11} = \begin{bmatrix} 0 & 0 \\ W_P G & W_P \end{bmatrix}, \; P_{12} = \begin{bmatrix} W_I \\ W_P G \end{bmatrix} \tag{8.13}$$

$$P_{21} = [\,-G \quad -I\,],\;\; P_{22} = -G \tag{8.14}$$

and then find $N = F_l(P, K)$ *using (8.2). We get*

$$N = \begin{bmatrix} -W_I KG(I+KG)^{-1} & -W_I K(I+GK)^{-1} \\ W_P G(I+KG)^{-1} & W_P(I+GK)^{-1} \end{bmatrix} \tag{8.15}$$

Alternatively, we can derive N *directly from Figure 78 by evaluating the closed-loop transfer function from inputs* $\begin{bmatrix} u_\Delta \\ w \end{bmatrix}$ *to outputs* $\begin{bmatrix} y_\Delta \\ z \end{bmatrix}$ *(*without *breaking the loop before and after* K*). For example, to derive* N_{12}*, which is the transfer function from* w *to* y_Δ*, we start at the output (*y_Δ*) and move backwards to the input (*w*) using the MIMO Rule (we first meet* W_I*, then* $-K$ *and we then exit the feedback loop and get the term* $(I + GK)^{-1}$*).*

The upper left block, N_{11}*, in (8.15) is the transfer function from* u_Δ *to* y_Δ*. This is the transfer function* M *needed in Figure 75 for evaluating robust stability. Thus, we have* $M = -W_I KG(I + KG)^{-1} = -W_I T_I$*.*

8.4 Definitions of robust stability and robust performance [8.4]

1. *Robust stability (RS) analysis*: with a given controller K we determine whether the system remains stable for all plants in the uncertainty set.
2. *Robust performance (RP) analysis*: if RS is satisfied, we determine how "large" the transfer function from exogenous inputs w to outputs z may be for all plants in the uncertainty set.

In Figure 74, w represents the exogenous inputs (normalized disturbances and references), and z the exogenous outputs (normalized errors). We have $z = F(\Delta)w$, where from (8.3)

$$F = F_u(N, \Delta) \stackrel{\Delta}{=} N_{22} + N_{21}\Delta(I - N_{11}\Delta)^{-1}N_{12} \quad (8.16)$$

We here use the $\mathcal{H}_\infty$ norm to define performance and require for RP that $\|F(\Delta)\|_\infty \leq 1$ for all allowed Δ's. A typical choice is $F = w_P S_p$ (the weighted sensitivity function), where w_P is the performance weight (capital P for performance) and S_p represents the set of perturbed sensitivity functions (lower-case p for perturbed).

In terms of the $N\Delta$-structure in Figure 74 our requirements for stability and performance are

$$\text{NS} \overset{\text{def}}{\Leftrightarrow} N \text{ is internally stable} \tag{8.17}$$

$$\text{NP} \overset{\text{def}}{\Leftrightarrow} \|N_{22}\|_\infty < 1; \quad \text{and NS} \tag{8.18}$$

$$\text{RS} \overset{\text{def}}{\Leftrightarrow} F = F_u(N, \Delta) \text{ is stable } \forall \Delta, \|\Delta\|_\infty \leq 1; \text{ and NS} \tag{8.19}$$

$$\text{RP} \overset{\text{def}}{\Leftrightarrow} \|F\|_\infty < 1, \quad \forall \Delta, \|\Delta\|_\infty \leq 1; \text{ and NS} \tag{8.20}$$

Remark 1 Allowed perturbations. For simplicity below we will use the shorthand notation

$$\forall \Delta \quad \text{and} \quad \max_{\Delta} \tag{8.21}$$

to mean "for all Δ's in the set of allowed perturbations", and "maximizing over all Δ's in the set of allowed perturbations". By *allowed perturbations* we mean that the $\mathcal{H}_\infty$ norm of Δ is less or equal to 1, $\|\Delta\|_\infty \leq 1$, and that Δ has a specified block-diagonal structure.

8.5 Robust stability of the $M\Delta$-structure [8.5]

Consider the uncertain $N\Delta$-system in Figure 74 for which the transfer function from w to z is, as in (8.16), given by

$$F_u(N, \Delta) = N_{22} + N_{21}\Delta(I - N_{11}\Delta)^{-1}N_{12} \quad (8.22)$$

Suppose that the system is nominally stable (with $\Delta = 0$), that is, N is stable. We also assume that Δ is stable. Thus, when we have nominal stability (NS), the stability of the system in Figure 74 is equivalent to the stability of the $M\Delta$-structure in Figure 75 where $M = N_{11}$.

Theorem 10 Determinant stability condition (Real or complex perturbations). *Assume that the nominal system $M(s)$ and the perturbations $\Delta(s)$ are stable. Consider the convex set of perturbations Δ, such that if Δ' is an allowed perturbation then so is $c\Delta'$ where c is any* real *scalar such that $|c| \leq 1$. Then the $M\Delta$-system in Figure 75 is stable for all allowed perturbations (we have RS) if and only if*

$$\text{Nyquist plot of } \det\left(I - M(s)\Delta(s)\right) \text{ does not encircle the origin, } \forall \Delta \tag{8.23}$$

$$\Leftrightarrow \quad \boxed{\det\left(I - M(j\omega)\Delta(j\omega)\right) \neq 0, \quad \forall \omega, \forall \Delta} \tag{8.24}$$

$$\Leftrightarrow \quad \lambda_i(M\Delta) \neq 1, \quad \forall i, \forall \omega, \forall \Delta \tag{8.25}$$

8.6 RS for complex unstructured uncertainty [8.6]

Theorem 11 RS for unstructured ("full") perturbations. *Assume that the nominal system $M(s)$ is stable (NS) and that the perturbations $\Delta(s)$ are stable. Then the $M\Delta$-system in Figure 75 is stable for all perturbations Δ satisfying $\|\Delta\|_\infty \leq 1$ (i.e. we have RS) if and only if*

$$\boxed{\bar{\sigma}(M(j\omega)) < 1 \quad \forall w} \qquad \Leftrightarrow \qquad \boxed{\|M\|_\infty < 1} \tag{8.26}$$

8.6.1 * Application of the unstructured RS-condition [8.6.1]

We will now present necessary and sufficient conditions for robust stability (RS) for each of the six single unstructured perturbations in Figure 77. with

$$E = W_2 \Delta W_1, \quad \|\Delta\|_\infty \le 1 \tag{8.27}$$

To derive the matrix M we simply "isolate" the perturbation, and determine the transfer function matrix

$$M = W_1 M_0 W_2 \tag{8.28}$$

from the output to the input of the perturbation, where M_0 for each of the six cases becomes (disregarding some negative signs which do not affect the subsequent robustness condition) is given by

$$\begin{aligned}
G_p = G + E_A: &\quad M_0 = K(I+GK)^{-1} = KS \\
G_p = G(I + E_I): &\quad M_0 = K(I+GK)^{-1}G = T_I \\
G_p = (I + E_O)G: &\quad M_0 = GK(I+GK)^{-1} = T \\
G_p = G(I - E_{iA}G)^{-1}: &\quad M_0 = (I+GK)^{-1}G = SG \\
G_p = G(I - E_{iI})^{-1}: &\quad M_0 = (I+KG)^{-1} = S_I \\
G_p = (I - E_{iO})^{-1}G: &\quad M_0 = (I+GK)^{-1} = S
\end{aligned} \tag{8.29}$$

Theorem 11 then yields

$$\text{RS} \quad \Leftrightarrow \quad \|W_1 M_0 W_2(j\omega)\|_\infty < 1, \forall\ w \qquad (8.30)$$

For instance, from second equation in (8.29) and (8.30) we get for multiplicative input uncertainty with a scalar weight:

$$\text{RS}\ \forall G_p = G(I+w_I\Delta_I),\ \|\Delta_I\|_\infty \le 1\ \Leftrightarrow\ \|w_I T_I\|_\infty < 1 \qquad (8.31)$$

Note that the SISO-condition (4.15) follows as a special case of (8.31). Similarly, (4.21) follows as a special case of the inverse multiplicative output uncertainty in (8.29):

$$\begin{aligned}&\text{RS}\ \forall G_p = (I - w_{iO}\Delta_{iO})^{-1}G,\\ &\|\Delta_{iO}\|_\infty \le 1 \Leftrightarrow \|w_{iO}S\|_\infty < 1\end{aligned} \qquad (8.32)$$

In general, the unstructured uncertainty descriptions in terms of a single perturbation are not "tight".

8.7 RS with structured uncertainty: Motivation [8.7]

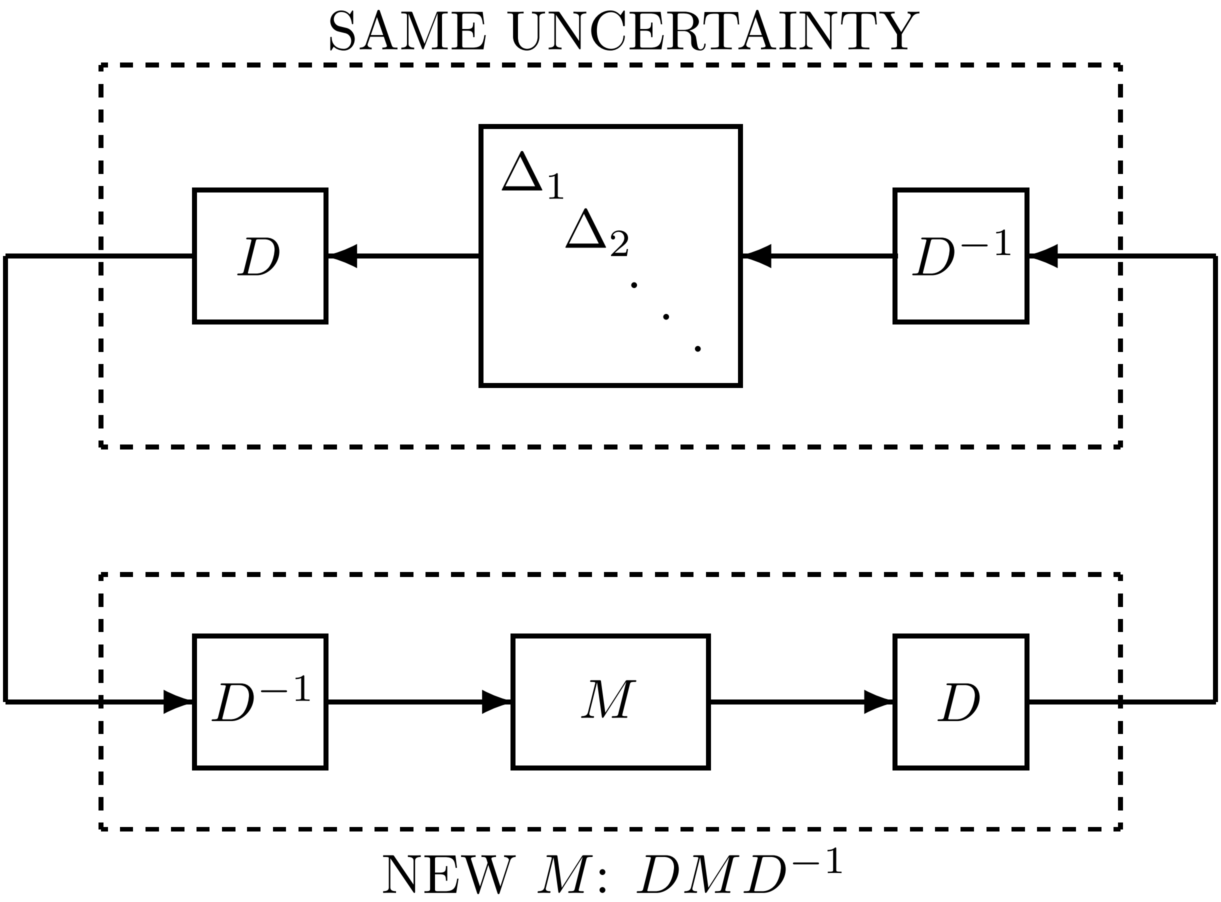


Figure 79: Use of block-diagonal scalings, $\Delta D = D\Delta$

Consider now the presence of structured uncertainty, where $\Delta = \text{diag}\{\Delta_i\}$ is block-diagonal. To test for robust stability we rearrange the system into the $M\Delta$-structure and we have from (8.26)

$$\text{RS} \quad \text{if} \quad \bar{\sigma}(M(j\omega)) < 1, \forall\omega \tag{8.33}$$

We have here written "if" rather than "if and only if" since this condition is only necessary for RS when Δ has "no structure" (full-block uncertainty). To reduce concervativism introduce the block-diagonal scaling matrix

$$D = \text{diag}\{d_i I_i\} \tag{8.34}$$

where d_i is a scalar and I_i is an identity matrix of the same dimension as the i'th perturbation block, Δ_i (Figure 79). This clearly has no effect on stability.

$$\text{RS} \quad \text{if} \quad \bar{\sigma}(DMD^{-1}) < 1, \forall\omega \tag{8.35}$$

This applies for any D in (8.34), and therefore the "most improved" (least conservative) RS-condition is obtained by minimizing at each frequency the scaled singular value, and we have

$$\boxed{\text{RS} \quad \text{if} \quad \min_{D(\omega)\in\mathcal{D}} \bar{\sigma}(D(\omega)M(j\omega)D(\omega)^{-1}) < 1, \forall\omega} \tag{8.36}$$

where $\mathcal{D}$ is the set of block-diagonal matrices whose structure is compatible to that of Δ, i.e, $\Delta D = D\Delta$.

8.8 The structured singular value [8.8]

The structured singular value (denoted Mu, mu, SSV or μ) is a function which provides a generalization of the singular value, $\bar{\sigma}$, and the spectral radius, ρ. We will use μ to get necessary and sufficient conditions for robust stability and also for robust performance. How is μ defined? A simple statement is:

> *Find the smallest structured Δ (measured in terms of $\bar{\sigma}(\Delta)$) which makes $\det(I - M\Delta) = 0$; then $\mu(M) = 1/\bar{\sigma}(\Delta)$.*

Mathematically,

$$\mu(M)^{-1} \triangleq \min_{\Delta}\{\bar{\sigma}(\Delta) | \det(I - M\Delta) = 0 \text{ for structured } \Delta\} \tag{8.37}$$

Clearly, $\mu(M)$ depends not only on M but also on the allowed structure for Δ. This is sometimes shown explicitly by using the notation $\mu_{\Delta}(M)$.

Remark. For the case where Δ is "unstructured" (a full matrix), the smallest Δ which yields singularity has $\bar{\sigma}(\Delta) = 1/\bar{\sigma}(M)$, and we have $\mu(M) = \bar{\sigma}(M)$.

Example 5 Full perturbation *(Δ is unstructured). Consider*

$$M = \begin{bmatrix} 2 & 2 \\ -1 & -1 \end{bmatrix} =$$
$$= \begin{bmatrix} 0.894 & 0.447 \\ -0.447 & 0.894 \end{bmatrix} \begin{bmatrix} 3.162 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0.707 & -0.707 \\ 0.707 & 0.707 \end{bmatrix}^H \quad (8.38)$$

The perturbation

$$\Delta = \frac{1}{\sigma_1} v_1 u_1^H = \frac{1}{3.162} \begin{bmatrix} 0.707 \\ 0.707 \end{bmatrix} [\, 0.894 \quad -0.447 \,] =$$
$$= \begin{bmatrix} 0.200 & 0.200 \\ -0.100 & -0.100 \end{bmatrix} \quad (8.39)$$

with $\bar{\sigma}(\Delta) = 1/\bar{\sigma}(M) = 1/3.162 = 0.316$ *makes* $\det(I - M\Delta) = 0$. *Thus* $\mu(M) = 3.162$ *when* Δ *is a full matrix.*

Note that the perturbation Δ in (8.39) is a full matrix. If we restrict Δ to be diagonal then we need a larger perturbation to make $\det(I - M\Delta) = 0$. This is illustrated next.

Example 5 continued. Diagonal perturbation *(Δ is structured). For the matrix M in (8.38), the smallest diagonal Δ which makes* $\det(I - M\Delta) = 0$ *is*

$$\Delta = \frac{1}{3}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{8.40}$$

with $\bar{\sigma}(\Delta) = 0.333$. Thus $\mu(M) = 3$ when Δ is a diagonal matrix.

Definition 2 Structured Singular Value. *Let M be a given complex matrix and let $\Delta = \mathrm{diag}\{\Delta_i\}$ denote a set of complex matrices with $\bar{\sigma}(\Delta) \le 1$ and with a given block-diagonal structure (in which some of the blocks may be repeated and some may be restricted to be real). The real non-negative function $\mu(M)$, called the structured singular value, is defined by*

$$\mu(M) \triangleq \frac{1}{\min\{k_m \mid \det(I - k_m M\Delta) = 0, \bar{\sigma}(\Delta) \le 1\}} \tag{8.41}$$

If no such structured Δ exists then $\mu(M) = 0$.

A value of $\mu = 1$ means that there exists a perturbation with $\bar{\sigma}(\Delta) = 1$ which is just large enough to make $I - M\Delta$ singular. A larger value of μ is "bad" as it means that a smaller perturbation makes $I - M\Delta$ singular, whereas a smaller value of μ is "good".

8.9 Robust stability with structured uncertainty [8.9]

Consider stability of the $M\Delta$-structure in Figure 75 for the case where Δ is a set of norm-bounded block-diagonal perturbations.

Theorem 12 RS for block-diagonal perturbations (real or complex). *Assume that the nominal system M and the perturbations Δ are stable. Then the $M\Delta$-system in Figure 75 is stable for all allowed perturbations with $\bar{\sigma}(\Delta) \leq 1, \forall\omega$,* <u>*if and only if*</u>

$$\boxed{\mu(M(j\omega)) < 1, \quad \forall\omega} \tag{8.42}$$

Condition (8.42) for robust stability may be rewritten as

$$\text{RS} \;\Leftrightarrow\; \mu(M(j\omega))\,\bar{\sigma}(\Delta(j\omega)) < 1, \quad \forall\omega \tag{8.43}$$

which may be interpreted as a "generalized small gain theorem" that also takes into account the *structure* of Δ.

Example 6 RS with diagonal input uncertainty

Consider robust stability of the feedback system in Figure 78 for the case when the multiplicative input uncertainty is diagonal. A nominal 2×2 *plant and the controller (which represents PI-control of a distillation process using the DV-configuration) is given by*

$$G(s) = \tfrac{1}{\tau s+1} \begin{bmatrix} -87.8 & 1.4 \\ -108.2 & -1.4 \end{bmatrix};$$

$$K(s) = \tfrac{1+\tau s}{s} \begin{bmatrix} -0.0015 & 0 \\ 0 & -0.075 \end{bmatrix} \qquad (8.44)$$

(time in minutes). The controller results in a nominally stable system with acceptable performance. Assume there is complex multiplicative uncertainty in each *manipulated input of magnitude*

$$w_I(s) = \frac{s+0.2}{0.5s+1} \qquad (8.45)$$

On rearranging the block diagram to match the $M\Delta$*-structure in Figure 75 we get* $M = w_I KG(I+KG)^{-1} = w_I T_I$ *(recall (8.15)), and the RS-condition* $\mu(M) < 1$ *in Theorem 12 yields*

$$\text{RS} \Leftrightarrow \mu_{\Delta_I}(T_I) < \frac{1}{|w_I(j\omega)|} \quad \forall\omega, \quad \Delta_I = \begin{bmatrix} \delta_1 & \\ & \delta_2 \end{bmatrix} \qquad (8.46)$$

This condition is shown graphically in Figure 80 so the *system is robustly stable. Also in Figure 80,* $\bar{\sigma}(T_I)$ *can be seen to be larger than* $1/|w_I(j\omega)|$ *over a wide frequency range. This shows that the system would be unstable for full-block input uncertainty (*Δ_I *full).*

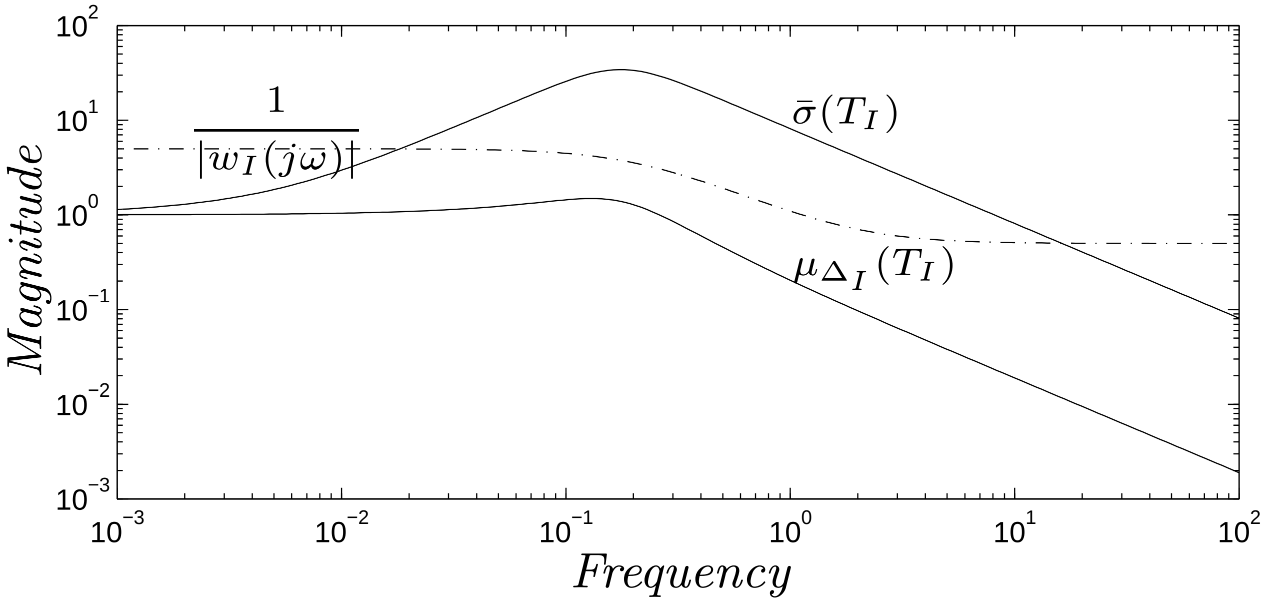


Figure 80: Robust stability for diagonal input uncertainty is guaranteed since $\mu_{\Delta_I}(T_I) < 1/|w_I|, \forall\omega$. *The use of unstructured uncertainty and* $\bar{\sigma}(T_I)$ *is conservative*

8.10 Robust performance [8.10]

With an $\mathcal{H}_\infty$ performance objective, the RP-condition is identical to a RS-condition with an additional perturbation block.

In Figure 81 step B is the key step.

Δ_P (where capital P denotes Performance) is always a full matrix. It is a fictitious uncertainty block representing the $\mathcal{H}_\infty$ performance specification.

8.10.1 Testing RP using μ [8.10.1]

Theorem 13 Robust performance. *Rearrange the uncertain system into the $N\Delta$-structure of Figure 81. Assume nominal stability such that N is (internally) stable. Then*

$$\begin{aligned} \text{RP} \quad & \overset{\text{def}}{\Leftrightarrow} \quad \|F\|_\infty = \|F_u(N,\Delta)\|_\infty < 1, \quad \forall \|\Delta\|_\infty \leq 1 \\ & \Leftrightarrow \quad \boxed{\mu_{\widehat{\Delta}}(N(j\omega)) < 1, \quad \forall w} \end{aligned} \tag{8.47}$$

where μ is computed with respect to the structure

$$\widehat{\Delta} = \begin{bmatrix} \Delta & 0 \\ 0 & \Delta_P \end{bmatrix} \tag{8.48}$$

and Δ_P is a full complex perturbation with the same dimensions as F^T.

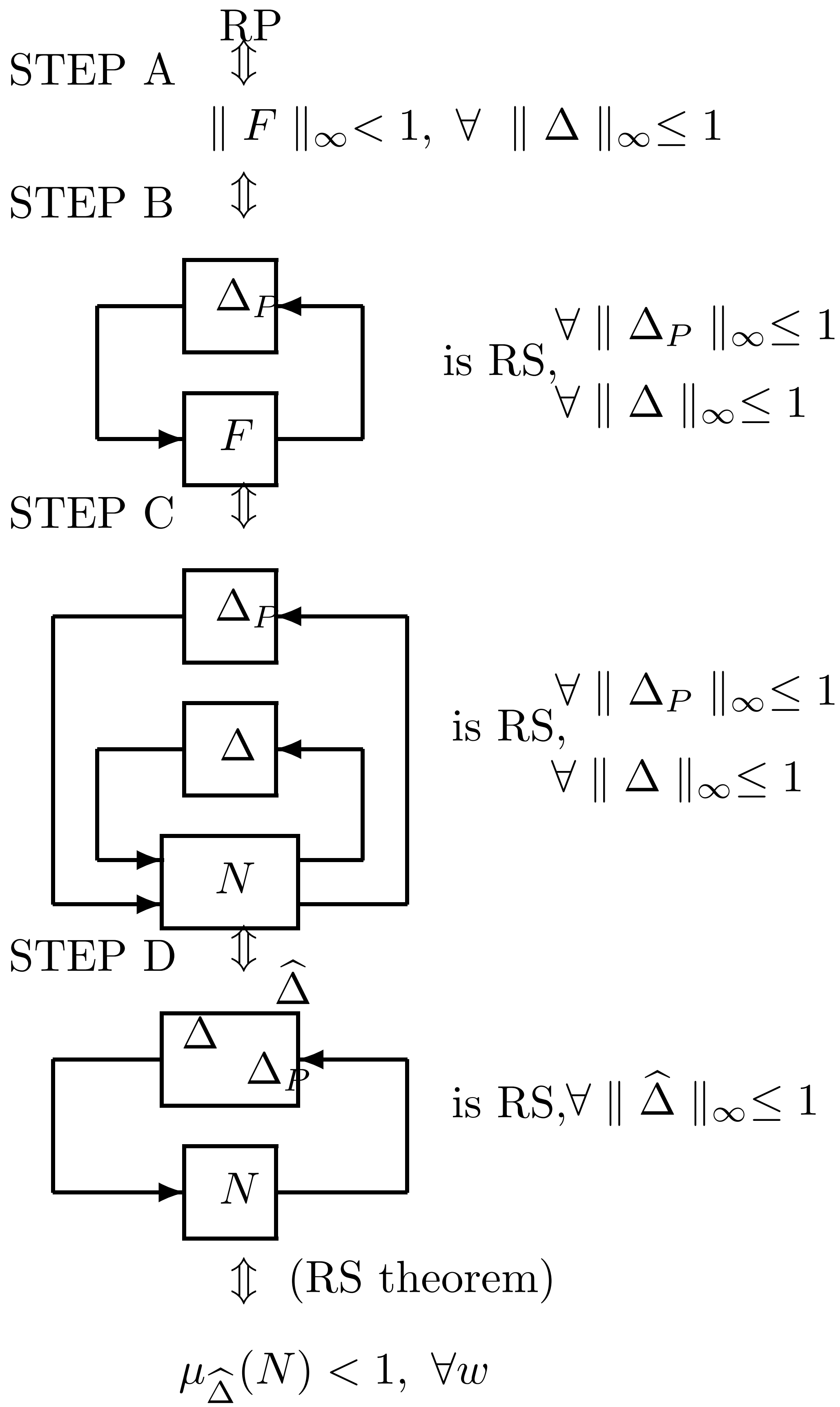


Figure 81: RP as a special case of structured RS. $F = F_u(N, \Delta)$

8.10.2 Summary of μ-conditions for NP, RS and RP [8.10.2]

Rearrange the uncertain system into the $N\Delta$-structure, where the block- diagonal perturbations satisfy $\|\Delta\|_\infty \leq 1$.

Introduce

$$F = F_u(N, \Delta) = N_{22} + N_{21}\Delta(I - N_{11}\Delta)^{-1}N_{12}$$

and let the performance requirement (RP) be $\|F\|_\infty \leq 1$ for all allowable perturbations. Then we have:

$$\text{NS} \Leftrightarrow N \text{ (internally) stable} \quad (8.49)$$

$$\text{NP} \Leftrightarrow \bar{\sigma}(N_{22}) = \mu_{\Delta_P} < 1,\ \forall\omega,\ \text{and NS} \quad (8.50)$$

$$\text{RS} \Leftrightarrow \mu_\Delta(N_{11}) < 1,\ \forall\omega,\ \text{and NS} \quad (8.51)$$

$$\text{RP} \Leftrightarrow \mu_{\widetilde{\Delta}}(N) < 1,\ \forall\omega,\ \widetilde{\Delta} = \begin{bmatrix} \Delta & 0 \\ 0 & \Delta_P \end{bmatrix}, \text{ and NS} \quad (8.52)$$

Here Δ is a block-diagonal matrix (its detailed structure depends on the uncertainty we are representing), whereas Δ_P always is a full complex matrix. Note that nominal stability (NS) must be tested separately in all cases.

What does $\mu = 1.1$ for RP mean?

Our RP-requirement would be satisfied exactly if we reduced *both* the performance requirement *and* the uncertainty by a factor of 1.1.

To find the worst-case weighted performance for a given uncertainty, one needs to keep the magnitude of the perturbations fixed ($\bar{\sigma}(\Delta) \leq 1$).

To find μ^s numerically, we scale the performance part of N by a factor $k_m = 1/\mu^s$ and iterate on k_m until $\mu = 1$. That is, at each frequency "skewed-μ" is the value $\mu^s(N)$ which solves

$$\mu(K_m N) = 1, \quad K_m = \begin{bmatrix} I & 0 \\ 0 & 1/\mu^s \end{bmatrix} \tag{8.53}$$

Note that μ underestimates how bad or good the actual worst-case performance is. This follows because $\mu^s(N)$ is always further from 1 than $\mu(N)$.

8.11 * Application: RP with input uncertainty [8.11]

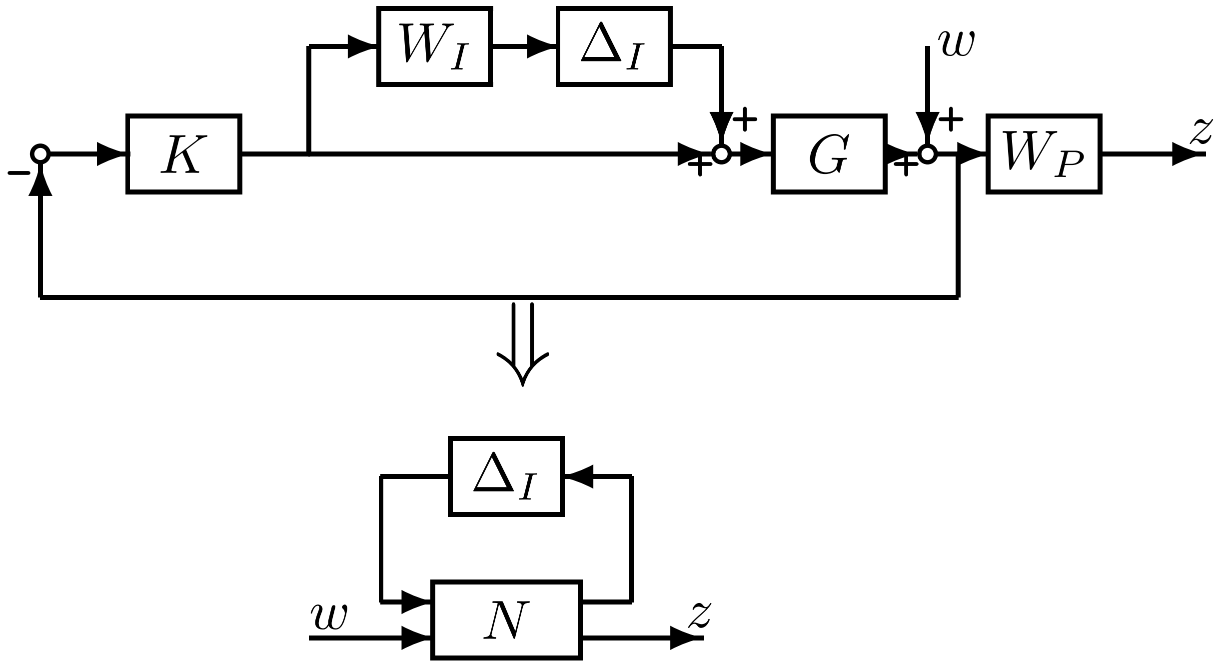


Figure 82: Robust performance of system with input uncertainty

8.11.1 Interconnection matrix [8.11.1]

On rearranging the system into the $N\Delta$-structure, as shown in Figure 82, we get

$$N = \begin{bmatrix} w_I T_I & w_I KS \\ w_P SG & w_P S \end{bmatrix} \qquad (8.54)$$

where $T_I = KG(I + KG)^{-1}$, $S = (I + GK)^{-1}$ and for simplicity we have omitted the negative signs in the 1,1 and 1,2 blocks of N, since $\mu(N) = \mu(UN)$ with unitary $U = \begin{bmatrix} -I & 0 \\ 0 & I \end{bmatrix}$.

For a given controller K we can now test for NS, NP, RS and RP using (8.49)-(8.52). Here $\Delta = \Delta_I$ may be a full or diagonal matrix (depending on the physical situation).

8.11.2 RP with input uncertainty for SISO system [8.11.2]

For a SISO system, conditions (8.49)-(8.52) with N as in (8.54) become

$$\text{NS} \quad \Leftrightarrow \quad S,\ SG,\ KS \text{ and } T_I \text{ are stable} \tag{8.55}$$

$$\text{NP} \quad \Leftrightarrow \quad |w_P S| < 1,\ \forall\omega \tag{8.56}$$

$$\text{RS} \quad \Leftrightarrow \quad |w_I T_I| < 1,\ \forall\omega \tag{8.57}$$

$$\text{RP} \quad \Leftrightarrow \quad |w_P S| + |w_I T_I| < 1,\ \forall\omega \tag{8.58}$$

8.11.3 Robust performance for 2×2 distillation process [8.11.3]

Consider again the distillation process example from Chapter 3 (Motivating Example No. 2) and the corresponding inverse-based controller:

$$G(s) = \frac{1}{75s+1}\begin{bmatrix} 87.8 & -86.4 \\ 108.2 & -109.6 \end{bmatrix}; \quad K(s) = \frac{0.7}{s}G(s)^{-1} \tag{8.59}$$

The controller provides a nominally decoupled system with

$$L = l\ I,\ S = \epsilon I \text{ and } T = tI \tag{8.60}$$

where

$$l = \frac{0.7}{s}, \epsilon = \frac{1}{1+l} = \frac{s}{s+0.7},$$
$$t = 1 - \epsilon = \frac{0.7}{s+0.7} = \frac{1}{1.43s+1}$$

We have used ϵ for the nominal sensitivity in each loop to distinguish it from the Laplace variable s.

Weights for uncertainty and performance:

$$w_I(s) = \frac{s+0.2}{0.5s+1}; \quad w_P(s) = \frac{s/2+0.05}{s} \tag{8.61}$$

The weight $w_I(s)$ may approximately represent a 20% gain error and a neglected time delay of 0.9 min. $|w_I(j\omega)|$ levels off at 2 (200% uncertainty) at high frequencies. The performance weight $w_P(s)$ specifies integral action, a closed-loop bandwidth of about 0.05 [rad/min] (which is relatively slow in the presence of an allowed time delay of 0.9 min) and a maximum peak for $\bar{\sigma}(S)$ of $M_s = 2$.

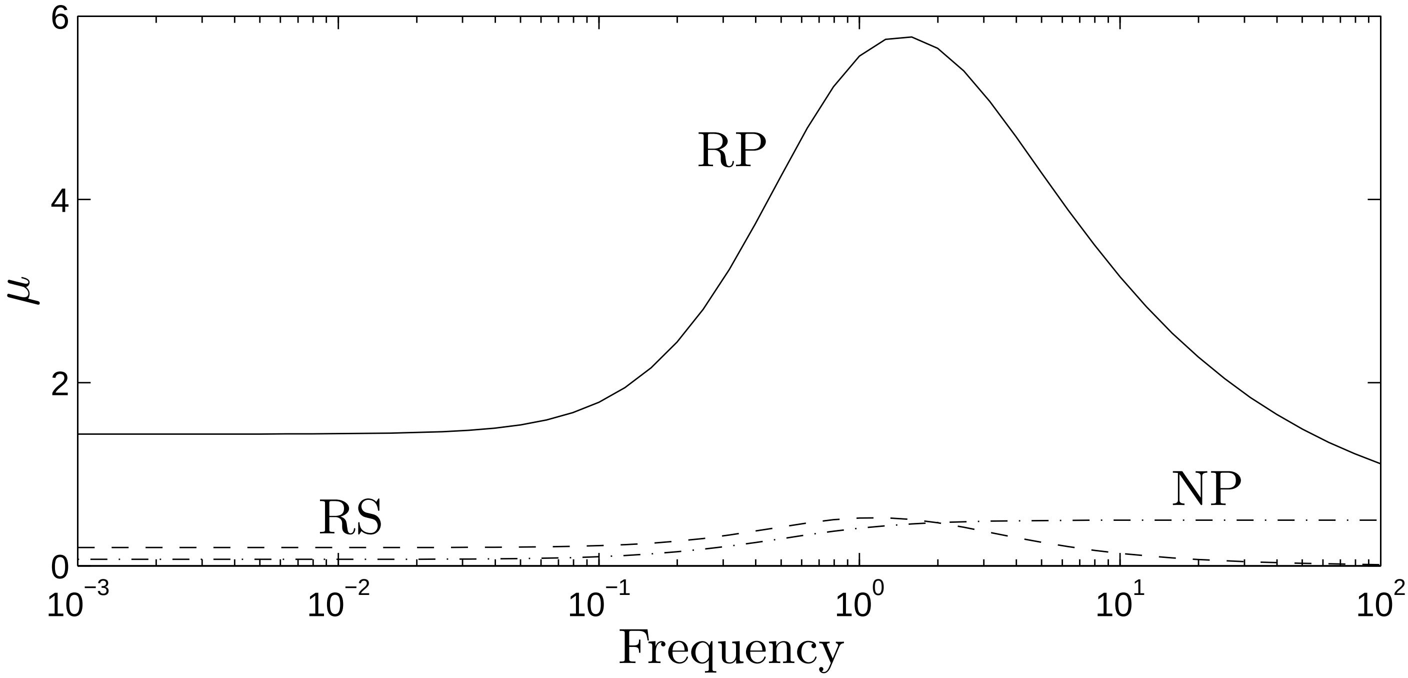


Figure 83: μ-plots for distillation process with decoupling controller

NS Yes.

NP With the decoupling controller we have

$$\bar{\sigma}(N_{22}) = \bar{\sigma}(w_P S) = \left| \frac{s/2 + 0.05}{s + 0.7} \right|$$

(dashed-dot line in Figure 83 $\Leftarrow$ NP is OK.)

RS Since in this case $w_I T_I = w_I T$ is a scalar times the identity matrix, we have, independent of the structure of Δ_I, that

$$\mu_{\Delta_I}(w_I T_I) = |w_I t| = \left| 0.2 \frac{5s + 1}{(0.5s + 1)(1.43s + 1)} \right|$$

and we see from the dashed line in Figure 83 that RS is OK.

RP Poor.

Table 3: **MATLAB program for μ-analysis**

```
% Uses the Mu toolbox
G0 = [87.8 -86.4; 108.2 -109.6];
dyn = nd2sys(1,[75 1]);
Dyn=daug(dyn,dyn); G=mmult(Dyn,G0);
%
% Inverse-based control.
%
dynk=nd2sys([75 1],[1 1.e-5],0.7);
Dynk=daug(dynk,dynk); Kinv=mmult(Dynk,minv(G0));
%
% Weights.
%
wp=nd2sys([10 1],[10 1.e-5],0.5); Wp=daug(wp,wp);
wi=nd2sys([1 0.2],[0.5 1]); Wi=daug(wi,wi);
%
% Generalized plant P.
%
systemnames = 'G Wp Wi';
inputvar = '[ydel(2); w(2) ; u(2)]';
outputvar = '[Wi; Wp; -G-w]';
input_to_G = '[u+ydel]';
input_to_Wp = '[G+w]'; input_to_Wi = '[u]';
sysoutname = 'P';
cleanupsysic = 'yes'; sysic;
%
N = starp(P,Kinv); omega = logspace(-3,3,61);
Nf = frsp(N,omega);
%
% mu for RP.
%
blk = [1 1; 1 1; 2 2];
[mubnds,rowd,sens,rowp,rowg] = mu(Nf,blk,'c');
muRP = sel(mubnds,':',1); pkvnorm(muRP)            % (ans = 5.7726).
%
```

Table 4: **MATLAB program for μ-analysis**

```
% Worst-case weighted sensitivity
%
[delworst,muslow,musup] = wcperf(Nf,blk,1); musup       % (musup = 44.93 for
%                                                       %    delta=1).
% mu for RS.
%
Nrs=sel(Nf,1:2,1:2);
[mubnds,rowd,sens,rowp,rowg]=mu(Nrs,[1 1; 1 1],'c');
muRS = sel(mubnds,':',1); pkvnorm(muRS)                 % (ans = 0.5242).
%
% mu for NP (= max.  singular value of Nnp).
%
Nnp=sel(Nf,3:4,3:4);
[mubnds,rowd,sens,rowp,rowg]=mu(Nnp,[2 2],'c');
muNP = sel(mubnds,':',1); pkvnorm(muNP)                 % (ans = 0.5000).
vplot('liv,m',muRP,muRS,muNP);
```

8.12 μ-synthesis and DK-iteration [8.12]

The structured singular value μ is a very powerful tool for the analysis of robust performance with a given controller. However, one may also seek to find the controller that minimizes a given μ-condition: this is the μ-synthesis problem.

8.12.1 DK-iteration [8.12.1]

At present there is no direct method to synthesize a μ-optimal controller. However, for complex perturbations a method known as DK-iteration is available. It combines $\mathcal{H}_\infty$-synthesis and μ-analysis, and often yields good results. The starting point is the upper bound on μ in terms of the scaled singular value

$$\mu(N) \leq \min_{D \in \mathcal{D}} \bar{\sigma}(DND^{-1})$$

The idea is to find the controller that minimizes the peak value over frequency of this upper bound, namely

$$\min_{K}(\min_{D\in\mathcal{D}} \|DN(K)D^{-1}\|_\infty) \qquad (8.62)$$

by alternating between minimizing $\|DN(K)D^{-1}\|_\infty$ with respect to either K or D (while holding the other fixed).

1. **K-step.** Synthesize an $\mathcal{H}_\infty$ controller for the scaled problem, $\min_K \|DN(K)D^{-1}\|_\infty$ with fixed $D(s)$.
2. **D-step.** Find $D(j\omega)$ to minimize at each frequency $\bar{\sigma}(DND^{-1}(j\omega))$ with fixed N.
3. Fit the magnitude of each element of $D(j\omega)$ to a stable and minimum phase transfer function $D(s)$ and go to Step 1.

8.12.2 * Example: μ-synthesis with DK-iteration [8.12.4]

Simplified distillation process

$$G(s) = \frac{1}{75s+1}\begin{bmatrix} 87.8 & -86.4 \\ 108.2 & -109.6 \end{bmatrix} \tag{8.63}$$

The uncertainty weight $w_I I$ and performance weight $w_P I$ are given in (8.61), and are shown graphically in Figure 84. The objective is to minimize the peak value of $\mu_{\widetilde{\Delta}}(N)$, where N is given in (8.54) and $\widetilde{\Delta} = \text{diag}\{\Delta_I, \Delta_P\}$. We will consider diagonal input uncertainty (which is always present in any real problem), so Δ_I is a 2×2 diagonal matrix. Δ_P is a full 2×2 matrix representing the performance specification.

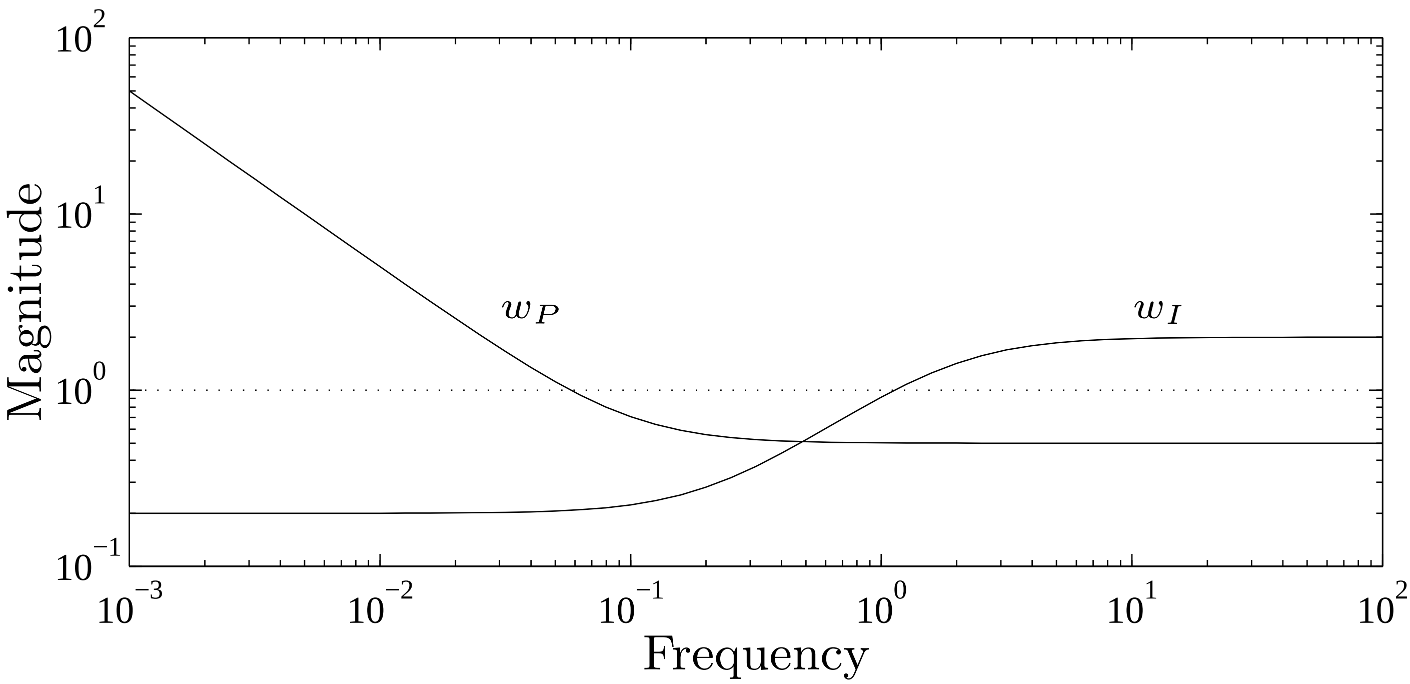


Figure 84: Uncertainty and performance weights. Notice that there is a frequency range ("window") where both weights are less than one in magnitude.

Table 5: **MATLAB program to perform DK-iteration**

```
% Uses the Mu toolbox
G0 = [87.8 -86.4; 108.2 -109.6];
dyn = nd2sys(1,[75 1]); Dyn = daug(dyn,dyn);
G = mmult(Dyn,G0);
%
% Weights.
%
wp = nd2sys([10 1],[10 1.e-5],0.5);             % Approximated
wi = nd2sys([1 0.2],[0.5 1]);                   % integrator.
Wp = daug(wp,wp); Wi = daug(wi,wi);
%
% Generalized plant P. %
systemnames = 'G Wp Wi';
inputvar = '[ydel(2); w(2) ; u(2)]';
outputvar = '[Wi; Wp; -G-w]';
input_to_G = '[u+ydel]';
input_to_Wp = '[G+w]'; input_to_Wi = '[u]';
sysoutname = 'P'; cleanupsysic = 'yes';
sysic;
%
```

Table 6: **MATLAB program to perform *DK*-iteration**

```
% Initialize.
%
omega = logspace(-3,3,61);
blk = [1 1; 1 1; 2 2];
nmeas=2; nu=2; gmin=0.9; gamma=2; tol=0.01; d0 = 1;
dsysl = daug(d0,d0,eye(2),eye(2)); dsysr=dsysl;
%
% START ITERATION.
%
% STEP 1:  Find H-infinity optimal controller
% with given scalings:
%
DPD = mmult(dsysl,P,minv(dsysr)); gmax=1.05*gamma;
[K,Nsc,gamma] = hinfsyn(DPD,nmeas,nu,gmin,gmax,tol);
Nf=frsp(Nsc,omega);                                     % (Remark:
%                                                       %  Without scaling:
%                                                       %  N=starp(P,K);).
% STEP 2:  Compute mu using upper bound:
%
[mubnds,rowd,sens,rowp,rowg] = mu(Nf,blk,'c');
vplot('liv,m',mubnds); murp=pkvnorm(mubnds,inf)
%
% STEP 3:  Fit resulting D-scales:
%
[dsysl,dsysr]=musynflp(dsysl,rowd,sens,blk,nmeas,nu);    % choose 4th order.
%
% New Version:
% [dsysL,dsysR]=msf(Nf,mubnds,rowd,sens,blk);           % order:  4, 4, 0.
% dsysl=daug(dsysL,eye(2)); dsysr=daug(dsysR,eye(2));
%
% GOTO STEP 1 (unless satisfied with murp).
%
```

Iteration No. 1.

Step 1: With the initial scalings, $D^0 = I$, the $\mathcal{H}_\infty$ software produced a 6 state controller (2 states from the plant model and 2 from each of the weights) with an $\mathcal{H}_\infty$ norm of $\gamma = 1.1823$.

Step 2: The upper μ-bound gave the μ-curve shown as curve "Iter. 1" in Figure 85, corresponding to a peak value of μ=1.1818.

Step 3: The frequency-dependent $d_1(\omega)$ and $d_2(\omega)$ from Step 2 were each fitted using a 4th order transfer function. $d_1(w)$ and the fitted 4th-order transfer function (dotted line) are shown in Figure 86 and labelled "Iter. 1".

Iteration No. 2.

Step 1: With the 8 state scaling $D^1(s)$ the $\mathcal{H}_\infty$ software gave a 22 state controller and $\|D^1 N (D^1)^{-1}\|_\infty = 1.0238$.

Iteration No. 3.

Step 1: With the scalings $D^2(s)$ the $\mathcal{H}_\infty$ norm was only slightly reduced from 1.024 to 1.019.

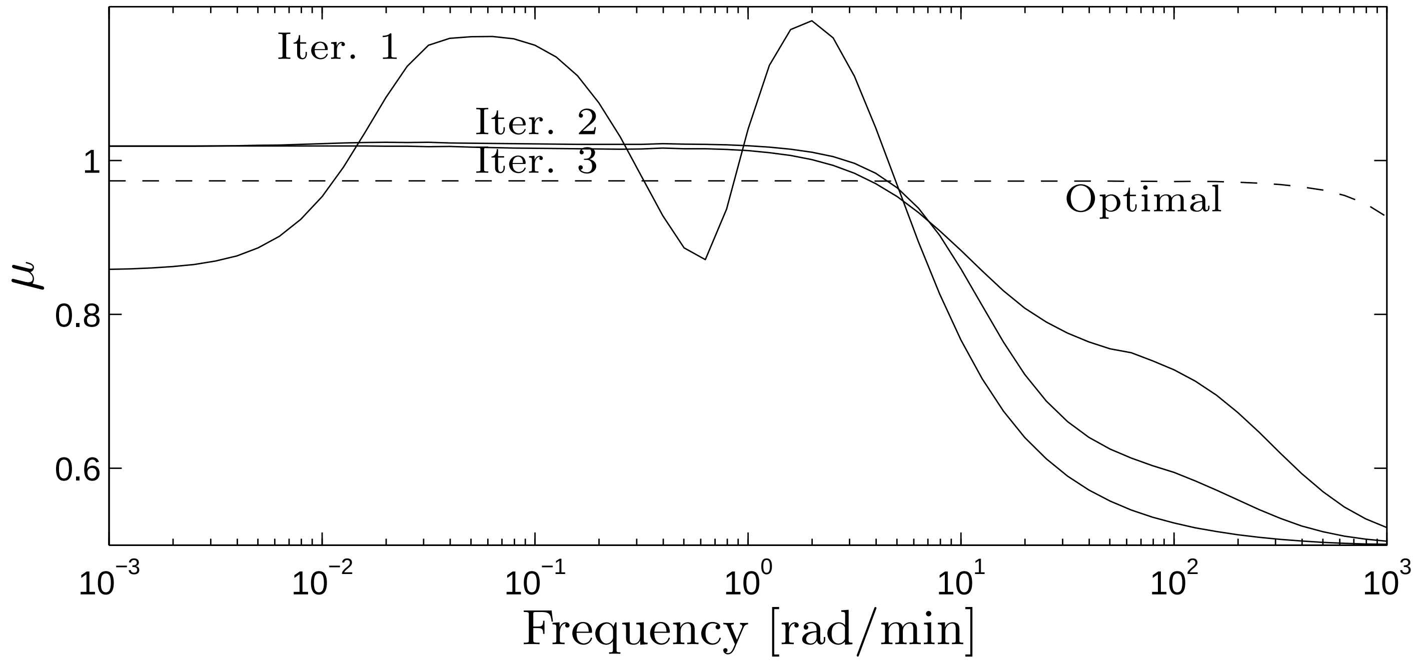


Figure 85: Change in μ during DK-iteration

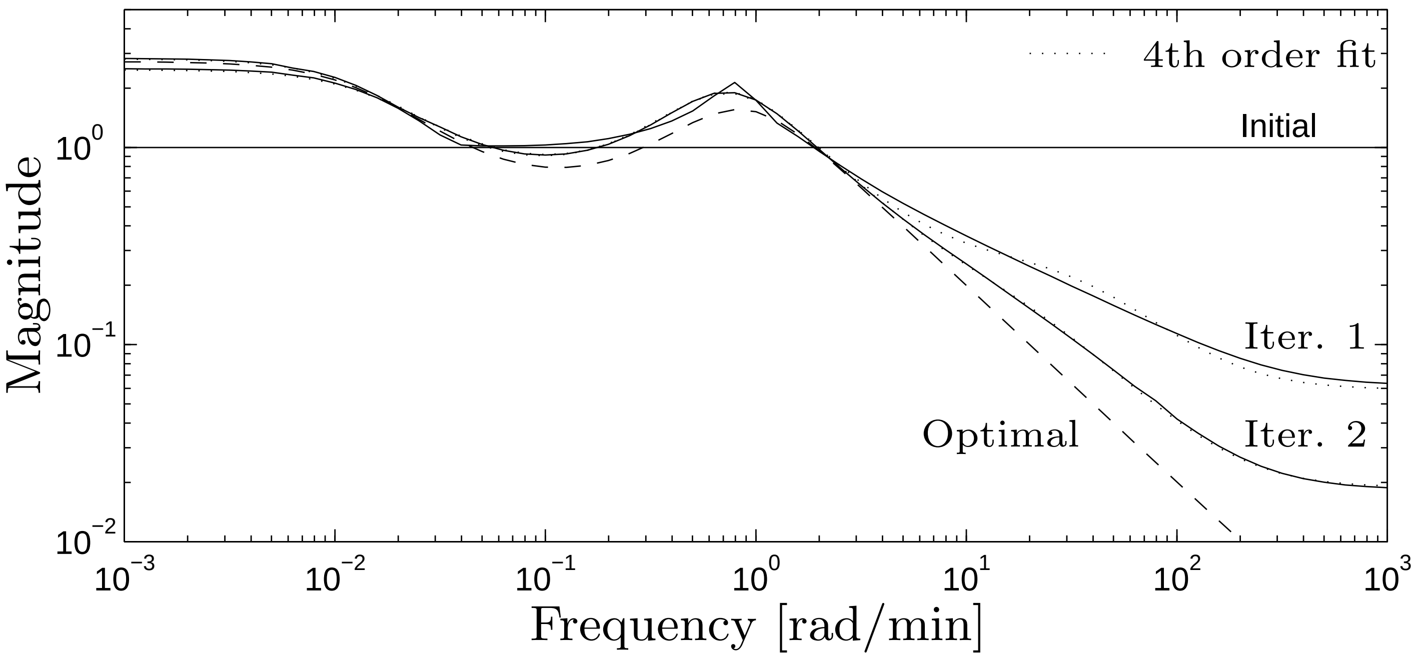


Figure 86: Change in D-scale d_1 during DK-iteration

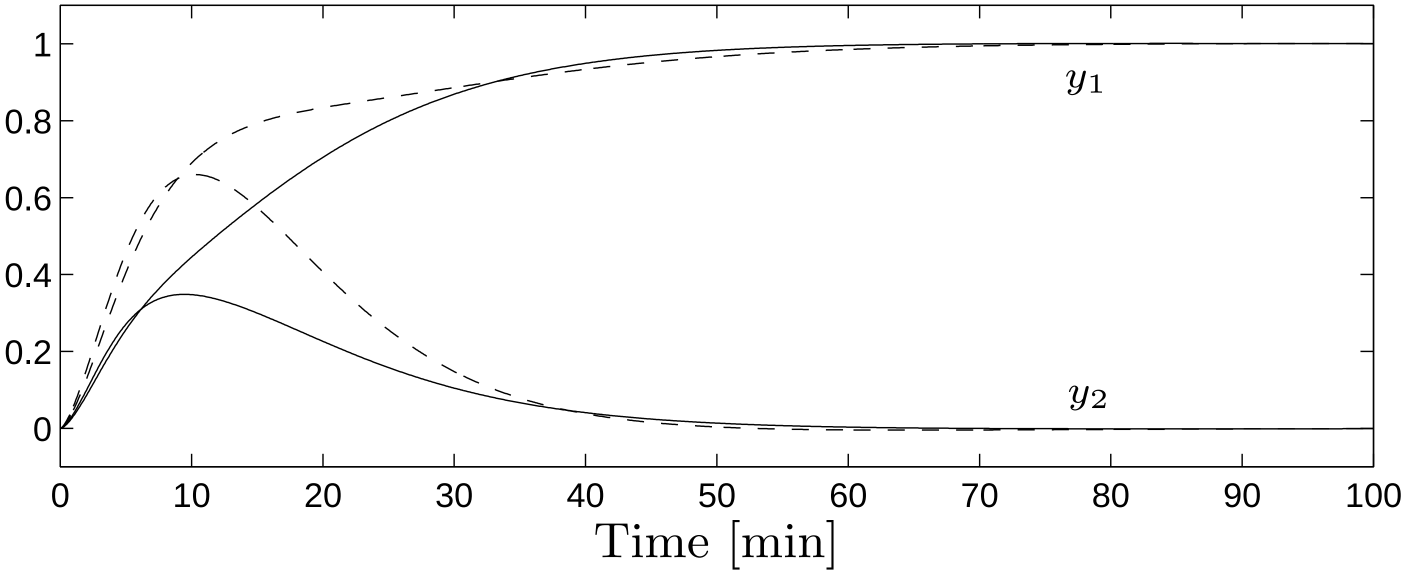


Figure 87: Setpoint response for μ-"optimal" controller K_3. Solid line: nominal plant. Dashed line: uncertain plant G_3'

9 Controller Design [9]

9.1 Trade-offs in MIMO feedback design [9.1]

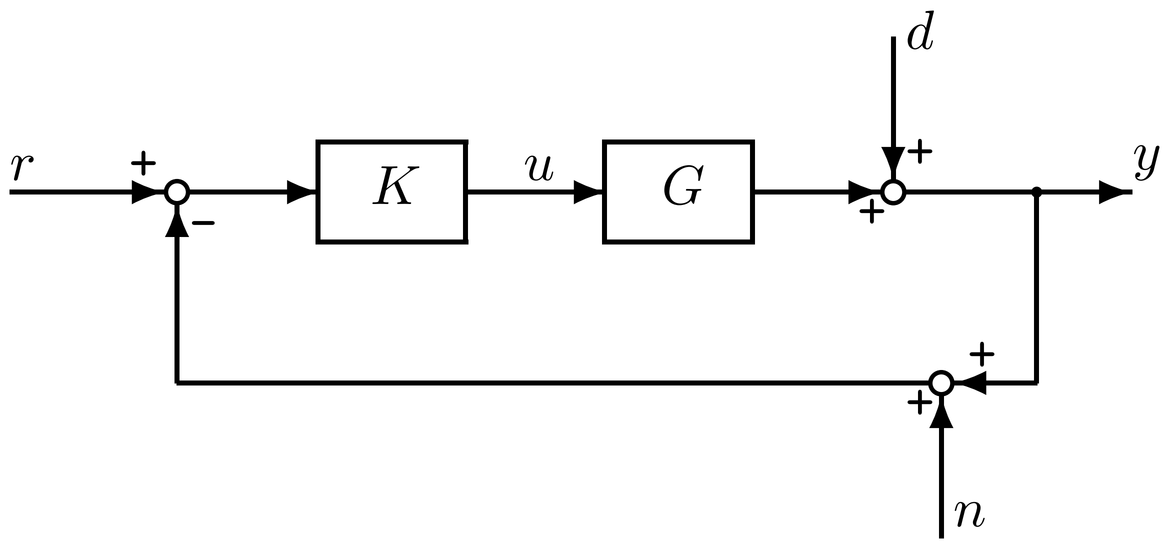


Figure 88: One degree-of-freedom feedback

$$y(s) = T(s)r(s) + S(s)d(s) - T(s)n(s) \quad (9.1)$$

$$u(s) = K(s)S(s)\left[r(s) - n(s) - d(s)\right] \quad (9.2)$$

Closed-loop objectives:

1. For *disturbance rejection* make $\bar{\sigma}(S)$ small.
2. For *noise attenuation* make $\bar{\sigma}(T)$ small.
3. For *reference tracking* make $\bar{\sigma}(T) \approx \underline{\sigma}(T) \approx 1$.
4. For *control energy reduction* make $\bar{\sigma}(KS)$ small.
5. For *robust stability* in the presence of an additive perturbation make $\bar{\sigma}(KS)$ small.
6. For *robust stability* in the presence of a multiplicative output perturbation make $\bar{\sigma}(T)$ small.

The closed-loop requirements 1 to 6 cannot all be satisfied simultaneously. Feedback design is therefore a trade-off over frequency of conflicting objectives.

Over specified frequency ranges, we can approximate the closed-loop requirements by the following open-loop objectives:

1. For *disturbance rejection* make $\underline{\sigma}(GK)$ large; valid for frequencies at which $\underline{\sigma}(GK) \gg 1$.

2. For *noise attenuation* make $\bar{\sigma}(GK)$ small; valid for frequencies at which $\bar{\sigma}(GK) \ll 1$.

3. For *reference tracking* make $\underline{\sigma}(GK)$ large; valid for frequencies at which $\underline{\sigma}(GK) \gg 1$.

4. For *control energy reduction* make $\bar{\sigma}(K)$ small; valid for frequencies at which $\bar{\sigma}(GK) \ll 1$.

5. For *robust stability to an additive perturbation* make $\bar{\sigma}(K)$ small; valid for frequencies at which $\bar{\sigma}(GK) \ll 1$.

6. For *robust stability to a multiplicative output perturbation* make $\bar{\sigma}(GK)$ small; valid for frequencies at which $\bar{\sigma}(GK) \ll 1$.

9.2 General control problem formulation [9.3.1]

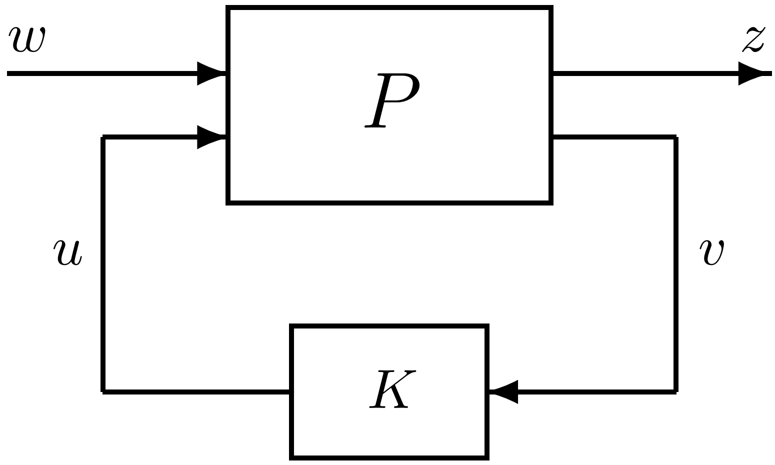


Figure 89: General control configuration

$$\begin{bmatrix} z \\ v \end{bmatrix} = P(s) \begin{bmatrix} w \\ u \end{bmatrix} = \begin{bmatrix} P_{11}(s) & P_{12}(s) \\ P_{21}(s) & P_{22}(s) \end{bmatrix} \begin{bmatrix} w \\ u \end{bmatrix} \tag{9.3}$$

$$u = K(s)v \tag{9.4}$$

The state-space realization of the generalized plant P is given by

$$P \overset{s}{=} \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right] \tag{9.5}$$

$$z = F_l(P, K)w \tag{9.6}$$

where

$$F_l(P, K) = P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21} \tag{9.7}$$

$\mathcal{H}_2$ and $\mathcal{H}_\infty$ control involve the minimization of the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms of $F_l(P, K)$ respectively.

9.3 $\mathcal{H}_2$ optimal control [9.3.2]

The standard $\mathcal{H}_2$ optimal control problem is to find a stabilizing controller K which minimizes

$$\| F(s)\|_2 = \sqrt{\frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\omega)F(j\omega)^T d\omega}; \quad F \triangleq F_l(P,K) \tag{9.8}$$

For a particular problem the generalized plant P will include the plant model, the interconnection structure, and the designer specified weighting functions. This is illustrated for the LQG problem in the next subsection.

Stochastic interpretation: suppose in the general control configuration that the exogenous input w is white noise of unit intensity. That is:

$$E\left\{w(t)w(\tau)^T\right\} = I\delta(t-\tau) \tag{9.9}$$

The expected power in the error signal z is then given by:

$$E\left\{\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} z(t)^T z(t)dt\right\} \tag{9.10}$$

$$= \quad \text{tr } E\left\{z(t)z(t)^T\right\}$$

$$= \quad \frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\omega)F(j\omega)^T d\omega$$

(by Parseval's Theorem)

$$= \quad \|F\|_2^2 = \|F_l(P,K)\|_2^2 \tag{9.11}$$

Thus, by minimizing the $\mathcal{H}_2$ norm, the output (or error) power of the generalized system, due to a unit intensity white noise input, is minimized; we are minimizing the root-mean-square (rms) value of z.

9.3.1 LQG: a special $\mathcal{H}_2$ optimal controller [9.3.3]

$$\begin{aligned} \dot{x} &= Ax + Bu + w_d \quad (9.12) \\ y &= Cx + w_n \quad (9.13) \end{aligned}$$

where:

$$E\left\{ \begin{bmatrix} w_d(t) \\ w_n(t) \end{bmatrix} \begin{bmatrix} w_d(\tau)^T & w_n(\tau)^T \end{bmatrix} \right\} = \begin{bmatrix} W & 0 \\ 0 & V \end{bmatrix} \delta(t-\tau) \quad (9.14)$$

The LQG problem is to find $u = K(s)y$ such that

$$J = E\left\{ \lim_{T\to\infty} \frac{1}{T} \int_0^T \left[x^T Q x + u^T R u \right] dt \right\} \quad (9.15)$$

is minimized with $Q = Q^T \geq 0$ and $R = R^T > 0$.

Define:

$$z = \begin{bmatrix} Q^{\frac{1}{2}} & 0 \\ 0 & R^{\frac{1}{2}} \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} \tag{9.16}$$

and represent the stochastic inputs w_d, w_n as

$$\begin{bmatrix} w_d \\ w_n \end{bmatrix} = \begin{bmatrix} W^{\frac{1}{2}} & 0 \\ 0 & V^{\frac{1}{2}} \end{bmatrix} w \tag{9.17}$$

where w is a white noise process of unit intensity. Then the LQG cost function is

$$J = E\left\{ \lim_{T\to\infty} \frac{1}{T} \int_0^T z(t)^T z(t) dt \right\} = \|F_l(P,K)\|_2^2 \tag{9.18}$$

where

$$z(s) = F_l(P,K)w(s) \tag{9.19}$$

and the generalized plant P is given by

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \stackrel{s}{=} \left[\begin{array}{c|cc:c} A & W^{\frac{1}{2}} & 0 & B \\ \hline Q^{\frac{1}{2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & R^{\frac{1}{2}} \\ \hdashline C & 0 & V^{\frac{1}{2}} & 0 \end{array}\right]$$

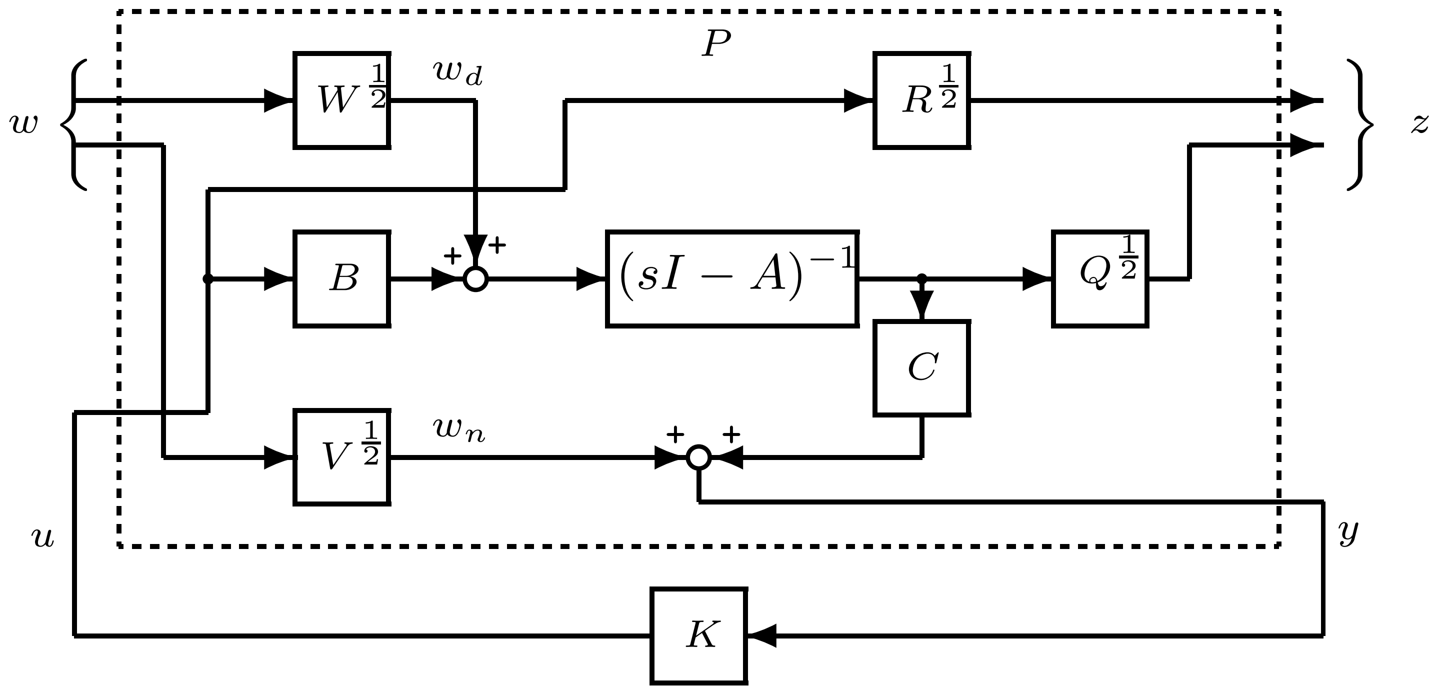


Figure 90: The LQG problem: general control configuration

9.4 $\mathcal{H}_\infty$ optimal control [9.3.4]

With reference to the general control configuration of Figure 89, the standard $\mathcal{H}_\infty$ optimal control problem is to find all stabilizing controllers K which minimize

$$\|F_l(P,K)\|_\infty = \max_\omega \bar{\sigma}(F_l(P,K)(j\omega)) \qquad (9.20)$$

This has a time domain interpretation as the induced (worst-case) 2-norm. Let $z = F_l(P,K)w$, then

$$\|F_l(P,K)\|_\infty = \max_{w(t)\neq 0} \frac{\|z(t)\|_2}{\|w(t)\|_2} \qquad (9.21)$$

where $\|z(t)\|_2 = \sqrt{\int_0^\infty \sum_i |z_i(t)|^2 dt}$ is the 2-norm of the vector signal.

It is often computationally (and theoretically) simpler to design a sub-optimal one (i.e. one close to the optimal controller in the sense of the $\mathcal{H}_\infty$ norm). Let $\gamma_{\min}$ be the minimum value of $\|F_l(P,K)\|_\infty$ over all stabilizing controllers K. Then the $\mathcal{H}_\infty$ sub-optimal control problem is: given a $\gamma > \gamma_{\min}$, find all stabilizing controllers K such that

$$\|F_l(P,K)\|_\infty < \gamma$$

9.4.1 Mixed-sensitivity $\mathcal{H}_\infty$ control [9.3.5]

To optimize performance, minimize $\|w_1 S\|_\infty$,

to minimize control inputs, minimize $\|w_2 KS\|_\infty$.

Compromise:

$$\left\| \begin{bmatrix} w_1 S \\ w_2 KS \end{bmatrix} \right\|_\infty \tag{9.22}$$

General setting: disturbance d as a single exogenous input, error signal $z = \begin{bmatrix} z_1^T & z_2^T \end{bmatrix}^T$, where $z_1 = W_1 y$ and $z_2 = -W_2 u$, (91).

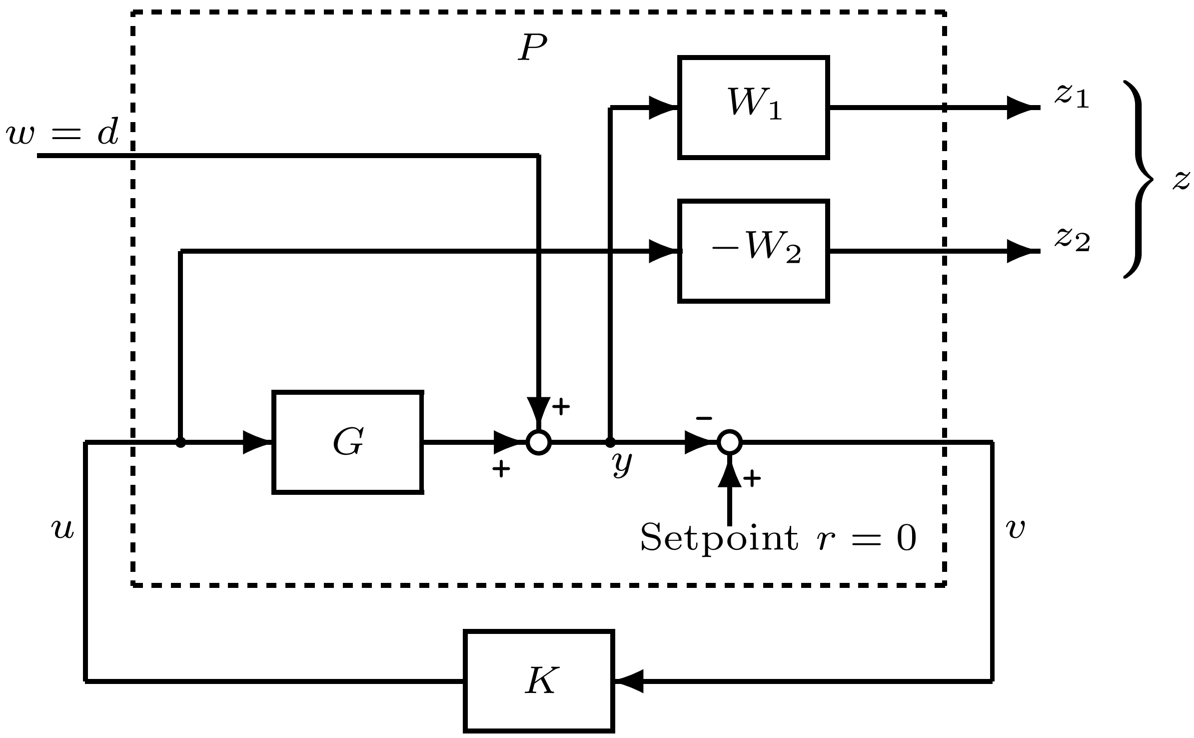


Figure 91: S/KS mixed-sensitivity optimization in standard form (regulation)

Thus $z_1 = W_1 S w$ and $z_2 = W_2 K S w$ and:

$$P_{11} = \begin{bmatrix} W_1 \\ 0 \end{bmatrix} \quad P_{12} = \begin{bmatrix} W_1 G \\ -W_2 \end{bmatrix} \tag{9.23}$$
$$P_{21} = -I \qquad P_{22} = -G$$

where the partitioning is such that

$$\begin{bmatrix} z_1 \\ z_2 \\ \hdashline v \end{bmatrix} = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} w \\ u \end{bmatrix} \tag{9.24}$$

and

$$F_l(P, K) = \begin{bmatrix} W_1 S \\ W_2 K S \end{bmatrix} \tag{9.25}$$

Another useful mixed sensitivity optimization problem, is to find a stabilizing controller which minimizes

$$\left\| \begin{bmatrix} W_1 S \\ W_2 T \end{bmatrix} \right\|_\infty \tag{9.26}$$

The S/T mixed-sensitivity minimization problem can be put into the standard control configuration as shown in Figure 92.

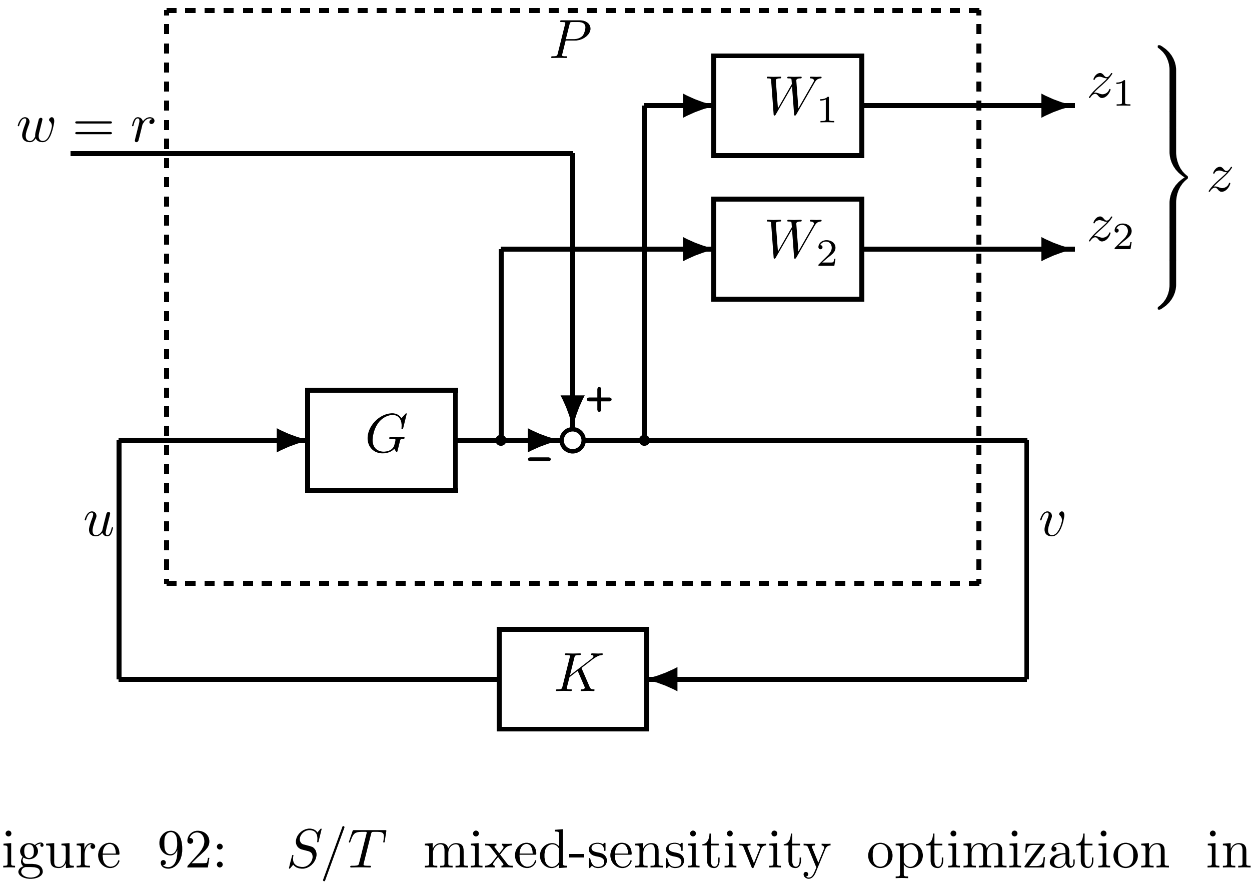


Figure 92: S/T mixed-sensitivity optimization in standard form

$$
\begin{aligned}
P_{11} &= \begin{bmatrix} W_1 \\ 0 \end{bmatrix} & P_{12} &= \begin{bmatrix} -W_1 G \\ W_2 G \end{bmatrix} \\
P_{21} &= I & P_{22} &= -G
\end{aligned}
\tag{9.27}
$$

9.5 Helicopter control [12.2]

9.5.1 Problem description [12.2.1]

Objective: reduce the effects of atmospheric turbulence on helicopters. The reduction of the effects of gusts is very important in reducing a pilot's workload, and enables aggressive maneuvers to be carried out in poor weather conditions. Also, as a consequence of decreased buffeting, the airframe and component lives are lengthened and passenger comfort is increased.

9.5.2 The helicopter model [12.2.2]

The aircraft model used in our work is representative of the Westland Lynx, a twin-engined multi-purpose military helicopter, approximately 9000 lbs gross weight, with a four-blade semi-rigid main rotor. The unaugmented aircraft is unstable, and exhibits many of the cross-couplings characteristic of a single main-rotor helicopter.

The equations governing the motion of the helicopter are complex and difficult to formulate with high levels of precision.

State	Description
θ	Pitch attitude
ϕ	Roll attitude
p	Roll rate (body-axis)
q	Pitch rate (body-axis)
ξ	Yaw rate
v_x	Forward velocity
v_y	Lateral velocity
v_z	Vertical velocity

Table 7: Helicopter state vector

The starting point for this study was to obtain an eighth-order differential equation modelling the small-perturbation rigid motion of the aircraft about hover.

Controlled outputs:

- Heave velocity $\dot{H}$
- Pitch attitude θ
- Roll attitude ϕ
- Heading rate $\dot{\psi}$

$\Big\}\ y_1$

together with two additional (body-axis) measurements

- Roll rate p
- Pitch rate q

$\Big\}\ y_2$

The controller (or pilot in manual control) generates four blade angle demands which are effectively the helicopter inputs. The blade angles are

- main rotor collective
- longitudinal cyclic
- lateral cyclic
- tail rotor collective

$\Big\}\ u$

Note: dynamics unstable, non-minimum phase.

Goal

full-authority controllers: the controller has total control over the blade angles of the main and tail rotors, and is interposed between the pilot and the actuation system.

One degree-of-freedom controllers as shown in Figure 93 are to be designed.

Notice that in the standard one degree-of-freedom configuration the pilot reference commands r_1 are augmented by a zero vector because of the rate feedback signals. These zeros indicate that there are no *a priori* performance specifications on $y_2 = [\, p \quad q \,]^T$.

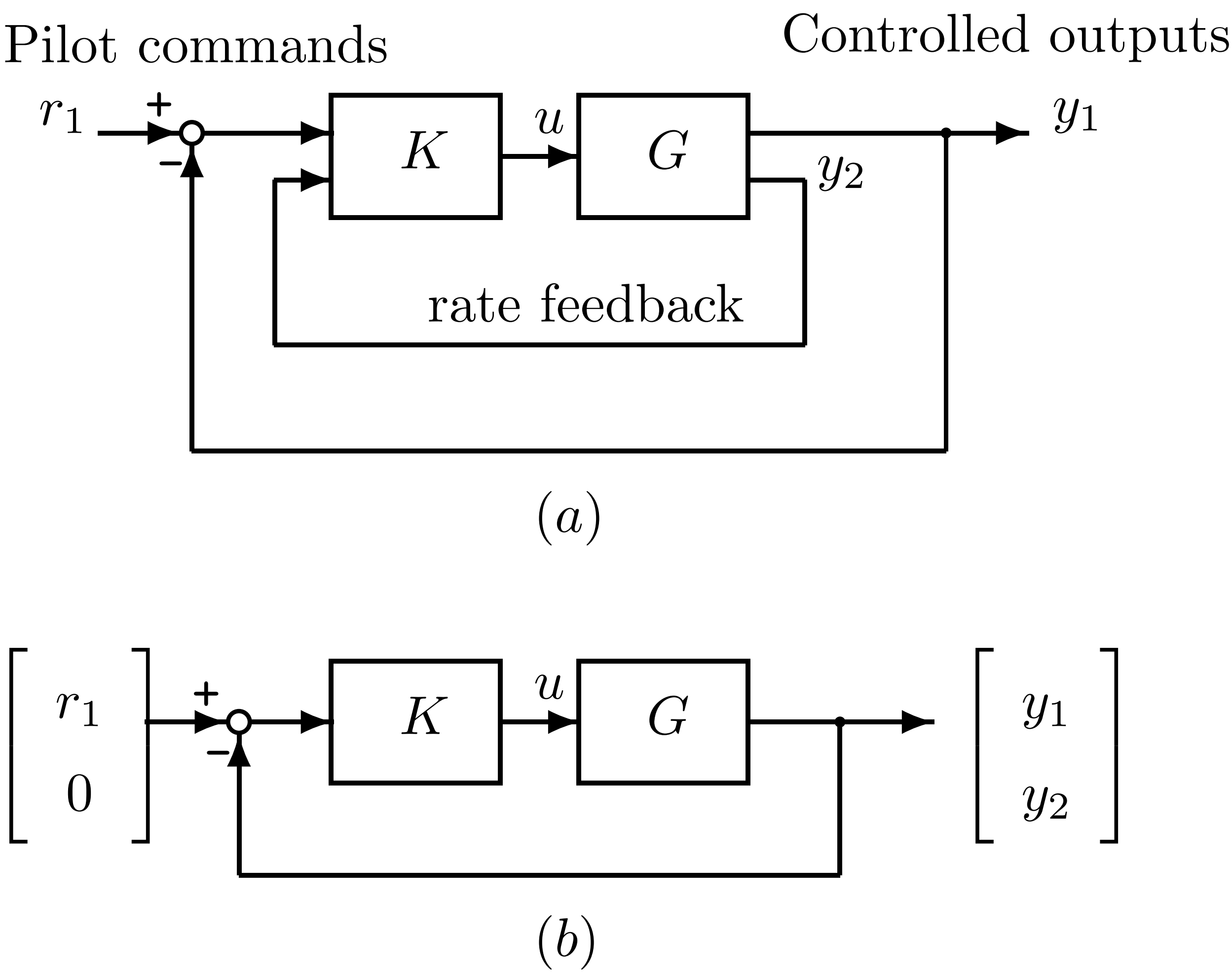


Figure 93: Helicopter control structure (a) as implemented, (b) in the standard one degree-of-freedom configuration

9.5.3 $\mathcal{H}_\infty$ mixed-sensitivity design [12.2.3]

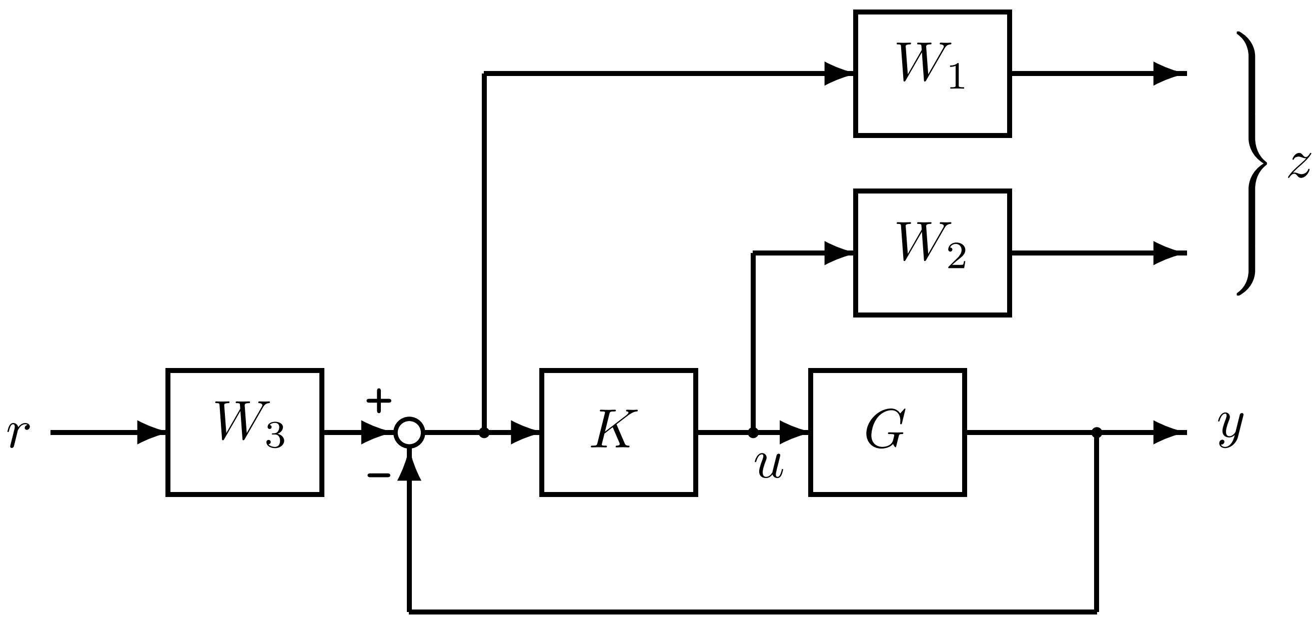


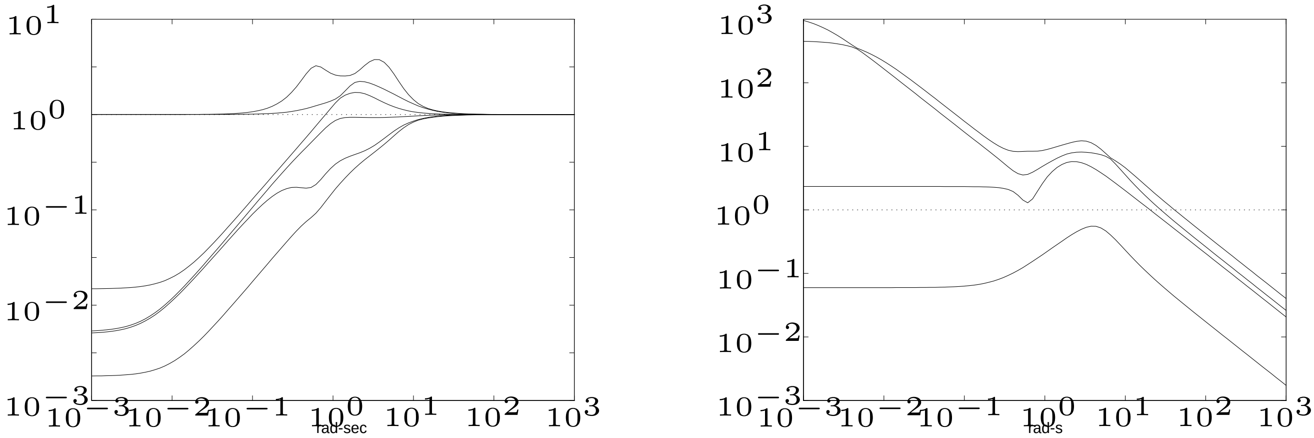
Figure 94: S/KS mixed-sensitivity minimization

Find a stabilizing controller K to minimize the cost function

$$\left\| \begin{bmatrix} W_1 S W_3 \\ W_2 K S W_3 \end{bmatrix} \right\|_\infty \tag{9.28}$$

(This cost was considered by Yue and Postlethwaite (1990) in the context of helicopter control. Their controller was successfully tested on a piloted flight simulator at DRA Bedford)

$$
\begin{aligned}
W_1 &= diag\left\{0.5\frac{s+12}{s+0.012},\ 0.89\frac{s+2.81}{s+0.005},\right. \\
&\quad 0.89\frac{s+2.81}{s+0.005}, 0.5\frac{s+10}{s+0.01}, \\
&\quad \left.\frac{2s}{(s+4)(s+4.5)},\ \frac{2s}{(s+4)(s+4.5)}\right\} && (9.29) \\
W_2 &= 0.5\frac{s+0.0001}{s+10}I_4 && (9.30) \\
W_3 &= diag\{1,\ 1,\ 1,\ 1,\ 0.1,\ 0.1\} && (9.31)
\end{aligned}
$$



(a) S (b) KS

Figure 95: Singular values of S and KS (S/KS design)

A MATRIX THEORY AND NORMS

A.1 Basics

$$\text{Complex Matrix} \quad A \in \mathcal{C}^{l\times m}$$
$$\text{Real Matrix} \quad A \in \mathcal{R}^{l\times m}$$

elements $a_{ij} = \text{Re } a_{ij} + j \text{ Im } a_{ij}$

$$\begin{aligned} l &= \text{number of rows} \\ &= \text{"outputs" when viewed as an } \textit{operator} \\ m &= \text{number of columns} \\ &= \text{"inputs" when viewed as an } \textit{operator} \end{aligned}$$

- A^T = transpose of A (with elements a_{ji}),
- $\bar{A}$ = conjugate of A (with elements Re $a_{ij} - j$ Im a_{ij}),
- $A^H \triangleq \bar{A}^T$ = conjugate transpose (or Hermitian adjoint) (with elements Re $a_{ji} - j$Im a_{ji}),

Matrix inverse:

$$A^{-1} = \frac{\text{adj}A}{\det A} \tag{A.1}$$

where adj A is the adjugate (or "classical adjoint") of A which is the transposed matrix of cofactors c_{ij} of A,

$$c_{ij} = [\text{adj}A]_{ji} \stackrel{\Delta}{=} (-1)^{i+j} \det A^{ij} \tag{A.2}$$

Here A^{ij} is a submatrix formed by deleting row i and column j of A.

Example:

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}; \quad \det A = a_{11}a_{22} - a_{12}a_{21}$$

$$A^{-1} = \frac{1}{\det A} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix} \tag{A.3}$$

Some matrix identities:

$$(AB)^T = B^T A^T, \quad (AB)^H = B^H A^H \tag{A.4}$$

Assuming the inverses exist,

$$(AB)^{-1} = B^{-1} A^{-1} \tag{A.5}$$

A is symmetric if $A^T = A$,

A is Hermitian if $A^H = A$,

A Hermitian matrix is positive definite if $x^H A x > 0$ for any non-zero vector x.

A.1.1 Some determinant identities

The determinant is defined as

$\det A = \sum_{i=1}^{n} a_{ij} c_{ij}$ (expansion along column j) or

$\det A = \sum_{j=1}^{n} a_{ij} c_{ij}$ (expansion along row i),

where c_{ij} is the ij'th cofactor given in (A.2).

1. Let A_1 and A_2 be square matrices of the same dimension. Then

$$\det(A_1 A_2) = \det(A_2 A_1) = \det A_1 \cdot \det A_2 \quad \text{(A.6)}$$

2. Let c be a complex scalar and A an $n \times n$ matrix. Then

$$\det(cA) = c^n \det(A) \quad \text{(A.7)}$$

3. Let A be a non-singular matrix. Then

$$\det A^{-1} = 1/\det A \quad \text{(A.8)}$$

4. Let A_1 and A_2 be matrices of compatible dimensions such that both matrices $A_1 A_2$ and $A_2 A_1$ are square (but A_1 and A_2 need not themselves be square). Then

$$\det(I + A_1 A_2) = \det(I + A_2 A_1) \quad \text{(A.9)}$$

(A.9) is useful in the field of control because it yields $\det(I + GK) = \det(I + KG)$.

5.

$$\det \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} = \det \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix} = \det(A_{11}) \cdot \det(A_{22}) \tag{A.10}$$

6. **Schur's formula** for the determinant of a partitioned matrix:

$$\det \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \det(A_{11}) \cdot \det(A_{22} - A_{21}A_{11}^{-1}A_{12}) = \det(A_{22}) \cdot \det(A_{11} - A_{12}A_{22}^{-1}A_{21}) \tag{A.11}$$

where it is assumed that A_{11} and/or A_{22} are non-singular.

A.2 Eigenvalues and eigenvectors

Definition

Eigenvalues and eigenvectors. Let A be a square $n \times n$ matrix. The eigenvalues λ_i, $i = 1, \ldots, n$, are the n solutions to the n'th order characteristic equation

$$\det(A - \lambda I) = 0 \tag{A.12}$$

The (right) eigenvector t_i corresponding to the eigenvalue λ_i is the nontrivial solution ($t_i \neq 0$) to

$$(A - \lambda_i I)t_i = 0 \quad \Leftrightarrow \quad At_i = \lambda_i t_i \tag{A.13}$$

The corresponding left eigenvectors q_i satisfy

$$q_i^H (A - \lambda_i I) = 0 \quad \Leftrightarrow \quad q_i^H A = \lambda_i q_i^H \tag{A.14}$$

When we just say *eigenvector* we mean the right eigenvector.

Remarks

- The left eigenvectors of A are the (right) eigenvectors of A^H.
- $\rho(A) \stackrel{\Delta}{=} \max_i |\lambda_i(A)|$ is the *spectral radius* of A.
- Eigenvectors corresponding to distinct eigenvalues are always linearly independent.
- Define

$$T = \{t_1, t_2, \ldots, t_n\}; \quad \Lambda = \text{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_n\} \tag{A.15}$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are distinct.

Then we may then write (A.13) in the following form

$$AT = T\Lambda \tag{A.16}$$

From (A.16) we then get that the eigenvector matrix diagonalizes A in the following manner

$$\Lambda = T^{-1}AT \tag{A.17}$$

A.2.1 Eigenvalue properties

1. $\mathrm{tr}A = \sum_i \lambda_i$ where $\mathrm{tr}A$ is the trace of A (sum of the diagonal elements).
2. $\det A = \prod_i \lambda_i$.
3. The eigenvalues of an upper or lower triangular matrix are equal to the diagonal elements of the matrix.
4. For a real matrix the eigenvalues are either real, or occur in complex conjugate pairs.
5. A and A^T have the same eigenvalues (but in general different eigenvectors).
6. The eigenvalues of A^{-1} are $1/\lambda_1, \ldots, 1/\lambda_n$.
7. The matrix $A + cI$ has eigenvalues $\lambda_i + c$.
8. The matrix cA^k where k is an integer has eigenvalues $c\lambda_i^k$.
9. Consider the $l \times m$ matrix A and the $m \times l$ matrix B. Then the $l \times l$ matrix AB and the $m \times m$ matrix BA have the same non-zero eigenvalues.

10. Eigenvalues are invariant under similarity transformations, that is, A and DAD^{-1} have the same eigenvalues.

11. The same eigenvector matrix diagonalizes the matrix A and the matrix $(I + A)^{-1}$.

12. *Gershgorin's theorem.* The eigenvalues of the $n \times n$ matrix A lie in the union of n circles in the complex plane, each with centre a_{ii} and radius $r_i = \sum_{j \neq i} |a_{ij}|$ (sum of off-diagonal elements in row i). They also lie in the union of n circles, each with centre a_{ii} and radius $r_i' = \sum_{j \neq i} |a_{ji}|$ (sum of off-diagonal elements in column i).

13. A symmetric matrix is positive definite if and only if all its eigenvalues are real and positive.

From the above we have, for example, that

$$\lambda_i(S) = \lambda_i((I + L)^{-1}) = \frac{1}{\lambda_i(I + L)} = \frac{1}{1 + \lambda_i(L)} \tag{A.18}$$

A.3 Singular Value Decomposition

Definition: Unitary matrix. A (complex) matrix U is unitary if

$$U^H = U^{-1} \tag{A.19}$$

Note:

$$\|\lambda(U)\| = 1 \;\; \forall i$$

Definition: SVD. Any complex $l \times m$ matrix A may be factorized into a singular value decomposition

$$A = U \Sigma V^H \tag{A.20}$$

where the $l \times l$ matrix U and the $m \times m$ matrix V are unitary, and the $l \times m$ matrix Σ contains a diagonal matrix Σ_1 of real, non-negative singular values, σ_i, arranged in a descending order as in

$$\Sigma = \begin{bmatrix} \Sigma_1 \\ 0 \end{bmatrix}; \quad l \geq m \tag{A.21}$$

or

$$\Sigma = [\, \Sigma_1 \quad 0 \,]; \quad l \leq m \tag{A.22}$$

where

$$\Sigma_1 = \text{diag}\{\sigma_1, \sigma_2, \ldots, \sigma_k\}; \quad k = \min(l, m) \tag{A.23}$$

and

$$\bar{\sigma} \triangleq \sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_k \triangleq \underline{\sigma} \tag{A.24}$$

- The unitary matrices U and V form orthonormal bases for the column (output) space and the row (input) space of A. The column vectors of V, denoted v_i, are called *right* or *input singular vectors* and the column vectors of U, denoted u_i, are called *left* or *output singular vectors*. We define $\bar{u} \equiv u_1$, $\bar{v} \equiv v_1$, $\underline{u} \equiv u_k$ and $\underline{v} \equiv v_k$.
- SVD is not unique since $A = U' \Sigma {V'}^H$, where $U' = US$, $V' = VS$, $S = \mathrm{diag}\{e^{j\theta_i}\}$ and θ_i is any real number, is also an SVD of A. However, the singular values, σ_i, *are* unique.

$$\sigma_i(A) = \sqrt{\lambda_i(A^H A)} = \sqrt{\lambda_i(A A^H)} \qquad \text{(A.25)}$$

The columns of U and V are unit eigenvectors of AA^H and $A^H A$, respectively. To derive (A.25) write

$$\begin{aligned} AA^H &= (U\Sigma V^H)(U\Sigma V^H)^H = (U\Sigma V^H)(V\Sigma^H U^H) \\ &= U\Sigma\Sigma^H U^H \end{aligned} \tag{A.26}$$

or equivalently since U is unitary and satisfies $U^H = U^{-1}$ we get

$$(AA^H)U = U\Sigma\Sigma^H \tag{A.27}$$

$\Rightarrow$ U is the matrix of eigenvectors of AA^H and $\{\sigma_i^2\}$ are its eigenvalues. Similarly, V is the matrix of eigenvectors of $A^H A$.

Definition: The **rank** of a matrix is equal to the number of non-zero singular values of the matrix. Let rank$(A) = r$, then the matrix A is called rank deficient if $r < k = \min(l, m)$, and we have singular values $\sigma_i = 0$ for $i = r+1, \ldots k$. A rank deficient square matrix is a singular matrix (non-square matrices are always singular).

A.3.3 SVD of a matrix inverse

Provided the $m \times m$ matrix A is non-singular

$$A^{-1} = V\Sigma^{-1}U^H \tag{A.28}$$

Let $j = m - i + 1$. Then it follows from (A.28) that

$$\begin{aligned} \sigma_i(A^{-1}) &= 1/\sigma_j(A), && \text{(A.29)} \\ u_i(A^{-1}) &= v_j(A), && \text{(A.30)} \\ v_i(A^{-1}) &= u_j(A) && \text{(A.31)} \end{aligned}$$

and in particular

$$\bar{\sigma}(A^{-1}) = 1/\underline{\sigma}(A) \tag{A.32}$$

A.3.4 Singular value inequalities

$$\underline{\sigma}(A) \le |\lambda_i(A)| \le \bar{\sigma}(A) \tag{A.33}$$

$$\bar{\sigma}(A^H) = \bar{\sigma}(A) \quad \text{and} \quad \bar{\sigma}(A^T) = \bar{\sigma}(A) \tag{A.34}$$

$$\bar{\sigma}(AB) \le \bar{\sigma}(A)\bar{\sigma}(B) \tag{A.35}$$

$$\underline{\sigma}(A)\bar{\sigma}(B) \le \bar{\sigma}(AB) \quad \text{or} \quad \bar{\sigma}(A)\underline{\sigma}(B) \le \bar{\sigma}(AB) \tag{A.36}$$

$$\underline{\sigma}(A)\underline{\sigma}(B) \le \underline{\sigma}(AB) \tag{A.37}$$

$$\max\{\bar{\sigma}(A), \bar{\sigma}(B)\} \le \bar{\sigma}\begin{bmatrix} A \\ B \end{bmatrix} \le \sqrt{2}\max\{\bar{\sigma}(A), \bar{\sigma}(B)\} \tag{A.38}$$

$$\bar{\sigma}\begin{bmatrix} A \\ B \end{bmatrix} \le \bar{\sigma}(A) + \bar{\sigma}(B) \tag{A.39}$$

$$\bar{\sigma}\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} = \max\{\bar{\sigma}(A), \bar{\sigma}(B)\} \tag{A.40}$$

$$\sigma_i(A) - \bar{\sigma}(B) \le \sigma_i(A+B) \le \sigma_i(A) + \bar{\sigma}(B) \tag{A.41}$$

Two special cases of (A.41) are:

$$|\bar{\sigma}(A) - \bar{\sigma}(B)| \le \bar{\sigma}(A+B) \le \bar{\sigma}(A) + \bar{\sigma}(B) \tag{A.42}$$

$$\underline{\sigma}(A) - \bar{\sigma}(B) \le \underline{\sigma}(A+B) \le \underline{\sigma}(A) + \bar{\sigma}(B) \tag{A.43}$$

(A.43) yields

$$\underline{\sigma}(A) - 1 \le \underline{\sigma}(I+A) \le \underline{\sigma}(A) + 1 \tag{A.44}$$

On combining (A.32) and (A.44) we get

$$\underline{\sigma}(A) - 1 \le \frac{1}{\bar{\sigma}(I+A)^{-1}} \le \underline{\sigma}(A) + 1 \tag{A.45}$$

A.4 Condition number

The **condition number** of a matrix is defined as the ratio

$$\gamma(A) = \sigma_1(A)/\sigma_k(A) = \bar{\sigma}(A)/\underline{\sigma}(A) \qquad \text{(A.46)}$$

where $k = \min(l, m)$.

A.5 Norms

Definition

A norm of e (which may be a vector, matrix, signal or system) is a real number, denoted $\|e\|$, that satisfies the following properties:

1. Non-negative: $\|e\| \geq 0$.

2. Positive: $\|e\| = 0 \Leftrightarrow e = 0$ (for semi-norms we have $\|e\| = 0 \Leftarrow e = 0$).

3. Homogeneous: $\|\alpha \cdot e\| = |\alpha| \cdot \|e\|$ for all complex scalars α.

4. Triangle inequality:

$$\|e_1 + e_2\| \leq \|e_1\| + \|e_2\| \tag{A.47}$$

We will consider the norms of four different objects (norms on four different vector spaces):

1. e is a constant vector.

2. e is a constant matrix.

3. e is a time dependent signal, $e(t)$, which at each fixed t is a constant scalar or vector.

4. e is a "system", a transfer function $G(s)$ or impulse response $g(t)$, which at each fixed s or t is a constant scalar or matrix.

A.5.1 Vector norms

General:

$$\|a\|_p = (\sum_i |a_i|^p)^{1/p}; \quad p \geq 1 \tag{A.48}$$

Vector 1-norm (or sum-norm)

$$\|a\|_1 \triangleq \sum_i |a_i| \tag{A.49}$$

Vector 2-norm (Euclidean norm).

$$\|a\|_2 \triangleq \sqrt{\sum_i |a_i|^2} \tag{A.50}$$

$$a^H a = \|a\|_2^2 \tag{A.51}$$

Vector ∞ -norm (or max norm)

$$\|a\|_{\max} \equiv \|a\|_\infty \stackrel{\Delta}{=} \max_i |a_i| \tag{A.52}$$

$$\|a\|_{\max} \leq \|a\|_2 \leq \sqrt{m}\ \|a\|_{\max} \tag{A.53}$$

$$\|a\|_2 \leq \|a\|_1 \leq \sqrt{m}\ \|a\|_2 \tag{A.54}$$

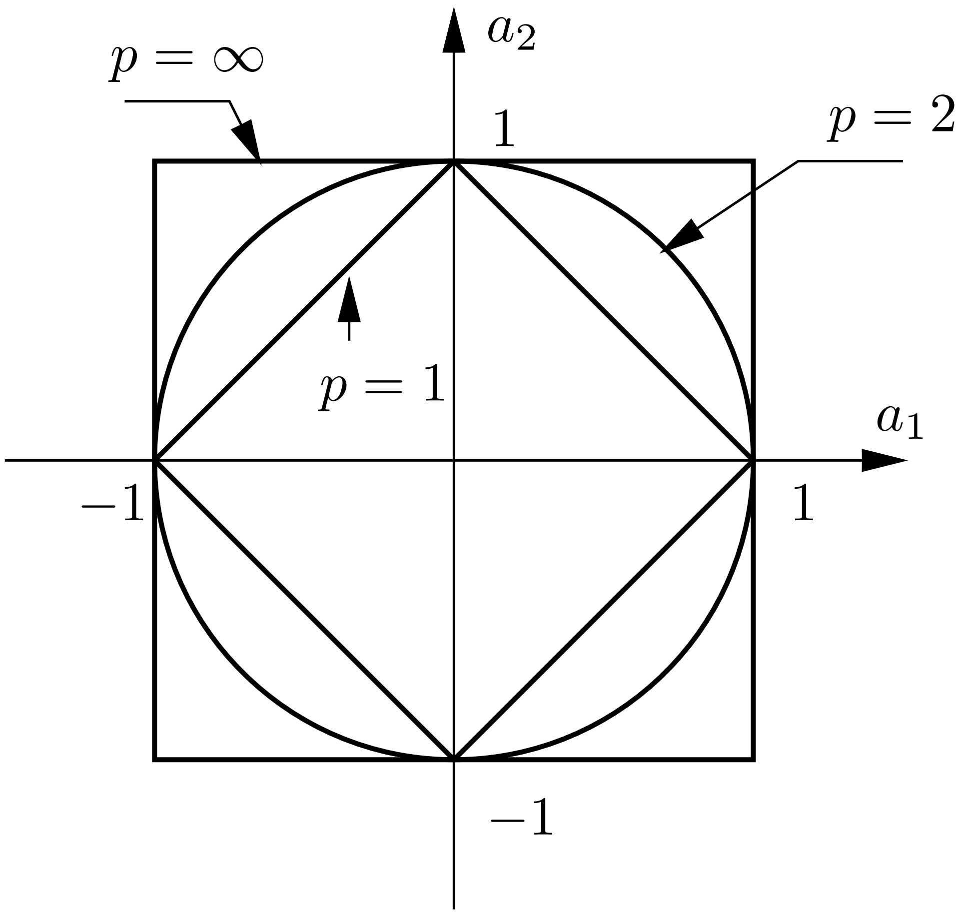


Figure 96: Contours for the vector p-norm, $\|a\|_p = 1$ for $p = 1, 2, \infty$

A.5.2 Matrix norms

Definition

A norm on a matrix $\|A\|$ is a **matrix norm** if, in addition to the four norm properties in Definition A.5, it also satisfies the multiplicative property (also called the consistency condition):

$$\|AB\| \leq \|A\| \cdot \|B\| \tag{A.55}$$

Sum matrix norm.

$$\|A\|_{\text{sum}} = \sum_{i,j} |a_{ij}| \tag{A.56}$$

Frobenius matrix norm (or Euclidean norm).

$$\|A\|_F = \sqrt{\sum_{i,j} |a_{ij}|^2} = \sqrt{\text{tr}(A^H A)} \tag{A.57}$$

Max element norm.

$$\|A\|_{\max} = \max_{i,j} |a_{ij}| \tag{A.58}$$

Not a matrix norm as it does not satisfy (A.55). However note that $\sqrt{lm}\ \|A\|_{\max}$ *is* a matrix norm.

Induced matrix norms

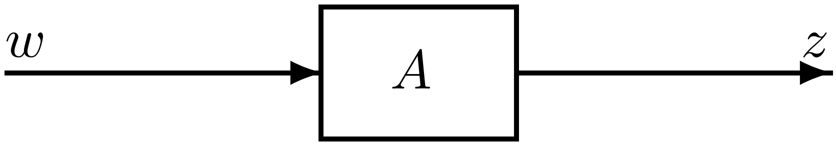


Figure 97: Representation of (A.59)

$$z = Aw \tag{A.59}$$

The *induced norm* is defined as

$$\|A\|_{ip} \triangleq \max_{w \neq 0} \frac{\|Aw\|_p}{\|w\|_p} \tag{A.60}$$

where $\|w\|_p = (\sum_i |w_i|^p)^{1/p}$ denotes the vector p-norm.

- We are looking for a direction of the vector w such that the ratio $\|z\|_p / \|w\|_p$ is maximized.
- The induced norm gives the largest possible "amplifying power" of the matrix. Equivalent definition is:

$$\|A\|_{ip} = \max_{\|w\|_p \leq 1} \|Aw\|_p = \max_{\|w\|_p = 1} \|Aw\|_p \tag{A.61}$$

$$\|A\|_{i1} = \max_j(\sum_i |a_{ij}|)$$
"maximum column sum"

$$\|A\|_{i\infty} = \max_i(\sum_j |a_{ij}|) \tag{A.62}$$
"maximum row sum"

$$\|A\|_{i2} = \bar{\sigma}(A) = \sqrt{\rho(A^H A)}$$
"singular value or spectral norm"

Theorem 14 *All induced norms $\|A\|_{ip}$ are matrix norms and thus satisfy the multiplicative property*

$$\|AB\|_{ip} \leq \|A\|_{ip} \cdot \|B\|_{ip} \tag{A.63}$$

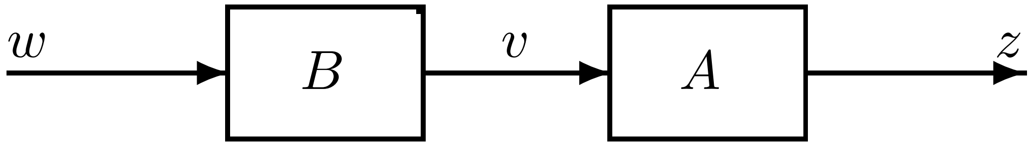


Figure 98:

Implications of the multiplicative property

1. Choose B to be a vector, i.e $B = w$.

$$\|Aw\| \leq \|A\| \cdot \|w\| \qquad \text{(A.64)}$$

   The "matrix norm $\|A\|$ is compatible with its corresponding vector norm $\|w\|$".

2. From (A.64)

$$\|A\| \geq \max_{w \neq 0} \frac{\|Aw\|}{\|w\|} \qquad \text{(A.65)}$$

   For induced norms we have equality in (A.65) $\|A\|_F \geq \bar{\sigma}(A)$ follows since $\|w\|_F = \|w\|_2$.

3. Choose both $A = z^H$ and $B = w$ as vectors. Then we derive the Cauchy-Schwarz inequality

$$|z^H w| \leq \|z\|_2 \cdot \|w\|_2 \qquad \text{(A.66)}$$

A.5.3 The spectral radius $\rho(A)$

$$\rho(A) = \max_i |\lambda_i(A)| \tag{A.67}$$

Not a norm!

Example:

$$A_1 = \begin{bmatrix} 1 & 0 \\ 10 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 10 \\ 0 & 1 \end{bmatrix} \tag{A.68}$$

$$\rho(A_1) = 1, \qquad \rho(A_2) = 1 \tag{A.69}$$

but

$$\rho(A_1 + A_2) = 12, \qquad \rho(A_1 A_2) = 101.99 \tag{A.70}$$

Theorem 15 *For any matrix norm (and in particular for any induced norm)*

$$\rho(A) \leq \|A\| \tag{A.71}$$

A.5.4 Some matrix norm relationships

$$\bar{\sigma}(A) \leq \|A\|_F \leq \sqrt{\min(l,m)}\ \bar{\sigma}(A) \tag{A.72}$$

$$\|A\|_{\max} \leq \bar{\sigma}(A) \leq \sqrt{lm}\ \|A\|_{\max} \tag{A.73}$$

$$\bar{\sigma}(A) \leq \sqrt{\|A\|_{i1}\|A\|_{i\infty}} \tag{A.74}$$

$$\frac{1}{\sqrt{m}}\|A\|_{i\infty} \leq \bar{\sigma}(A) \leq \sqrt{l}\ \|A\|_{i\infty} \tag{A.75}$$

$$\frac{1}{\sqrt{l}}\|A\|_{i1} \leq \bar{\sigma}(A) \leq \sqrt{m}\ \|A\|_{i1} \tag{A.76}$$

$$\max\{\bar{\sigma}(A), \|A\|_F, \|A\|_{i1}, \|A\|_{i\infty}\} \leq \|A\|_{\text{sum}} \tag{A.77}$$

- All these norms, except $\|A\|_{\max}$, are matrix norms and satisfy (A.55).
- The inequalities are tight.
- $\|A\|_{\max}$ can be used as a simple estimate of $\bar{\sigma}(A)$.

The Frobenius norm and the maximum singular value (induced 2-norm) are invariant with respect to unitary transformations.

$$\|U_1 A U_2\|_F = \|A\|_F \tag{A.78}$$

$$\bar{\sigma}(U_1 A U_2) = \bar{\sigma}(A) \tag{A.79}$$

Relationship between Frobenius norm and singular values, $\sigma_i(A)$

$$\|A\|_F = \sqrt{\sum_i \sigma_i^2(A)} \tag{A.80}$$

Perron-Frobenius theorem

$$\min_D \|DAD^{-1}\|_{i1} = \min_D \|DAD^{-1}\|_{i\infty} = \rho(|A|) \tag{A.81}$$

where D is a diagonal "scaling" matrix.

Here:

- $|A|$ denotes the matrix A with all its elements replaced by their magnitudes.
- $\rho(|A|) = \max_i |\lambda_i(|A|)|$ is the Perron root (Perron-Frobenius eigenvalue). Note: $\rho(A) \leq \rho(|A|)$

A.5.5 Matrix and vector norms in MATLAB

$\bar{\sigma}(A) = \|A\|_{i2}$	`norm(A,2)` or `max(svd(A))`
$\|A\|_{i1}$	`norm(A,1)`
$\|A\|_{i\infty}$	`norm(A,'inf')`
$\|A\|_F$	`norm(A,'fro')`
$\|A\|_{\mathrm{sum}}$	`sum (sum(abs(A)))`
$\|A\|_{\max}$	`max(max(abs(A)))` (which is not a matrix norm)

$\rho(A)$	`max(abs(eig(A)))`
$\rho(\|A\|)$	`max(eig(abs(A)))`
$\gamma(A) = \bar{\sigma}(A)/\underline{\sigma}(A)$	`cond(A)`

For vectors:

$\|a\|_1$	`norm(a,1)`
$\|a\|_2$	`norm(a,2)`
$\|a\|_{\max}$	`norm(a,'inf')`

A.5.6 Signal norms

Contrary to spatial norms (vector and matrix norms), choice of temporal norm makes big difference for signals.

Example:

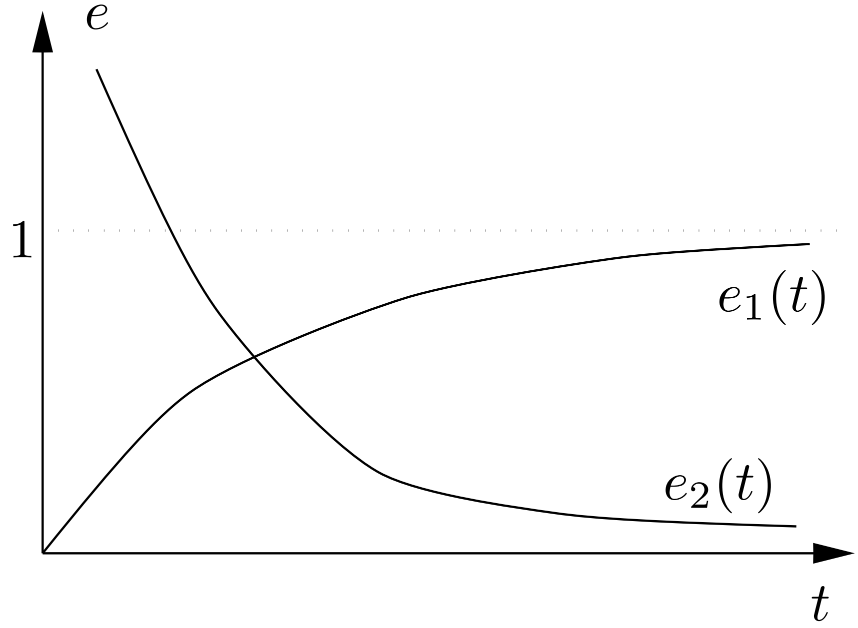


Figure 99: Signals with entirely different 2-norms and ∞-norms.

$$
\begin{aligned}
\|e_1(t)\|_\infty = 1, \quad & \|e_1(t)\|_2 = \infty \\
\|e_2(t)\|_\infty = \infty, \quad & \|e_2(t)\|_2 = 1
\end{aligned}
\tag{A.82}
$$

Compute norm in two steps:

1. "Sum up" the channels at a given time or frequency using a vector norm.
2. "Sum up" in time or frequency using a temporal norm.

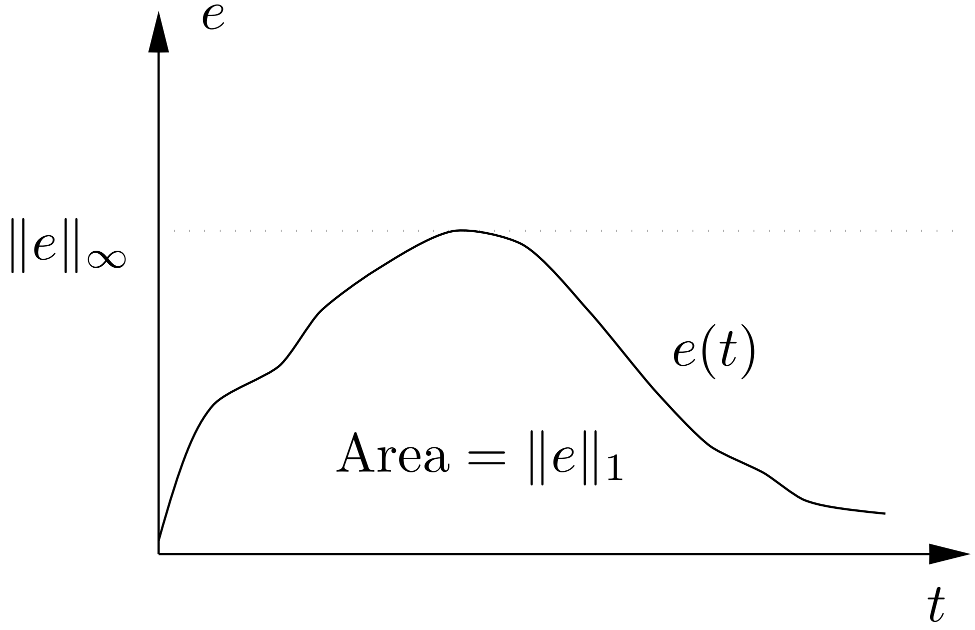


Figure 100: Signal 1-norm and ∞-norm.

General:

$$l_p \text{ norm:} \quad \|e(t)\|_p = \left(\int_{-\infty}^{\infty} \sum_i |e_i(\tau)|^p d\tau \right)^{1/p} \tag{A.83}$$

1-norm in time (integral absolute error (IAE), see Figure 100):

$$\|e(t)\|_1 = \int_{-\infty}^{\infty} \sum_i |e_i(\tau)| d\tau \tag{A.84}$$

2-norm in time (quadratic norm, integral square error (ISE), "energy" of signal):

$$\|e(t)\|_2 = \sqrt{\int_{-\infty}^{\infty} \sum_i |e_i(\tau)|^2 d\tau} \tag{A.85}$$

∞-norm in time (peak value in time, see Figure 100):

$$\|e(t)\|_\infty = \max_\tau \left(\max_i |e_i(\tau)| \right) \tag{A.86}$$

Power-norm or RMS-norm (semi-norm since it does not satisfy property 2)

$$\|e(t)\|_{\text{pow}} = \lim_{T\to\infty} \sqrt{\frac{1}{2T} \int_{-T}^{T} \sum_i |e_i(\tau)|^2 d\tau} \tag{A.87}$$